Exponential Decay for a Transmission-Contact Problem with Frictional Damping

Eugenio Cabanillas Lapa *
Instituto de Investigación, Facultad de Ciencias Matemáticas-UNMSM, Lima-Perú
(Received 27 November 2011, accepted 21 January 2012)

Abstract: In this article we study the evolution of displacement in a body constituted by two different types of materials: one part is simply elastic while the other has a frictional damping and may come into contact with a rigid foundation. Under this condition we have a transmission-contact problem. We prove that the solution of the problem decays exponentially to zero as time goes to infinity.

Keywords: Contact problem; exponential decay; frictional damping
M.S Classification A: 35B40, 35L70, 45K05

1 Introduction

In this work we study the existence and the asymptotic behavior of weak solutions of the system

$$\begin{align*}
\rho_1 u_{tt} - bu_{xx} &= 0 \quad \text{in } [0, L_0] \times \mathbb{R}^+, \\
\rho_2 v_{tt} - av_{xx} + \alpha v_t &= 0 \quad \text{in } [L_0, L] \times \mathbb{R}^+, \\
u(0, t) &= 0, \\
v(L_0, t) &= v(L_0, t), \quad bu_x(L_0, t) = av_x(L_0, t), \quad t > 0, \\
(v(L, t) - g)v_x(L, t) &= 0, \\
u(x, 0) &= u^0(x), \quad u_t(x, 0) = u^1(x), \quad x \in [0, L_0], \\
v(x, 0) &= v^0(x), \quad v_t(x, 0) = v^1(x), \quad x \in [L_0, L],
\end{align*}$$

(1)

where $\rho_1, \rho_2$ are different densities of the material; $a, b$ are the elastic coefficients, $\alpha, g$ are positive constants and $0 < L_0 < L$.

Equations (1) model the evolution of displacement in a elastic body consisting of two different types of materials one of them is simply elastic while, the other is subject to external forces and that may come into contact with the rigid foundation. The mathematical model which deals with the above situation is called transmission-contact problem. Here we are interested to study the resulting properties of a mixed material when one of its components is conservative and the other has a frictional damping and that come into contact with a rigid obstacle fixed at a distance $g$ from the end $x = L$. The main question is about the asymptotic behavior; we may ask whether the sole dissipation produced by the frictional part is strong enough to produce an uniform rate of decay.

The transmission problem have been a subject of intense studies from different mathematical points of view. The transmission problem to hyperbolic equations was studied by Dautray and Lions [5] who proved existence and regularity of solutions for the linear problem. While in Lions [10] was proved the exact controllability. Later, Lagnese [11] extended this result; he showed the exact controllability for a class hyperbolic systems which include the transmission problem for homogeneous anisotropic materials. In [4] Cabanillas Lapa and Muñoz Rivera proved exponential decay for the system with time dependent coefficients. The exact controllability for the plate equations was proved by Liu and Williams [11]. For the memory condition on the boundary we can cite the works in Andrade, Fatori and Muñoz Rivera [1], Muñoz Rivera and Portillo Oquendo [15], Bae [3] and references therein. They used multiplier techniques and compactness arguments to...
get exponential stability. Also, concerning to contact problems related to the evolution equations, we have very important results (see e.g. [2, 6, 13, 14, 16, 17] among others). They used perturbation techniques and the construction of suitable Lyapunov functionals to obtain exponential and polynomial decay. Some kind of Signorini transmission problem have been studied in [18]. Here, the authors used an iterative method to solve a variational inequality associated to the problem. The goal of this paper is to show that the solution of the transmission-contact problem decays exponentially to zero as time goes to infinity, no matter how small is the difference $L - L_0$. The main difficulties are that we have a more complicated situation involving transmission condition in $x = L_0$ and Signorini’s contact condition in $x = L$, the dissipation only works in $[L_0, L]$ and we need estimates over the whole domain $[0, L]$. The situation for the variational inequalities arising in transmission-contact problem is worse because we lead with weak solutions. So, we have neither regularity nor nice boundary conditions. We overcome this problem combining arguments of [4] and [13] and introducing suitable multipliers which allow us to control the energy only estimating over $[L_0, L]$.

The rest of this article is developed as follows. In section 2 we present notations, preliminaries concepts, weak formulation and statement of the existence result for the penalized problem. In section 3 we obtain the solution of (1) as limit of solutions of the penalized problem. Finally, in section 4 we prove the exponential stability of the energy associated with the system.

