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# Positive Solutions for a Nonlocal Boundary-Value Problem of a Class of Arbitrary (Fractional) Orders Differential Equations

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**Abstract:** In this paper we study the existence of positive solution for the boundary value problem of the arbitrary (fractional ) orders differential equation $u''(t) + f(t, {}^cD^{\alpha+1}u(t)) = 0, \alpha \in (0, 1)$, a.e. $t \in (0, 1)$ with the boundary conditions $u'(0) = 0,\ u(1) = \sum_{k=1}^{m} a_k\, u(\tau_k),\ \tau_k \in (a, b) \subset (0, 1)$. The cases of $\alpha = 0$ and $\alpha = 1$ will be deduced. The corresponding integral condition problem will be considered.

**Keywords:** fractional calculus; boundary value problem; nonlocal condition; integral condition; positive solution

## 1 Introduction

Problems with boundary conditions have been extensively studied by several authors in the last two decades. The reader is referred to ([1]-[3]) and ([6]-[20]), and references therein.
In this work we study the existence of at least one solution for the nonlocal problem of the arbitrary (fractional) order differential equation

$$u''(t) + f(t, {}^cD^{\alpha+1}u(t)) = 0, \alpha \in (0, 1),\ \text{a.e. } t \in (0, 1) \tag{1}$$

with the nonlocal (multi-points) boundary conditions

$$u'(0) = 0,\ u(1) = \sum_{k=1}^{m} a_k\, u(\tau_k),\ \tau_k \in (a, b) \subset (0, 1). \tag{2}$$

Also we deduce the same results for the nonlocal problems of the two differential equations

$$u''(t) + f(t, u'(t)) = 0. \tag{3}$$

$$u''(t) + f(t, u''(t)) = 0. \tag{4}$$

The integral conditions problem for the differential equations (2),(3) and (4) are also studied.

## 2 Preliminaries

Let $L^1(I)$ denotes the class of Lebesgue integrable functions on the interval $I = [0, 1]$, where $0 \leq a < b < \infty$ and let $\Gamma(.)$ denotes the gamma function.

**Definition 2.1** *The fractional-order integral of the function $f \in L^1[a, b]$ of order $\beta > 0$ is defined by (see [22])*

$$I_a^\beta f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds,$$

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**Definition 2.2** *The Caputo fractional-order derivative of order* $\alpha \in (0,1]$ *of the absolutely continuous function* $f(t)$ *is defined by (see [21] and [22]).*

$$D_a^{\alpha} f(t) = I_a^{1-\alpha} \frac{d}{dt} f(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} f(s)\, ds.$$

**Theorem 2.1** ***(Schauder fixed point theorem)*** *[4]*
*Let* $E$ *be a Banach space and* $Q$ *be a convex subset of* $E$*, and* $T : Q \longrightarrow Q$ *is compact, continuous map, Then* $T$ *has at least one fixed point in* $Q$*.*

**Theorem 2.2** ***(Kolmogorov compactness criterion)*** *[5]*
*Let* $\Omega \subseteq L^p(0,1),\ 1 \leq p < \infty$*. If*

*(i)* $\Omega$ *is bounded in* $L^p(0,1)$*, and*

*(ii)* $u_h \to u$ *as* $h \to 0$ *uniformly with respect to* $u \in \Omega$*, then* $\Omega$ *is relatively compact in* $L^p(0,1)$*, where*

$$u_h(t) = \frac{1}{h} \int_t^{t+h} u(s)\, ds.$$

# 3 Main results

Consider firstly the fractional-order functional integral equation

$$y(t) = f(t,\ -I^{1-\alpha} y(t)),\ \alpha \in (0,\ 1). \tag{5}$$

**Definition 3.1** *The function* $y$ *is called a solution of the fractional-order functional integral equation (5), if* $y \in L^1[0,1]$ *and satisfies (5).*

Consider the following assumptions:

(i) $f : [0,1] \times R \to R^+$ be a function with the following properties:

- (a) $u \longrightarrow f(t,u)$ is continuous for almost all $t \in [0,1]$,
- (b) $t \longrightarrow f(t,u)$ is measurable for each $u \in R$,

