Soliton and Periodic Solutions to the Generalized Hirota-Satsuma Coupled System Using Trigonometric and Hyperbolic Function Methods

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Abstract: In this paper, the Hirota-Satsuma coupled KdV equation and its generalized form are discussed. Trigonometric and hyperbolic function methods such as sine-cosine method, rational sine-cosine method, rational sinh-cosh method and rational tanh-coth method, are used for analytical treatment of these systems. By means of these methods, we have the advantage of reducing the nonlinear problem to a system of algebraic equations that can be solved by any computerized packages. The proposed methods are straightforward, concise and effective, and can be applied to other nonlinear equations and evolutionary systems which arise in mathematical physics.

Keywords: Trigonometric-function method; Hyperbolic-function method; generalized Hirota-Satsuma KdV system

1 Introduction

Many models in physics are described by nonlinear partial differential equations (PDEs). To understand the nature of such applications, many researchers have stressed their goals to search for explicit solutions. Many powerful methods, such as Hirota bilinear method [1, 2], the rational sine-cosine function method [6], the Tanh method [9], the Tanh-Coth method [11], the sine-cosine method [7, 10], the extended Tanh method [4, 12], and many other techniques were used to investigate these types of equations and obtained interesting class of solutions: solitons, kinks and periodic. Solitons are the solutions in the form \( \text{sech} \) and \( \text{sech}^p \), the graph of soliton is a wave that goes up only. It is not like periodic solutions \( \sin, \cos \), etc, as in trigonometric functions, that goes above and below the horizontal. Kink is also called a soliton; it is in the form \( \tanh \) not \( \tanh^2 \). In kink the limit as \( x \to \infty \), the answer is a nonzero constant, unlike solitons where the limit goes to 0.

In this work, we consider the Hirota-Satsuma coupled KdV equation

\[
\begin{align*}
    u_t &= \frac{1}{4} u_{xxx} + 3uu_x - 6vv_x, \\
    v_t &= -\frac{1}{2} v_{xxx} - 3uv_x,
\end{align*}
\]

and the generalized Hirota-Satsuma coupled KdV

\[
\begin{align*}
    u_t &= \frac{1}{2} u_{xxx} - 3uu_x + 3(vv)_x, \\
    v_t &= -v_{xxx} + 3uv_x, \\
    w_t &= -w_{xxx} + 3uw_x.
\end{align*}
\]

Hirota and Satsuma [11] presented equation (1) which describes an interaction of two long waves with different dispersion relations. Wu et al. [14] introduced equation (2) which can be obtained from the four-reduction spectral problem with three potentials. In the literature, Dianchen et al. [5] obtained combined formal solitary wave solutions and combined formal periodic wave solutions to equation (1) by using new improved projective Riccati equations method. Based on the
Hirota method and the perturbation technique, the N-soliton solution of equation (2) is obtained by WU Lian-Ping et al. [13]. The extended tanh method was used in investigating the soliton solutions of (1) and (2), see [4] and [12]. Finally, the Homotopy perturbation method was applied [8] to handle (2).

This paper is organized as follows: In Section 2, we give a description of the proposed methods. In Section 3, we apply the methods to equation (1). In Section 4, we apply the methods to equation (2).

2 Trigonometric and hyperbolic function methods

It is appropriate to assume different ansatzes that involve hyperbolic and trigonometric functions. Some of these ansatzes provide real solutions, whereas others do not. A criterion to specify which ansatze works for real solutions cannot be determined. Only direct substitution can show the existence of real solutions for some ansatzes. In what follows, we present the main steps of such ansatzes.

Since we restrict our attention to traveling waves, we use the transformation \( u(x, t) = u(\zeta) \), where the wave variable \( \zeta = x - ct \), converts the nonlinear PDE to an equivalent ODE.