![Figure 1: Schematic of the elastic body consisting of two different types of materials and the rigid foundation](image)

2 Penalized problem

Let us the following notations

$$V = \{(w, z) \in H^1(0, L_0) \times H^1(L_0, L) : w(L_0) = z(L_0), w(0) = 0\}$$

$$K = \{(w, z) \in V : z(L) \leq g\}.$$

To obtain the solution of penalized problem, we firstly define what we mean by a weak solution to (1).

**Definition 2.1** We say that the pair $\{u(x, t), v(x, t)\}$ is a weak solution of (1) when

$$\{u, v\} \in L^\infty(0, T; K) \cap W^{1,\infty}(0, T; L^2(0, L_0) \times L^2(L_0, L))$$

$$\cap C^{1,\infty}(0, T; H^{-1/2}(0, L_0) \times H^{-1/2}(L_0, L))$$

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and satisfies

\[-\rho_1 \int_0^{L_0} u^1(x)(\varphi(x,0) - u^0)dx + \rho_1 \langle u_t(T), \varphi(T) - u(T) \rangle_{H^{-1/2} \times H^{1/2}} \]

\[-\rho_2 \int_{L_0}^L v^1(x)(\psi(x,0) - v^0)dx + \rho_2 \langle v_t(T), \psi(T) - v(T) \rangle_{H^{-1/2} \times H^{1/2}} \]

\[-\rho_1 \int_0^T \int_{L_0}^L u_t(\varphi_t - u_t) \, dx \, dt - \rho_2 \int_0^T \int_{L_0}^L v_t(\psi_t - v_t) \, dx \, dt + \]

\[b \int_0^T \int_{L_0}^L u_x(\varphi_x - u_x) \, dx \, dt + a \int_0^T \int_{L_0}^L v_x(\psi_x - v_x) \, dx \, dt \]

\[\geq -\alpha \int_0^T \int_{L_0}^L (\psi - v) \, dx \, dt \]

for any \( \varphi, \psi \in L^2(0, T; K) \cap W^{1, \infty}(0, T; L^2(0, L_0) \times L^2(L_0, L)) \)

To show the existence of weak solutions to problem (1), we follows similar ideas as in [6]. We first show existence of a strong solutions to the perturbed system

\[\rho_1 u_{tt} - bu_{xx} = 0 \quad \text{in } ]0, L_0[ \times \mathbb{R}^+, \]

\[\rho_2 v_{tt} - av_{xx} + \alpha v_t = 0 \quad \text{in } ]0, L[ \times \mathbb{R}^+, \]

\[u(0, t) = 0, \quad t > 0, \]

\[u(L_0, t) = v(L_0, t), \quad bu_x(L_0, t) = av_x(L_0, t), \quad t > 0, \]

\[av_x(L, t) = -\frac{1}{\epsilon} [v(L, t) - g]^+ - \epsilon v_t(L, t) \]

\[u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad x \in ]0, L_0[, \]

\[v(x, 0) = v^0(x), \quad v_t(x, 0) = v^1(x), \quad x \in ]L_0, L[. \]

for any \( \epsilon > 0 \). Letting \( \epsilon \to 0^+ \) we will show that the corresponding limit is the weak solution to our original problem.

**Lemma 2.1** Let us denote by \( u^k \) a sequence satisfying

\[u^k \to u \quad \text{weak star in } L^\infty(0, T; H^0(0, L)), \]

\[u_{tt}^k \to u_t \quad \text{weak star in } L^\infty(0, T; H^0(0, L))\]

when \( k \to +\infty \) and \(-1 \leq \alpha < \beta \leq 1.\) Then for any \( r < \beta, \) we have

\[u^k \to u \quad \text{strongly in } C(0, T; H^r(0, L))\]

**Proof.** See [6]. ■

To show the existence result, let us introduce the following functionals

\[E(t) \equiv E(t, u, v) = E_1(t, u) + E_2(t, v).\]

\[E_1(t, u) = \frac{1}{2} \int_0^{L_0} (\rho_1 |u_t|^2 + b |u_x|^2) \, dx\]

\[E_2(t, v) = \frac{1}{2} \int_0^L (\rho_2 |v_t|^2 + a |v_x|^2) \, dx\]

