Numerical Solution of the MRLW Equation Using Finite Difference Method

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Abstract: Two finite difference approximations for the space discretization and a multi-time step method for the time discretization are proposed for the modified regularized long wave equations. The efficiency of the proposed two algorithms is examined by evaluating the $L_{\infty}$ norm and the three invariants of the motion. The results arising from the experiments are compared with the theoretical solution.

Keywords: Finite difference method; Soliton wave; Modified regularized long wave equation.

1 Introduction

The modified regularized long-wave (MRLW) equation has the form [1]

$$u_t + u_x + \varepsilon u^2 u_x - \mu u_{xxx} = 0,$$

(1)

where $u = u(x, t)$ is an appropriately differentiable function of the space and the time variables, respectively. This equation has a major role in the propagation of nonlinear dispersive waves. The boundary conditions on the region $a \leq x \leq b$ are

$$u(a, t) = u(b, t) = 0,$$
$$u_x(a, t) = u_x(b, t) = 0, \quad t \in (0, T)$$
$$u_{xx}(a, t) = u_{xx}(b, t) = 0,$$

(2)

and the initial condition is

$$u(x, 0) = f(x).$$

(3)

The MRLW equation (1) was considered by Gardner et al. [1] using collocation method with quintic B-splines finite element. The equation was solved numerically using the finite difference method and interaction of solitary waves and other properties of the MRLW equation were also studied in [2]. An algorithm for the numerical solution of the MRLW equation was proposed using quadratic B-spline finite element within collocation method and four test problems were worked out to examine the performance of the algorithm in the paper [3]. The collocation method using cubic B-splines was applied to study the solitary waves of the MRLW equation [4]. Hassan and Alamery [5] proposed the collocation methods using sextic B-splines for solving numerically the MRLW equation. Raslan and Hassan [6] presented a computational comparison study of quadratic, cubic, quartic and quintic splines for solving the MRLW equation. A numerical method based on collocation method using quintic B-spline finite elements within the collocation method leads to a system of first order differential equations was solved by fourth order Runge–Kutta method [7]. Collocation method based on quartic B-splines was applied to investigate propagation of nonlinear dispersive solitary waves of the MRLW equation [8]. A new 10-point multisymplectic scheme was proposed for the numerical solution of the MRLW equation [9].

In this study, MRLW equation is solved numerically using the finite difference method. The aim of the paper is to investigate the accuracy of the three steps method when it is used for the time discretization of the equation. After the new time discretization of the Eq. (1) is performed, three and five-point stencils approximating of the first and second derivatives for the space discretization are used to obtain a system of algebraic equation. The nonlinear part of the resulting system of the method is handled by using an inner iteration. In the numerical experiments section, the propagation of a

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solitary wave and the interaction of two solitary waves are investigated. The numerical results obtained in this study demonstrate that the proposed two algorithms are a remarkably successful numerical technique for solving the M RLW equation and also can be efficiently applied to the similarly important nonlinear evolution equations.

2 Time discretization

For computational work, the space-time plane is discretized by grid with space step length \( h \) and the time step \( \Delta t \). The exact solution of unknown function at the grid point is denoted by

\[
u(x_m, t_n) = u_m^n, \quad m = 0, 1, \ldots, N; \quad n = 0, 1, 2, \ldots
\]

and the notation \( U_m^n \) is used to represent the numerical value of \( u_m^n \).

If we take \( v = u - \mu u_{xx} \), the Eq. (1) can be rewritten as

\[
v_t = -(u_x + \varepsilon u^2 u_x).
\]

Consider the time discretization of the form

\[
v^{n+1} + \theta_1 v^n + \theta_2 v^{n-1} = \theta_3 e_i^{n+1} + \theta_4 e_i^n + \theta_5 e_i^{n-1}
\]

where \( v^{n+i} = v(x, t_n + i\Delta t), i = -1, 0, 1 \) and \( \theta_1, \ldots, \theta_5 \) are real constants determined in later to have higher accuracy of the proposed method with respect to time discretization. Eq. (4) subject to Eq. (5) is written as

\[
U^{n+1} + \theta_3 \left( 1 + \varepsilon \left( U^{n+1} \right)^2 \right) U_x^{n+1} - \mu U_{xx}^{n+1} = -\theta_1 U^n - \theta_4 \left( 1 + \varepsilon \left( U^n \right)^2 \right) U_x^n + \mu \theta_1 U_{xx}^n
\]

\[
-\theta_2 U^{n-1} - \theta_5 \left( 1 + \varepsilon \left( U^{n-1} \right)^2 \right) U_x^{n-1} + \mu \theta_2 U_{xx}^{n-1}
\]

It can be easily verified that the local truncation error error arising from Eq. (6) is \( O((\Delta t)^3) \) when \( \theta_1 = -1, \theta_2 = \theta_5 = 0, \theta_3 = \theta_4 = \frac{\Delta t}{2} \) and it is \( O((\Delta t)^5) \) when \( \theta_1 = 0, \theta_2 = -1, \theta_4 = \frac{4\Delta t}{3}, \theta_3 = \theta_5 = \frac{\Delta t}{3} \).