2.1 The sine-cosine method (Ansatz I)

The sine-cosine algorithm admits the use of the ansatze [3, 10]

\[
    u(x, t) = \lambda \cos(\mu \zeta), \quad |\zeta| \leq \frac{\pi}{2\mu},
\]

(3)

and the ansatze,

\[
    u(x, t) = \lambda \sin(\mu \zeta), \quad |\zeta| \leq \frac{\pi}{\mu},
\]

(4)

where \( \lambda, \epsilon, \mu \) and \( \gamma \) are parameters that will be determined later. Substituting (3) or (4) into the reduced ODE gives a polynomial equation of cosine or sine terms. We then collect the coefficients of the resulting triangle functions and set them to zeros to get a system of algebraic equations among the unknowns \( \lambda, \epsilon, \mu, \) and \( \gamma \). The problem is now completely reduced to an algebraic one. Having determined \( \lambda, \epsilon, \mu \) and \( \gamma \) by algebraic calculations or by using any software, the solutions proposed in (3) and in (4) follow immediately.

2.2 The second rational sine-cosine function method (Ansatz II)

The second rational triangle functions method can be expressed in the form [3]

\[
    u(x, t) = \frac{a_0 + b_0 \sin(\mu \zeta)}{1 + a_1 \sin(\mu \zeta)}
\]

(5)

and

\[
    u(x, t) = \frac{a_0 + b_0 \cos(\mu \zeta)}{1 + a_1 \cos(\mu \zeta)}
\]

(6)

where \( a_0, a_1, b_0, \) and \( \mu \) are parameters that will be determined.

2.3 The first rational hyperbolic function method (Ansatz III)

The first rational hyperbolic function method has the form [3]

\[
    u(x, t) = \frac{a}{1 + \lambda f(\mu \zeta)},
\]

(7)

where \( f(\mu \zeta) \) takes any of the hyperbolic functions \( sech, csch, \sinh, \cosh \). The approach is simply used by applying the equation, setting the coefficients of the resulting hyperbolic functions to zero, and solving the resulting equations to determine the parameters \( a, \lambda \) and \( \mu \).
2.4 The second rational hyperbolic function method (Ansätze IV)

The second rational hyperbolic function method has the form [3]

\[ u(x, t) = f(\mu \zeta) \frac{1}{1 + \lambda f(\mu \zeta)} \]  

where \( f(\mu \zeta) \) takes any of the hyperbolic functions.

3 The coupled Hirota Satsuma KdV equation

The Hirota Satsuma system is given by

\[ \begin{align*}
  u_t & = \frac{1}{4} u_{xxx} + 3uu_x - 6vv_x, \\
  v_t & = -\frac{1}{2} v_{xxx} - 3uv_x. 
\end{align*} \tag{9} \]

Using the wave variable \( \zeta = x - \epsilon t \) transforms (9) into the ODEs

\[ \begin{align*}
  \epsilon u + \frac{1}{4} u'' + \frac{3}{2} u^2 - 3v^2 & = 0 \quad \text{(10)} \\
  u & = \frac{v'''}{3v} - \epsilon \frac{c}{3}. 
\end{align*} \tag{11} \]

Substituting (11) into (10) gives

\[ \begin{align*}
  0 & = - (v')^2v(5) + 2v'v''v(4) + 2v'(v(3)) - 8\epsilon(v')^2v(3) \\
  & \quad - 2(v'')^2v(3) + 12\epsilon^2v(3) - 72(v')^3v. 
\end{align*} \tag{12} \]

First, we solve (12) by using the second rational sine method; Ansätze (III). Substituting (5) into (12) results in

\[ \begin{align*}
  0 & = 72\alpha_0^2 - 12\epsilon^2 - 8\epsilon\mu^2 + 48\alpha_1^2\epsilon\mu^2 - \mu^4 - 12\alpha_1^4 \\
  & \quad + \sin(\mu \zeta)(144\alpha_0 b_0 - 24\alpha_1 \epsilon^2 + 32\alpha_1 \epsilon^2 + 4\alpha_1 \mu^4) \\
  & \quad + \sin^2(\mu \zeta)(72 b_0 - 12\alpha_1^2 \epsilon^2 - 8\alpha_1^2 \epsilon^2 - \alpha_1^2 \mu^4). 
\end{align*} \tag{13} \]