The energy of the penalized problem (3)-(9) is

\[E_\epsilon(t) = E(t, u^\epsilon, v^\epsilon) + \frac{1}{2\epsilon} [v^\epsilon(L, t) - g]^+\]

To simplify notation, we will avoid the superindex \( \epsilon \) in our calculations.
Lemma 2.2 Let us take initial data satisfying \( \{u^0, v^0\} \in (H^2(0, L_0) \times H^2(L_0, L)) \cap V \), \( \{u^1, v^1\} \in V \), and the compatibility condition
\[
bu^0_x(L_0) = av^0_x(L_0),
\]
\[
av^0_x(L) = -\frac{1}{\epsilon}v^0(L) - g^+ + \epsilon v^1(L) \tag{10}
\]
Then there exists one solution \( \{u, v\} \) to (3)–(9) satisfying
\[
\{u, v\} \in \bigcap_{k=0}^{2} W^{k,\infty}(0, T; H^{2-k}(0, L_0) \times H^{2-k}(L_0, L))
\]

Proof. We employ the Faedo-Galerkin method. Indeed, we will choose a basis of \( V \), \( B = \{\{\varphi^i, \psi^i\}, i \in \mathbb{N}\} \) such that
\[
\{u^0, v^0\}, \{u^1, v^1\} \text{ are in the span of } \{\varphi^0, \psi^0\}, \{\varphi^1, \psi^1\}\}
\]
Therefore, \( \{u^n(0), v^n(0)\} = \{u^0, v^0\} \) and \( \{u^n_m(0), v^n_m(0)\} = \{u^1, v^1\} \).

Standard results about ordinary differential equations guarantee that there exists only one solution
\[
\{u^m(t), v^m(t)\} = \sum_{i=1}^{m} h_{im}(t)\{\varphi^i, \psi^i\}
\]
of the approximated system
\[
\frac{d}{dt}E^m(t) + \alpha \int_{L_0}^{L} |v^n|^2 dx = -\epsilon |u^n(L, t)|^2 + 2\epsilon \int_{L_0}^{L} v^n dx
\]
(11)
where \( i = 1, 2, \ldots \) with initial data
\[
\{u^m(0), v^m(0)\} = \{u^0, v^0\},
\]
\[
\{u^n_m(0), v^n_m(0)\} = \{u^1, v^1\}. \tag{12}
\]

On some interval \([0, T_m]\), To extend this solution to the whole interval \([0, T]\), for all \( T > 0 \) it is enough to show that approximated solutions are bounded independently of \( m \) and \( t \). In order to do so, we first multiply equation (11) by \( h'_{jm}(t) \), integrating by parts and then summing up \( j \), to obtain
\[
\frac{d}{dt}E^m(t) + \alpha \int_{L_0}^{L} |v^n|^2 dx = -\epsilon |u^n(L, t)|^2. \tag{13}
\]
and integrating from 0 to \( t \), we get
\[
E^m(t) + \alpha \int_{0}^{t} \int_{L_0}^{L} |v^n|^2 dx ds + \int_{0}^{t} |v^n(L, s)|^2 ds \leq E^m(0). \tag{14}
\]
Differentiating the approximated equation (11) with respect to \( t \) and then multiplying the result using equation by \( h''_{jm}(t) \), summing up in \( j \), we obtain
\[
\frac{d}{dt}E(t, u^n, v^n) + \alpha \int_{L_0}^{L} |v^n|^2 dx = -\epsilon |u^n(L, t)|^2 + \frac{1}{\epsilon} \left[|v^n(L, t)|^2 + \epsilon |v^n(L, t)|^2\right] \tag{15}
\]
To get the second order estimate we have to show that \( E(0, u^n, v^n) \) is bounded. Here we will use the special basis chosen above. In fact, letting \( t \to 0^+ \) in equation (11), multiplying the limit result by \( h''_{jm}(0) \), taking into account the compatibility conditions and using the transmission conditions we get
\[
\rho_1 \int_{L_0}^{L} |u^n|^2 dx + \rho_2 \int_{L_0}^{L} |v^n|^2 dx = b \int_{L_0}^{L} v^n_0 \frac{v^n}{t^2} dx + a \int_{L_0}^{L} v^n x_0 v^n dx - \alpha \int_{L_0}^{L} v^n v^n_L(0) dx
\]
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from which it follows that
\[ \int_0^L |u_t^m(0)|^2 dx + \int_0^L |v_t^m(0)|^2 dx \leq \epsilon \left( \int_0^L |u_{xx}^0|^2 dx + \int_0^L (|v_{xx}^0|^2 + |v|^2) dx \right) \]  
(16)