3 Finite difference method I (FDM1)

For the space discretization, we take three-point stencils approximating first and second derivatives

\[
(U_x)_m = \frac{U_{m+1} - U_{m-1}}{2h} + O(h^2)
\]

\[
(U_{xx})_m = \frac{U_{m-1} - 2U_m + U_{m+1}}{h^2} + O(h^2)
\]

Then, substituting (7-8) into Eq. (6), the resulting algebraic system of equations takes the form

\[
U_{m+1}^{n+1} \left( \frac{\alpha_{m1}}{2h} - \frac{\mu}{h^2} \right) + U_m^{n+1} \left( 1 + \frac{2\mu}{h^2} \right) + U_{m+1}^{n+1} \left( \frac{\alpha_{m1}}{2h} - \frac{\mu}{h^2} \right) =
\]

\[
U_{m-1}^{n} \left( \frac{\alpha_{m2}}{2h} + \frac{\mu}{h^2} \theta_3 \right) + U_m^{n} \left( -\theta_1 - \frac{2\mu}{h^2} \theta_1 \right) + U_{m+1}^{n} \left( -\frac{\alpha_{m2}}{2h} + \frac{\mu}{h^2} \theta_1 \right) +
\]

\[
U_{m-1}^{n} \left( \frac{\alpha_{m3}}{2h} + \frac{\mu}{h^2} \theta_2 \right) + U_m^{n-1} \left( -\theta_2 - \frac{2\mu}{h^2} \theta_2 \right) + U_{m+1}^{n-1} \left( -\frac{\alpha_{m3}}{2h} + \frac{\mu}{h^2} \theta_2 \right)
\]

where

\[
\alpha_{m1} = \theta_3 \left( 1 + \varepsilon \left( U_m^{n+1} \right)^2 \right),
\]

\[
\alpha_{m2} = \theta_4 \left( 1 + \varepsilon \left( U_m^{n} \right)^2 \right),
\]

\[
\alpha_{m3} = \theta_5 \left( 1 + \varepsilon \left( U_m^{n-1} \right)^2 \right).
\]
Whereas the finite difference scheme (9) with $\theta_1 = -1, \theta_2 = \theta_5 = 0, \theta_3 = \theta_4 = \frac{\Delta t}{2}$ has a truncation error of $O\left((\Delta t)^3 + \Delta t h^2\right)$, the scheme with $\theta_1 = 0, \theta_2 = -1, \theta_4 = \frac{4\Delta t}{3}, \theta_3 = \theta_5 = \frac{\Delta t}{3}$ has a truncation error of $O\left((\Delta t)^3 + \Delta t h^2\right)$.

This set of equations is the recurrence relationship for unknown parameters vector $d^{n+1} = \left(U_0^{n+1}, \ldots, U_N^{n+1}\right)$. This tridiagonal matrix system is made up $N - 1$ equations including $N + 1$ unknown for $m = 1, \ldots, N - 1$. Application of the boundary conditions

$$u(a, t) = u(b, t) = 0$$

enables the elimination of the variables $U_0^{n+1}$ and $U_N^{n+1}$ from the system (9). Then, the system is reduced to $(N - 1) \times (N - 1)$ matrix system, which can be solved by using the Thomas algorithm. After $d^n = \left(U_0^n, \ldots, U_N^n\right)$ unknown vector are found from the initial condition, taking $\theta_1 = -1, \theta_2 = \theta_5 = 0, \theta_3 = \theta_4 = \frac{\Delta t}{2}$ in the system (9), we can find $d^1 = \left(U_0^1, \ldots, U_N^1\right)$. Once the initial vectors $d^0$ and $d^1$ are computed, $d^{n+1}, n = 1, 2, 3, \ldots$ unknown vectors are found repeatedly by solving the recurrence relation (9) with $\theta_1 = 0, \theta_2 = -1, \theta_4 = \frac{4\Delta t}{3}, \theta_3 = \theta_5 = \frac{\Delta t}{3}$. Note that since the system (9) is an implicit system, we have taken $(U_m^{n+1})^2 \equiv (U_m^n)^2$ in the coefficient $\alpha_m$ and done an inner iteration for 5 times to increase the accuracy of the system.