The above equation is satisfied only if the following system of algebraic equations hold and \( \epsilon = 1, \mu = 1 \)

\[ \begin{align*}
  0 & = 72\alpha_0^2 - 31 + 48\alpha_1^2 - 12\alpha_1^4 \\
  0 & = 144\alpha_0 b_0 + 12\alpha_1 \\
  0 & = 72 b_0 - 31\alpha_1. 
\end{align*} \tag{14} \]

Solving this system gives

\[ \begin{align*}
  \alpha_0 & = \mp \frac{1}{\sqrt{42}}, \quad b_0 = \pm \frac{\sqrt{5}}{4\sqrt{2}}, \quad \alpha_1 = \pm \frac{\sqrt{15}}{2\sqrt{7}}. 
\end{align*} \tag{15} \]

Thus, the solution of (9) is

\[ v_1(x, t) = \frac{\mp \frac{1}{\sqrt{42}} \pm \frac{\sqrt{5}}{4\sqrt{2}} \sin(x - \cdot)}{1 \pm \frac{\sqrt{15}}{2\sqrt{7}} \sin(x - \cdot)}. \tag{16} \]

Substituting (16) into (11), we get

\[ u_1(x, t) = \frac{-7 \left( -11 + 15 \cos(2(x - \cdot)) \right)}{4 \left( \mp \frac{1}{\sqrt{42}} + \frac{\sqrt{15}}{2\sqrt{7}} \sin(x - \cdot) \right)^2}. \tag{17} \]
Now, we apply the second rational cosine method. Substituting (6) into (12) gives the same system (14). Therefore, the solution of (9) is

$$v_2(x, y) = \frac{\pm i \sqrt{2} \, \text{sech}(\sqrt{2}x) + 2 \sqrt{2} \, \text{sech}(\sqrt{2}x)}{1 + \sqrt{2} \, \cos(x - y)}.$$

(18)

Substituting (18) into (11), we get

$$u_2(x, y) = \frac{7(-11 + 15 \cos(2(x - y)))}{4(\mp 14 + \sqrt{105} \cos(x - y))^2}.$$  

(19)

Second, we apply the cosine method Ansätze (I). Substituting (3) into (12) gives

$$0 = -8\mu^4 + 6\gamma^2 \mu^4 + 7\gamma^2 \mu^4 - 6\gamma^3 \mu^4 + \gamma^4 \mu^4$$

$$+ \cos^2(\mu)(-16e\mu^2 + 24\gamma \mu^2 - 8\gamma^2 \mu^2 + 8\mu^4 + 8\gamma^2 \mu^2 + 6\gamma^3 \mu^4 - 2\gamma^4 \mu^4 - 12\gamma \mu^4)$$

$$+ \cos^4(\mu)(12e^2 + 8\gamma^2 \mu^2 + \gamma^4 \mu^4)$$

$$- 72\lambda^2 \cos^4(\mu).$$

(20)

The above equation is satisfied only if the following system of algebraic equations holds and $e = 1$

$$0 = -8 + 6\gamma + 7\gamma^2 - 6\gamma^3 + \gamma^4$$

$$0 = -16\mu^2 + 24\gamma \mu^2 - 8\gamma^2 \mu^2 + 8\mu^4 + 8\gamma^2 \mu^2 + 6\gamma^3 \mu^4 - 2\gamma^4 \mu^4 - 12\gamma \mu^4 - 72\lambda^2$$

$$0 = 12 + 8\gamma^2 \mu^2 + \gamma^4 \mu^4$$

$$0 = 2 + 2\gamma.$$  

(21)

Solving the above system gives two solution sets

$$\lambda = \mp \sqrt{2}, \quad \mu = \pm i \sqrt{2}, \quad \gamma = -1,$$

$$\lambda = \mp \sqrt{10}, \quad \mu = \pm i \sqrt{6}, \quad \gamma = -1.$$  

(22)