From the above inequality we conclude that \( E(0, u_t^m, v_t^m) \) is bounded. In this manner, combining (15) and (16) we derive that
\[ E(t, u_t^m, v_t^m) \text{ is bounded}. \]  
(17)

**Passage to limit as** \( m \to \infty \). From the estimates (14) and (17) we can guarantee that there exists a subsequence of \( \{u^{e,m}, v^{e,m}\} \) still denoted in the same form, such that

\[ \{u^{e,m}, v^{e,m}\} \to \{u^*, v^*\} \text{ weak in } L^\infty(0, T; V) \]
\[ \{u_t^{e,m}, v_t^{e,m}\} \to \{u_t^*, v_t^*\} \text{ weak * in } L^\infty(0, T; H^1(0, L_0) \times H^1(L_0, L)) \]
\[ \{u_{tt}^{e,m}, v_{tt}^{e,m}\} \to \{u_{tt}^*, v_{tt}^*\} \text{ weak * in } L^\infty(0, T; L^2(0, L_0) \times L^2(L_0, L)). \]

Using Lemma 2.1 we have
\[ v^{e,m}(L, t) \to v^*(L, t) \text{ uniformly on } [0, T] \]

The remaining part of the proof is a matter of routine. ■

## 3 Contact problem

This section is devoted to obtain the solution for the system (1). We are interested in solving the problem as limit (when \( \epsilon \to 0 \)) of the penalized problem.

We first introduce the sets
\[ \mathcal{W} = \{(w, z) \in H^2(0, L_0) \times H^2(L_0, L) \cap V : bw_x(L_0) = az_x(L_0), z_x(L) = 0\}. \]
\[ \mathcal{W}_g = \{(w, z) \in H^2(0, L_0) \times H^2(L_0, L) \cap V : bw_x(L_0) = az_x(L_0), z_x(L) = 0, z(L) = g\}. \]
\[ V_L = \{(w, z) \in H^1(0, L_0) \times H^1(L_0, L) : w(L_0) = z(L_0), w(0) = 0 = z(L)\}. \]

We establish now the result that treats the existence of solutions for the transmission-contact problem associated with the wave equation.

**Theorem 3.1** Suppose that \( \{u^0, v^0\} \in K \), \( \{u^1, v^1\} \in L^2(0, L_0) \times L^2(L_0, L) \), then there exists a weak solution of (1).

**Proof.** Let us take a sequence \( \{u^{e,0}, v^{e,0}\} \in \mathcal{W} \) (or \( \{u^{e,0}, v^{e,0}\} \in \mathcal{W}_g \) when \( \{u^0, v^0\} \in \mathcal{W}_g \)) and \( \{u^{e,1}, v^{e,1}\} \in V_L \) such that
\[ \{u^{e,0}, v^{e,0}\} \to \{u^0, v^0\} \text{ in } V \]
\[ \{u^{e,1}, v^{e,1}\} \to \{u^1, v^1\} \text{ in } L^2(0, L_0) \times L^2(L_0, L) \]
when \( \epsilon \to 0 \). Observe that \( \{u^{e,0}, v^{e,0}\}, \{u^{e,1}, v^{e,1}\} \) satisfies the compatibility conditions (10). Let us denote by \( \{u^\epsilon, v^\epsilon\} \) the strong solution of (3)-(9) with initial data
\[ \{u^\epsilon(x, 0), v^\epsilon(x, 0)\} = \{u^{0, 0}(x), v^{0, 0}(x)\}, \{u^\epsilon_t(x, 0), v^\epsilon_t(x, 0)\} = \{u^{0, 1}(x), v^{0, 1}(x)\}. \]

Multiplying equation (3) by \( u^\epsilon_t \), equation (4) by \( v^\epsilon_t \) and performing an integration by parts we have
\[ \frac{d}{dt} \mathcal{E}_\epsilon(t) + \alpha \int_0^L |v^\epsilon|^2 \geq \epsilon |v^\epsilon|^2 \]
and integrating from 0 to \( t \), we get
\[ \mathcal{E}_\epsilon(t) \leq \mathcal{E}_\epsilon(0). \]
(19)

Thus, we conclude that
\[ \mathcal{E}_\epsilon(t) \text{ is bounded} \]
(20)
which implies that
\[ \{u^\epsilon, v^\epsilon\} \to \{u, v\} \text{ weak * in } L^\infty(0, T; V) \]
\[ \{u_t^\epsilon, v_t^\epsilon\} \to \{u_t, v_t\} \text{ weak * in } L^\infty(0, \infty) \times L^2(\Omega, L) \].