### 4 Finite difference method II (FDM2)

For the space discretization, we take five-point stencils approximating first and second derivatives

$$\begin{align*}
(U_x)_m &= \frac{U_{m-2} - 8U_{m-1} + 8U_{m+1} - U_{m+2}}{12h} + O(h^4), \\
(U_{xx})_m &= -\frac{U_{m-2} + 16U_{m-1} - 30U_m + 16U_{m+1} - U_{m+2}}{12h^2} + O(h^4).
\end{align*}$$

(10) (11)

Then, substituting (10-11) into Eq. (6), we have

$$\begin{align*}
U_m^{n+1} \left(\frac{\alpha_{m1}}{12h} + \frac{\mu}{12h^2}\right) + U_m^{n-1} \left(-\frac{2\alpha_{m1}}{3h} - \frac{4\mu}{3h^2}\right) + U_m^{n+1} \left(1 + \frac{5\mu}{2h^2}\right) + \\
U_m^{n+1} \left(-\frac{2\alpha_{m1}}{3h} - \frac{4\mu}{3h^2}\right) + U_m^{n-1} \left(-\frac{\alpha_{m1}}{12h} + \frac{\mu}{12h^2}\right) = \\
U_m^n \left(\frac{\alpha_{m2}}{12h} + \frac{\mu}{12h^2}\theta_1\right) + U_{m-1}^{n-1} \left(\frac{2\alpha_{m2}}{3h} + \frac{4\mu}{3h^2}\theta_1\right) + U_m^n \left(-\theta_1 - \frac{5\mu}{2h^2}\theta_1\right) + \\
U_m^{n+1} \left(-\frac{2\alpha_{m2}}{3h} + \frac{4\mu}{3h^2}\theta_1\right) + U_m^{n+2} \left(-\frac{\alpha_{m2}}{12h} + \frac{\mu}{12h^2}\theta_1\right) = \\
U_m^{n-1} \left(-\frac{\alpha_{m3}}{12h} - \frac{\mu}{12h^2}\theta_2\right) + U_{m-1}^{n-2} \left(-\frac{2\alpha_{m3}}{3h} + \frac{4\mu}{3h^2}\theta_2\right) + U_m^{n-1} \left(-\theta_2 - \frac{5\mu}{2h^2}\theta_2\right) + \\
U_m^{n+1} \left(-\frac{\alpha_{m3}}{12h} - \frac{\mu}{12h^2}\theta_2\right) + U_m^{n+2} \left(-\frac{\alpha_{m3}}{12h} - \frac{\mu}{12h^2}\theta_2\right)
\end{align*}$$

(12)

where

$$\begin{align*}
\alpha_{m1} &= \theta_3 \left(1 + \varepsilon (U_m^{n+1})^2\right), \\
\alpha_{m2} &= \theta_4 \left(1 + \varepsilon (U_m^n)^2\right), \\
\alpha_{m3} &= \theta_5 \left(1 + \varepsilon (U_m^{n-1})^2\right).
\end{align*}$$

Whereas the finite difference scheme (12) with $\theta_1 = -1, \theta_2 = \theta_5 = 0, \theta_3 = \theta_4 = \frac{\Delta t}{2}$ has a truncation error of $O\left((\Delta t)^3 + \Delta t h^4\right)$, the scheme with $\theta_1 = 0, \theta_2 = -1, \theta_4 = \frac{4\Delta t}{3}, \theta_3 = \theta_5 = \frac{\Delta t}{3}$ has a truncation error of $O\left((\Delta t)^3 + \Delta t h^4\right)$. 

LINS homepage: http://www.nonlinearscience.org.uk/
This set of equations is recurrence relationship for unknown parameters vector \( d^{n+1} = (U_{n-2}^{n+1}, U_{n-1}^{n+1}, \ldots, U_{N+1}^{n+1}, U_{N+2}^{n+1}) \). This pentadiagonal matrix system is made up \( N + 1 \) equations including \( N + 5 \) unknown for \( m = 0, 1, \ldots, N - 1, N \).