Thus, the solutions of (9) are

$$v_3(x, y) = \pm \sqrt{2} \, \text{sech}(\sqrt{2}(x - y)),$$

$$v_4(x, y) = \pm \sqrt{10} \, \text{sech}(\sqrt{6}(x - y)).$$

(23)

Substituting (23) into (11) gives

$$u_3(x, y) = 2 \text{sech}^2(\sqrt{2}(x - y)),$$

$$u_4(x, y) = 6 \text{sech}^2(\sqrt{6}(x - y)) - \frac{2}{3}.$$

(24)

By the sine method, substituting (4) into (12) gives the same system (21). Therefore, two more solutions follow

$$v_5(x, y) = \pm i \sqrt{2} \, \text{sech}(\sqrt{2}(x - y)),$$

$$v_6(x, y) = \pm i \sqrt{10} \, \text{sech}(\sqrt{6}(x - y)).$$

(25)

Substituting (25) into (11) gives

$$u_5(x, y) = -2 \text{sech}^2(\sqrt{2}(x - y)),$$

$$u_6(x, y) = -6 \text{sech}^2(\sqrt{6}(x - y)) - \frac{2}{3}.$$

(26)

Third, we apply Ansätze (IV) with $f(\zeta) = \text{sech}(\zeta)$. Substituting (8) into (12) gives

$$0 = -72 + 12e^2 \lambda^2 - 8e\lambda^2 \mu^2 + 48e\mu^2 + 12\lambda^4 + \lambda^2 \mu^4$$

$$+ \cosh(\mu)(24\lambda^2 e^2 + 32e\lambda^2 \mu^2 - 4\lambda^4 \mu^4)$$

$$+ \cosh^2(\mu)(12e^2 - 8e^2 \mu^2 + \mu^4).$$

(27)
The above equation is satisfied only if the following system of algebraic equations holds
\[ 0 = -72 + 12\epsilon^2 \lambda^2 - 8\epsilon^2 \mu^2 + 48\epsilon \mu^2 + 12\lambda^2 + \lambda^2 \mu^4 \]
\[ 0 = 24\lambda^2 \epsilon^2 + 32\epsilon \lambda^2 \mu^2 - 4\lambda^2 \mu^4 \]
\[ 0 = 12\epsilon^2 - 8\epsilon \mu^2 + \mu^4. \] (28)

Solving the above system gives four solution sets
\[ \epsilon = -\frac{1}{\sqrt{2}}, \quad \mu = \pm i\sqrt{2}, \quad \lambda = 0, \]
\[ \epsilon = \frac{1}{\sqrt{2}}, \quad \mu = \pm i\sqrt{2}, \quad \lambda = 0, \]
\[ \epsilon = -\frac{1}{\sqrt{10}}, \quad \mu = \pm i\sqrt{3}\sqrt[4]{\frac{2}{5}}, \quad \lambda = 0, \]
\[ \epsilon = \frac{1}{\sqrt{10}}, \quad \mu = \pm i\sqrt{3}\sqrt[4]{\frac{2}{5}}, \quad \lambda = 0. \] (29)

Thus, the solutions of (9) are
\[ v_7(x, \tau) = \sec(\sqrt[4]{2}\zeta), \]
\[ v_8(x, \tau) = \text{sech}(\sqrt[4]{3}\sqrt[5]{2}\zeta), \]
\[ v_9(x, \tau) = \sec(\sqrt[4]{3}\sqrt[5]{2}\zeta), \]
\[ v_{10}(x, \tau) = \text{sech}(\sqrt[4]{2}\zeta), \quad \zeta = x - \epsilon. \] (30)