Our next step is to show that \( \{u_t, v_t\} \) satisfies the variational inequality (2). To do this, let us multiply equation (3) by \( z - v^\epsilon \) with \( \{w, z\} \in W^{1,\infty}(0, T; L^2(\Omega, L)) \cap L^\infty(0, T; K) \) and performing an integration by parts we have
\[
- \rho_1 \int_0^T \int_\Omega u_t^\epsilon(x,w(x,0) - u^\epsilon(0))\,dx\,dt + \rho_1 \int_0^T \int_\Omega u_t^\epsilon(T)\,dx
d - \rho_2 \int_0^T \int_\Omega v_t^\epsilon(x,z(x,0) - v^\epsilon(0))\,dx\,dt + \rho_2 \int_0^T \int_\Omega v_t^\epsilon(T)\,dx + b \int_0^T \int_\Omega u_x^\epsilon w_x^\epsilon - u_x^\epsilon z_x^\epsilon \,dx\,dt + \alpha \int_0^T \int_\Omega v_x^\epsilon(z_x^\epsilon - v_x^\epsilon)\,dx\,dt
\geq - \epsilon \int_0^T v_t(L, t)(z(L, t) - v^\epsilon(L, t))\,dt - \alpha \int_0^T \int_\Omega v_t^\epsilon(z - v^\epsilon)\,dx\,dt
\]
from (21)-(22) our conclusion follows.

4 Exponential decay

In this section we will prove that the solution of the transmission-contact problem (1) decays exponentially as time goes to infinity. To do it we first show that the energy associated to the penalized problem (3)–(9) decays exponentially as time goes to infinity. We used the perturbed energy method developed by Komornik and Zuaţua [7] and the method of the integral inequalities introduced by Martínez [12], so we construct a suitable Lyapunov functional which allow us to get our result. From the convergence of solutions \( \{u^\epsilon, v^\epsilon\} \) we obtain that the solution of the problem (1) decays exponentially too. For the sake of simplicity we omit the superscript \( \epsilon \).

Note that (13) still holds
\[
\frac{d}{dt} \mathcal{E}_\epsilon(t) = -\alpha |v_t|^2 - \epsilon |v_t(L, t)|^2.
\]
and we have
\[
\mathcal{E}_\epsilon(0) = \mathcal{E}_\epsilon(T) + \alpha \int_0^T |v_t|^2\,dt + \epsilon \int_0^T |v_t(L, t)|^2.
\]
In the following lemmas we will prove some technical inequalities which will be useful to show the exponential decay of the solution.

**Lemma 4.1** The functional defined by
\[
J_1(t) = \int_0^\Omega (x - L_0)u_t u_x\,dx
\]
satisfies
\[
\frac{d}{dt} J_1(t) = - E_1(t, u) + \frac{b L_0}{2} \int_0^\Omega |u_x(0, t)|^2.
\]
Proof. Multiplying equation (3) by \((x - L_0)u_x\) and performing an integration in \((0, L_0)\) we get

\[\rho_1 \int_0^{L_0} (x - L_0)u_x u_{tt} \, dx - b \int_0^{L_0} (x - L_0)u_x u_{xx} \, dx = 0.\]  (25)

Note that

\[\frac{d}{dt}(x - L_0)u_x u_t = (x - L_0)u_x u_{tt} + (x - L_0)u_x u_t.\]  (26)

Now using (26) in (25) we get

\[\rho_1 \frac{d}{dt} \int_0^{L_0} (x - L_0)u_x u_t \, dx = \rho_1 \int_0^{L_0} (x - L_0) \frac{1}{2} \frac{d}{dx} |u_t|^2 \, dx + b \int_0^{L_0} (x - L_0) \frac{1}{2} \frac{d}{dx} |u_x|^2 \, dx\]