Application of the boundary conditions

\[
\begin{align*}
    u_x(a,t) &= u_x(b,t) = 0, \\
    u_{xx}(a,t) &= u_{xx}(b,t) = 0,
\end{align*}
\]

enables the elimination of the variables \( U_{n+1}^{n+1}, U_{n+1}^{n+1}, U_{N+1}^{n+1} \) and \( U_{N+2}^{n+1} \) from the system (12). Then, the system is reduced to \( (N + 1) \times (N + 1) \) matrix system, which can be solved by using the Thomas algorithm. After \( d^0 \) and \( d^1 \) unknown vectors are computed, \( d^{n+1} \) unknown vector can be found by solving the system (12) with \( \theta_1 = 0, \theta_2 = -1, \theta_4 = \frac{4\Delta t}{3} \), \( \theta_3 = \frac{\Delta t}{3} \) as the previous section.

5 Numerical experiments

In this section, to illustrate the effectiveness of the presented numerical scheme, two test problems are studied for the MRLW equation. Since the analytical solution of the MRLW equation is known for the first test problem, the \( L_\infty \) error norm is used to measure the error between the analytical and numerical results at the node points:

\[
L_\infty = \max_m |u_m - U_m|.
\]

Since an accurate numerical scheme must keep the conservation properties of evolution equations, we will monitor the three invariants of numerical solution for the MRLW equation corresponding to conservation of mass, momentum and energy given by the following integrals [1]:

\[
I_1 = \int_{-\infty}^{\infty} u dx, \quad I_2 = \int_{-\infty}^{\infty} (u^2 + \mu (u_x)^2) dx, \quad I_3 = \int_{-\infty}^{\infty} \left( u^4 - \mu \left( \frac{6}{\varepsilon} \right) (u_x)^2 \right) dx
\]

(13)

5.1 Motion of single solitary wave for MRLW equation

It is known [1] that Eq. (1) has a solitary wave solution of the form

\[
u(x,t) = \sqrt{\frac{6\varepsilon}{\mu}} \text{sech}(k[x - x_0 - (c + 1)t])
\]

(14)

where \( k = \sqrt{\frac{\mu}{\varepsilon}} \). This solution corresponds to a solitary wave of amplitude \( A = \sqrt{\frac{6\varepsilon}{\mu}} \), initially centered on the peak position \( x_0 \) propagating towards the right without change of shape at a steady velocity \( c + 1 \).

For this problem the analytical values of the invariants by using the solitary wave solution (14) when \( t = 0 \) can be found as

\[
I_1 = \frac{\pi A}{k}, \quad I_2 = \frac{2A^2}{k} + \frac{2\mu k A^2}{3}, \quad I_3 = \frac{4A^2}{3k \varepsilon} \left( A^2 \varepsilon - 3\mu k^2 \right).
\]

(15)

Numerical experiments of this motion are performed for the amplitudes of \( A = \sqrt{0.3}, 1 \) and parameters \( \varepsilon = 6, \mu = 1 \). In the first case for the first test problem, we have used various space and time steps, peak position \( x_0 = 40 \) and amplitude 1 over the region \( 0 \leq x \leq 100 \). After the computation is done until time \( t = 10 \) to find error norm \( L_\infty \) and numerical invariants \( I_1, I_2, I_3 \), results of the proposed two algorithms together with the analytical values of the invariants are documented in Table 1. It is clearly seen that the result obtained by the FDM2 are more accurate then obtained by the FDM1. We can also say that when we use smaller time and space steps, numerical invariants for the FDM2 are almost the same as the their exact values. In the second case after the program is rerun up to time \( t = 10 \) with various space and time steps and \( \varepsilon = 6, \mu = 1, c = 0.3, x_0 = 40 \), the computed invariants and error norm for the two proposed algorithms are given in Table 2. According to the Table 2, FDM2 gives more accurate results then FDM1. It can also be seen from the table that the computed values of invariants for the FDM2 are in good agreement with the analytical values of the invariants when we use small time and time steps.

Absolute error (analytical-numerical) distributions for the two algorithms with amplitude 1 and \( x_0 = 40, h = \Delta t = 0.1 \) is drawn at time \( t = 10 \) in Fig1 Fig2. It can be easily observed that maximum error is taken place around the middle of the space interval.

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Table 1: Invariants and error norms for a single solitary wave at time $t = 10, \varepsilon = 6, \mu = 1, c = 1, x_0 = 40, [a, b] = [0, 100]$.