Substituting (30) into (11) gives
\[ u_7(x, \tau) = -\sqrt{2}\sec^2(\sqrt[4]{2}\zeta), \]
\[ u_8(x, \tau) = -\frac{1}{3\sqrt{2}} + \frac{1}{\sqrt{10}} + 3\sqrt[5]{2}\text{sech}(\sqrt[4]{3}\sqrt[5]{2}\zeta), \]
\[ u_9(x, \tau) = -\frac{1}{3\sqrt{2}} + \frac{1}{\sqrt{10}} - 3\sqrt[5]{2}\sec(\sqrt[4]{3}\sqrt[5]{2}\zeta), \]
\[ u_{10}(x, \tau) = \frac{\sqrt{2}}{3} \left( 3\text{sech}^2(\sqrt[4]{2}\zeta) - 1 \right), \quad \zeta = x - \epsilon. \] (31)

4 Generalized coupled Hirota Satsuma KdV equation

In this section we consider a three-component evolutionary system of KdV equation of order 3. Such system is called the generalized Hirota Satsuma system given by
\[ u_t = \frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x, \]
\[ v_t = -v_{xxx} + 3uv_x, \]
\[ w_t = -w_{xxx} + 3uw_x. \] (32)

Using the wave variable \( \zeta = x - \epsilon \) transforms (32) into the ODEs
\[ 0 = \epsilon w' + \frac{1}{2}w''' - \frac{3}{2}(v^2)' + 3(vw)', \]
\[ 0 = \epsilon v' + v''' + 3uv', \]
\[ 0 = \epsilon w' - w''' + 3uv'. \] (33)
Considering the second and third equations of system (33), we obtain

\[(v'w'')_x = (w'v'')_x.\]  
(34)

Integrating the obtained ODE in (34) twice and setting the constant of integration to zero, we get

\[v' = w'.\]  
(35)

Now, we simplify the second equation in (33) to get

\[u = \frac{v^{(3)}}{3v'} - \frac{\epsilon}{3}.\]  
(36)

Substituting (36) into the first equation in (33) and then integrating and setting the constant of integration to zero, yield

\[0 = (v')^2v^{(5)} - 2v'v''v^{(4)} - 2v'(v^{(3)})^2 + 4\epsilon (v')^2v^{(3)} + 2(v'')^2v^{(3)} - 3\epsilon^2(v')^3 + 18(v')^3vw.\]  
(37)

First, we solve (37) by using Ansätze (V). Suppose

\[v(x, t) = a + \lambda \tanh(\mu \zeta), \quad w(x, t) = b + \lambda \tanh(\mu \zeta).\]  
(38)

Now substituting (38) into (37), we get

\[0 = 18 \phi - 3\epsilon^2 - 8\epsilon \mu^2 + 8\mu^4 + (18\alpha \lambda + 18 \Phi \lambda) \tanh(\mu \zeta) + (18\lambda^2 + 24\epsilon \mu^2 - 24\mu^4) \tanh^2(\mu \zeta).\]  
(39)

The above equation is satisfied only if the following system of algebraic equations holds

\[0 = 18 \phi - 3\epsilon^2 - 8\epsilon \mu^2 + 8\mu^4, \quad 0 = 18\alpha \lambda + 18 \Phi \lambda, \quad 0 = 18\lambda^2 + 24\epsilon \mu^2 - 24\mu^4.\]  
(40)

The above system gives

\[\alpha = \pm \left(\frac{-3\epsilon^2 - 8\epsilon \mu^2 + 8\mu^4}{3\sqrt{2}}\right), \quad b = \pm \frac{-3\epsilon^2 - 8\epsilon \mu^2 + 8\mu^4}{3\sqrt{2}}, \quad \lambda = \pm \frac{2(\sqrt{-\epsilon \mu^2 + \mu^4})}{\sqrt{3}}.\]  
(41)

where \(\epsilon\) and \(\mu\) are arbitrary constants. Thus, the solutions of (32) are

\[v_1(x, t) = \pm \left(\frac{-3\epsilon^2 - 8\epsilon \mu^2 + 8\mu^4}{3\sqrt{2}}\right) \pm \frac{2(\sqrt{-\epsilon \mu^2 + \mu^4})}{\sqrt{3}} \tanh(\mu(x - \epsilon)), \quad w_1(x, t) = \pm \frac{-3\epsilon^2 - 8\epsilon \mu^2 + 8\mu^4}{3\sqrt{2}} \pm \frac{2(\sqrt{-\epsilon \mu^2 + \mu^4})}{\sqrt{3}} \tanh(\mu(x - \epsilon)).\]  
(42)