Then, performing integration by parts and using the boundary conditions we get

\[\rho_1 \frac{d}{dt} \int_0^{L_0} (x - L_0)u_x u_t \, dx = \frac{-\rho_1}{2} \int_0^{L_0} |u_t|^2 \, dx - \frac{b}{2} \int_0^{L_0} |u_x|^2 \, dx + bL_0 \frac{1}{2} |u_x(0, t)|^2\]

from which it follows that

\[\frac{d}{dt} J_1(t) = -E_1(t, u) + bL_0 \frac{1}{2} |u_x(0, t)|^2.\]

In order to control the punctual term \(|u_x(0, t)|^2\) we implement the following lemma

**Lemma 4.2** Let us take \(q \in C^1(0, L_0)\) with \(q(0) > 0\) and \(q(L_0) = 0\). Then, there exist positive constants \(C_0, N_0\) independent of initial data, such that the functional defined by

\[J_2(t) = N_0 J_1(t) + \int_0^{L_0} qu_x u_x \, dx\]

satisfies

\[\frac{d}{dt} J_2(t) \leq -C_0 E_1(t, u)\]

**Proof.** Multiplying equation (3) by \(q u_x\) and performing an integration in \((0, L_0)\) we get

\[\frac{d}{dt} \int_0^{L_0} q u_x u_t \, dx = -\frac{1}{2} \int_0^{L_0} q' (\rho_1 |u_t|^2 + b|u_x|^2) \, dx - \frac{bq(0)}{2} |u_x(0, t)|^2,\]

from which it follows that

\[\frac{d}{dt} \int_0^{L_0} q u_x u_t \, dx \leq -\frac{bq(0)}{2} |u_x(0, t)|^2 + C_1 E_1(t, u).\]  (27)

Adding \(N_0 J_1(t)\) to the above inequality, then taking \(N_0 > C_1\) and choosing \(q(0) > N_0 L_0\) we arrive at our result. ■

**Lemma 4.3** There exists a positive constant \(C_2\), independent of initial data, such that the functional defined by

\[J_3(t) = \int_{L_0}^{L} (x - L_0)v_x v_x \, dx\]

satisfies

\[\frac{d}{dt} J_3(t) \leq -\frac{1}{2} E_2(t, v) + \frac{(L - L_0)}{2} I(L, t) + C_2 \int_0^{L_0} |v_x|^2 \, dx.\]

where \(I(L, t) = \rho_2 |v_x(L, t)|^2 + a |v_x(L, t)|^2\)
Proof. Multiplying equation (4) by \((x - L_0)v_x\) and performing an integration in \((L_0, L)\) we get

\[
\int_{L_0}^{L} \rho_2 (x - L_0) v_x v_t \, dx - \int_{L_0}^{L} (x - L_0) v_x v_{xx} \, dx + \alpha \int_{0}^{L_0} (x - L_0) v_x v_t \, dx = 0. \tag{28}
\]

Performing integration by parts we get

\[
\frac{d}{dt} \int_{L_0}^{L} (x - L_0) v_x v_t \, dx = - E_2(t, v) + \frac{(L - L_0)}{2} \left( \rho_2 |v_t(L)|^2 + a|v_x(L)|^2 \right) - \alpha \int_{0}^{L_0} (x - L_0) v_x v_t \, dx
\]

thus we conclude our proof. \(\blacksquare\)

Lemma 4.4 Let us take \(q \in C^1(L_0, L)\) with \(q(L_0) = 0\) and \(q(L) < 0\). Then, there exist positive constants \(C_4, C_5, \alpha_2\) and \(N_1\) independent of initial data such that the functional defined by

\[
J_4(t) = N_1 J_3(t) + \int_{L_0}^{L} q v_x \, dx
\]

satisfies

\[
\frac{d}{dt} J_4(t) \leq -C_4 E_2(t, v) + \frac{1}{2} \left( C_5 + q(L) \right) I(L, T) + \alpha_2 \int_{L_0}^{L} |v_t|^2 \, dx.
\]