<table>
<thead>
<tr>
<th>$h = \Delta t$</th>
<th>$L_\infty$</th>
<th>$I_1$</th>
<th>$I_2$</th>
<th>$I_3$</th>
<th>$L_\infty$</th>
<th>$I_1$</th>
<th>$I_2$</th>
<th>$I_3$</th>
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<td>1.4348608</td>
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<td>1.5635449</td>
</tr>
<tr>
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<td>3.2962654</td>
<td>1.4192589</td>
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</tr>
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<td>2.4$\times$10$^{-4}$</td>
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</tr>
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</tr>
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Analytical values of invariants

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<tr>
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<th>FDM2</th>
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<tr>
<td>4.4428829</td>
<td>3.2998316</td>
</tr>
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</table>

Table 2: Invariants and error norms for a single solitary wave at time $t = 10, \varepsilon = 6, \mu = 1, c = 0.3, x_0 = 40, [a, b] = [0, 100]$.

<table>
<thead>
<tr>
<th>$h = \Delta t$</th>
<th>$L_\infty$</th>
<th>$I_1$</th>
<th>$I_2$</th>
<th>$I_3$</th>
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<tr>
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<td>0.1537294</td>
</tr>
<tr>
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Analytical values of invariants

<table>
<thead>
<tr>
<th>FDM1</th>
<th>FDM2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5819667</td>
<td>1.3450765</td>
</tr>
</tbody>
</table>

Figure 1: Absolute error distribution at $t = 10$ for FDM1

Figure 2: Absolute error distribution at $t = 10$ for FDM

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5.2 Interaction of two solitary waves for MRLW equation

In the second test problem, interaction of two positive solitary waves for MRLW equation is studied by using the initial condition given by the linear sum of two separate solitary waves of various amplitudes

\[ u(x, 0) = A_1 \text{sech}(k_1[x - x_1]) + A_2 \text{sech}(k_2[x - x_2]) \] (16)

where\( A_i = \sqrt{\frac{6c_i}{\varepsilon}} \), \( k_i = \sqrt{\frac{c_i}{\mu(c_i + 1)}} \), \( i = 1, 2 \), and \( x_i, c_i \) are arbitrary constants. Calculations are carried about with \( \varepsilon = 6, \mu = 1, c_1 = 1, c_2 = 1/4, x_1 = 30, x_2 = 60, h = 0.1, \Delta t = 0.01 \) over the space interval \([0, 200]\) from \( t = 0 \) to \( t = 60 \). These parameters provide two solitary waves of amplitude 1 and 0.5 and peak positions of them are located at \( x = 30 \) and 60. The analytical invariants can be found as

\[ I_1 = \frac{\pi}{k_1 k_2} (k_2 A_1 + k_1 A_2) \simeq 7.955290304, \]
\[ I_2 = \frac{2}{k_1 k_2} (k_2 A_1^2 + k_1 A_2^2) + \frac{2\mu}{3k_1 k_2} (k_1^2 k_2 A_1^2 + k_1 k_2^2 A_2^2) \simeq 4.492401234, \]
\[ I_3 = \frac{4}{3k_1 k_2 \varepsilon} (\varepsilon k_1 A_2^4 - 3\mu k_1 k_2^2 A_2^2 + \varepsilon k_2 A_1^4 - 3\mu k_1^2 k_2 A_1^2) \simeq 1.526016961. \]

Simulations of interaction of two solitary waves at times \( t = 0, t = 35 \) and \( t = 60 \) are plotted in Figure 2 for the FDM2. Since solitary waves don’t obey superposition, when a taller wave overtakes a shorter wave, they don’t combine and add together. As seen in Fig3, two solitary waves at the time \( t = 0 \) are propagated to the right with velocities dependent upon their magnitudes and reached a stage that the larger solitary wave has passed through the smaller solitary wave and emerged in their original positions.

The absolute value of the difference between exact and numerical values of the invariants for the both algorithms are plotted in Fig4 and Fig5. We can easily seen that the first invariant remain almost constant while the second and third invariants are affected more from the interaction of the solitary waves during run of the two algorithms. We can also say that FDM2 gives more accurate results than FDM1 too.

6 Conclusion

The MRLW equation is solved numerically using finite difference method with multi-time step discretization. The performance of the algorithms has been examined by studying the propagation of the single solitary wave and interaction of two solitary waves. Numerical results show that since the proposed method is an accurate and efficient numerical technique and also application of the method is easier than many other numerical techniques, proposed scheme especially FDM2 can be efficiently applied to the MRLW and this type of nonlinear problems.
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References


IJNS homepage: http://www.nonlinearscience.org.uk/