Substituting (42) into (36), we get

\[u_1(x, t) = \frac{1}{3}(4\mu^2 - \epsilon - 6\mu^2 \sech^2(\mu(x - \epsilon))).\]  
(43)
If we use the coth instead of tanh, we get the same system and also one more solution follows

\[ v_2(x, t) = \pm \left( -3e^2 - 8\mu^2 + 8\mu^4 \right) \pm \frac{2(\sqrt{-\mu^2 + \mu^4})}{\sqrt{3}} \coth(\mu(x - c)), \]
\[ w_2(x, t) = \pm \left( -3e^2 - 8\mu^2 + 8\mu^4 \right) \pm \frac{2(\sqrt{-\mu^2 + \mu^4})}{\sqrt{3}} \coth(\mu(x - c)). \quad (44) \]

Substituting (44) into (36), we get

\[ v_2(x, t) = \frac{1}{3}(4\mu^2 + \epsilon + 6\mu^2 \coth^2(\mu(x - c))). \quad (45) \]

Second, we apply Ansätze (V). Suppose that

\[ v(x, t) = a + \lambda \sech(\mu\zeta), \]
\[ w(x, t) = b + \lambda \sech(\mu\zeta). \quad (46) \]

Substituting (46) into (37) yields

\[ 0 = 18\lambda^2 - 24\epsilon\mu^2 - 12\mu^4 + (18a\lambda + 18b\lambda) \cosh(\mu\zeta) + (18\lambda^2 - 3e^2 + 4\epsilon\mu^2 + \mu^4) \cosh^2(\mu\zeta). \quad (47) \]

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The above equation is satisfied only if the following system of algebraic equations holds

\[
\begin{align*}
0 &= 18\lambda^2 - 24\epsilon\mu^2 - 12\mu^4 \\
0 &= 18a + 18b \\
0 &= 18\phi - 3\epsilon^2 + 4\epsilon\mu^2 - \mu^4.
\end{align*}
\] (48)

Solving the above system yields

\[
\begin{align*}
a &= \pm \left(\frac{-3\epsilon^2 + 4\epsilon\mu^2 - \mu^4}{3\sqrt{2}}\right), \\
b &= \pm \frac{-3\epsilon^2 + 4\epsilon\mu^2 - \mu^4}{3\sqrt{2}}, \\
\lambda &= \pm \frac{\sqrt{4\epsilon\mu^2 + 2\mu^4}}{3\sqrt{3}},
\end{align*}
\] (49)

where \(\epsilon\) and \(\mu\) are arbitrary constants. Thus, the solutions of (32) are

\[
\begin{align*}
v_3(x, t) &= \pm \left(\frac{-3\epsilon^2 + 4\epsilon\mu^2 - \mu^4}{3\sqrt{2}}\right) \pm \frac{\sqrt{4\epsilon\mu^2 + 2\mu^4}}{3\sqrt{3}} \sech(\mu(x - \epsilon)), \\
w_3(x, t) &= \pm \frac{-3\epsilon^2 + 4\epsilon\mu^2 - \mu^4}{3\sqrt{2}} \pm \frac{\sqrt{4\epsilon\mu^2 + 2\mu^4}}{3\sqrt{3}} \sech(\mu(x - \epsilon)).
\end{align*}
\] (50)

Substituting (50) into (36), we get

\[
u_3(x, t) = \frac{1}{3}(\mu^2 - \epsilon - 6\mu^2 \sech^2(\mu(x - \epsilon))).\] (51)