Proof. Multiplying equation (4) by \(q v_x\) and performing an integration in \((L_0, L)\), after simple calculations we get

\[
\frac{d}{dt} \int_{L_0}^{L} q v_x v_t \, dx = - \frac{1}{2} \int_{L_0}^{L} q \left( \rho_2 |v_t|^2 + a|v_x|^2 \right) \, dx + \frac{q(L)}{2} (a|v_x(L)|^2 + \rho_2 |v_t(L)|^2) - \alpha \int_{L_0}^{L} q v_x v_t \, dx
\]

from which, using Young’s inequality, it follows that

\[
\frac{d}{dt} \int_{L_0}^{L} q v_x v_t \, dx \leq \frac{q(L)}{4 \alpha} (a|v_x(L)|^2 + \rho_2 |v_t(L)|^2) + C_4 E_2(t, v) + \alpha_1 \int_{L_0}^{L} |v_t|^2 \, dx.
\]

Then, we have

\[
\frac{d}{dt} J_4(t) \leq - \left( \frac{N_1}{2} - C_4 \right) E_2(t, v) + \left( \frac{N_1 (L - L_0)}{2} + \frac{q(L)}{2} \right) I(L, t) + (\alpha_1 + N_1 C_2) \int_{L_0}^{L} |v_t|^2 \, dx.
\]

Now taking \(N_1\) such that \(N_1 > 2C_4\), this lemma is proved. \(\blacksquare\)

Theorem 4.5 Let us denote by \(\{u, v\}\) the solution of (3)–(9). Then there exist positive constants \(C\) and \(\gamma\), such that

\[
E_\varepsilon(t) \leq C E_\varepsilon(0) e^{-\gamma t}.
\]

Proof. From (23), Lemmas (4.2) and (4.4) we get

\[
\frac{d}{dt} \left( N E_\varepsilon(t) + J_2(t) + J_4(t) \right) \leq -C_6 E(t, u, v) - (N\alpha - \alpha_2) \int_{L_0}^{L} |v_t|^2 \, dx - N E|v_t(L, t)|^2 + (N_2 + q(L)) I(L, t)
\]

for \(N > 0\).

From which it follows

\[
C_6 E(t, u, v) + a|v_x(L, t)|^2 \leq
- (N\alpha - \alpha_2) \int_{L_0}^{L} |v_t|^2 \, dx - N E|v_t(L, t)|^2 + (N_2 + 1 + q(L)) I(L, t)
- \frac{d}{dt} \left( N E_\varepsilon(t) + J_2(t) + J_4(t) \right)
\]

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Now, using the inequality
\[ a|v_x(L, t)|^2 = | - \frac{1}{\epsilon} [v(L, t) - g]_+ - \epsilon v_t(L, t)|^2 \geq \frac{1}{2\epsilon} [v(L, t) - g]_+^2 - \epsilon |v_t(L, t)|^2 \]
and denoting by \( L(t) \) the functional
\[ L(t) = N E_\epsilon(t) + J_2(t) + J_4(t) \]
we get
\[ C_6 E(t, u, v) + \frac{1}{2\epsilon} [v(L, t) - g]_+^2 \leq -(N \alpha - \alpha_2) \int_{L_0}^L |v_t|^2 \, dx + (N_2 + a + 2 + q(L)) I(L, t) \]
\[ - \frac{d}{dt} L(t). \]
Integrating from \( S \) to \( T \), \( 0 \leq S \leq T < +\infty \), choosing \( q(L) = -2(N_2 + a + 2) \), \( N > \alpha_2 \alpha^{-1} \) and observing that \( L(t) \) is equivalent to \( E_\epsilon(t) \) we arrive at
\[ \int_S^T E_\epsilon(t) \, dt \leq C (L_\epsilon(T) + L_\epsilon(S)) \]
\[ \leq C E_\epsilon(S). \]
for some \( C > 0 \).
From the above inequality, our conclusion follows.

We are in position to show the main result of this paper.

**Theorem 4.6** Let \( \{u, v\} \) be the solution of (1). Then there are exist positive constants \( C \) and \( \gamma \) independents of \( \epsilon \) and \( t \) such that
\[ E(t) \leq C E(0) e^{-\gamma t}. \]

**Proof.** We have from theorem 4.5, the convergence of \( \{u^\epsilon, v^\epsilon\} \) and the lower semicontinuity of the energy that
\[ E(t, u, v) \leq \liminf_{\epsilon \to 0} E_\epsilon(t) \leq C \left( \liminf_{\epsilon \to 0} E_\epsilon(0) \right) e^{-\gamma t} \leq CE(0) e^{-\gamma t}. \]
This completes the proof.

**Acknowledgements**

The authors are grateful to the anonymous referee for giving us valuable suggestions that improved our paper.

**References**