Third, we apply the cosine method; Ansätze (I). From (35), suppose \(v(x, t) = w(x, t)\) and then substitute (3) into (36) to get

\[
\begin{align*}
0 &= -8\mu^4 + 6\gamma\mu^3 + 7\gamma^2\mu^2 - 6\gamma^3\mu^2 + \gamma^4\mu^4 + \cos^2(\mu\xi)(-8\epsilon\mu^4 + 12\epsilon\mu^2 - 4\epsilon\gamma^2\mu^2 + 8\mu^4 - 12\gamma\mu^4 + 6\gamma^3\mu^4 - 2\gamma^4\mu^4) \\
&\hspace{2cm}+ \cos^4(\mu\xi)(3\epsilon^2 + 4\epsilon\gamma^2\mu^2 + \gamma^4\mu^4) \\
&\hspace{2cm}+ 18\lambda^2 \cos^4(2\gamma)(\mu\xi).
\end{align*}
\] (52)

The above equation is satisfied only if the following system of algebraic equations holds and \(\mu\) is any nonzero constant

\[
\begin{align*}
0 &= -8\mu^4 + 6\gamma\mu^3 + 7\gamma^2\mu^2 - 6\gamma^3\mu^2 + \gamma^4\mu^4, \\
0 &= -8\epsilon\mu^4 + 12\epsilon\gamma\mu^2 - 4\epsilon\gamma^2\mu^2 + 8\mu^4 - 12\gamma\mu^4 + 6\gamma^3\mu^4 - 2\gamma^4\mu^4 - 18\lambda^2, \\
0 &= 3\epsilon^2 + 4\epsilon\gamma^2\mu^2 + \gamma^4\mu^4, \\
0 &= 2 + 2\gamma.
\end{align*}
\] (53)
Solving the above system gives

\[ \lambda = \pm \sqrt{2} \mu^2, \quad \epsilon = -\mu^2, \quad \gamma = -1, \]
\[ \lambda = \pm \frac{\sqrt{10} \mu^2}{3}, \quad \epsilon = -\frac{\mu^2}{3}, \quad \gamma = -1. \]  

(54)

Therefore, the solutions of (32) are

\[ v_4(x, t) = w_4(x, t) = \pm \sqrt{2} \mu^2 \sec(\mu(x + \mu^2)), \]
\[ v_5(x, t) = w_5(x, t) = \pm \frac{\sqrt{10} \mu^2}{3} \sec(\mu(x + \frac{\mu^2}{3})). \]  

(55)

Substituting (55) into (36) gives

\[ u_4(x, t) = \sqrt{2} \mu^2 \sec^2(\mu(x + \mu^2)) \]
\[ u_5(x, t) = \frac{2\mu^2}{9}(-1 + 9 \sec^2(\mu(x + \frac{\mu^2}{3}))). \]  

(56)

Now, by the sine method, we obtain the same system (53) and thus two more solutions follows

\[ v_6(x, t) = w_6(x, t) = \pm \sqrt{2} \mu^2 \csc(\mu(x + \mu^2)), \]
\[ v_7(x, t) = w_7(x, t) = \pm \frac{\sqrt{10} \mu^2}{3} \csc(\mu(x + \frac{\mu^2}{3})). \]  

(57)

Substituting (57) into (36) gives

\[ u_6(x, t) = \sqrt{2} \mu^2 \csc^2(\mu(x + \mu^2)), \]
\[ u_7(x, t) = \frac{2\mu^2}{9}(-1 + 9 \csc^2(\mu(x + \frac{\mu^2}{3}))). \]  

(58)

Figure 4: The second obtained solution \( v(x, t) \) for equation 4.1. \( \lambda = \pm \sqrt{2}, \mu = 1, \beta = 1, \epsilon = -1; -5 \leq x \leq 5; t = 0.1, 0.2, 0.3, 0.4, 2.1, 2.2, 2.3, 2.4 \)

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Figure 5: The second obtained solution $u(x,t)$ for equation 4.1. $\lambda = -\sqrt{2}, \mu = 1, \beta = 1, c = -1; -5 \leq x \leq 5; t = 0.1, 0.2, 0.3, 0.4, 2.1, 2.2, 2.3, 2.4$

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