(ii) there exists an integrable function $a \in L^1[0,1]$ and a constant $b > 0$, such that

$$|f(t,u)| \leq a(t) + b\,|u|, \ \text{a.e } t \in [0,1],$$

**Theorem 3.1** *Let the assumptions (i) and (ii) are satisfied.*

$$\text{If} \quad B = \frac{b}{\Gamma(2-\alpha)} < 1, \tag{6}$$

*then the fractional-order functional integral equation (5) has at least one positive solution* $y \in L^1[0,1]$*.*

**Proof.** Define the operator $T$ associated with equation (5) by

$$Ty(t) = f(t,\ -I^{1-\alpha} y(t))$$

Let $Q_r^+ = \{y \in R^+ : \|y\| < r,\ r > 0\},\ r = \frac{\|a\|}{1-B}$
Let $y$ be an arbitrary element in $Q_r^+$, then from assumptions (i) and (ii), we obtain

$$\begin{aligned}
||Ty||_{L^1} &= \int_0^1 |Ty(t)|\,dt \\
&= \int_0^1 |f(t,\ -\ I^{1-\alpha}y(t))|dt \\
&\leq \int_0^1 |a(t)|dt \ +\ b\int_0^1\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}|y(s)|\,ds\,dt \\
&= \|a\|_{L^1}\ +\ b\int_0^1\int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}dt|y(s)|\,ds \\
&\leq \|a\|_{L^1}\ +\ b\int_0^1 \frac{(t-s)^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}|_s^1|y(s)|\,ds \\
&\leq \|a\|_{L^1}\ +\ b\int_0^1 \frac{1}{\Gamma(2-\alpha)}|y(s)|\,ds \\
&\leq \|a\|_{L^1}\ +\ \frac{b}{\Gamma(2-\alpha)}\|y\|_{L^1} = r.
\end{aligned}$$

which implies that the operator $T$ maps $Q_r^+$ into it self.
Assumption (i) implies that $T$ is continuous. Now let $\Omega$ be a bounded subset of $Q_r^+$, then $T(\Omega)$ is bounded in $L^1[0,1]$, i.e. condition (i) of Theorem 2.2 is satisfied.
Let $y \in\ \Omega$, then

$$\begin{aligned}
||(Ty)_h \ - \ Ty|| &= \int_0^1 |(Ty)_h(t) - (Ty)(t)|\,dt \\
&= \int_0^1 |\frac{1}{h}\int_t^{t+h} (Ty)(s)\,ds\ -\ (Ty)(t)|\,dt \\
&\leq \int_0^1 \left(\frac{1}{h}\int_t^{t+h} |(Ty)(s)\ - (Ty)(t)|\,ds\right)\,dt \\
&\leq \int_0^1 \frac{1}{h}\int_t^{t+h} |f(s,\ -\ I^{1-\alpha}y(t))\ -\ f(t,\ -\ I^{1-\alpha}y(t))|\,ds\,dt.
\end{aligned}$$

the assumption (ii) implies that $f \in L^1(0,1)$ and

$$\frac{1}{h}\int_t^{t+h} |f(s,\ -\ I^{1-\alpha}y(t))\ -\ f(t,\ -\ I^{1-\alpha}y(t))|\,ds\ \rightarrow\ 0$$

Therefore, by Theorem 2.2, we deduce that $T(\Omega)$ is relatively compact, that is, $T$ is a compact operator, then the operator $T$ has a fixed point $Q_r^+$, which proves the existence of positive solution $y \in\ L^1[0,1]$ of equation (5). ■

**Theorem 3.2** *Let the assumptions of Theorem 3.1 are satisfied. If* $0 < \sum_{k=1}^m\ a_k\ < 1$, *then nonlocal problem (1)- (2) has at least one positive solution* $u \in C[0,1],\ u' \in AC[0,1]$.

**Proof.** Consider the nonlocal problem of the differential equation

$$u''(t)\ +\ f(t,\ {}^cD^{\alpha\,+\,1}u(t)\ ) = 0\ ,\alpha\ \in (1,\ 2),\ \text{a.e. } t \in (0,\ 1)$$

with the conditions

$$u'(0) = 0, \quad u(1) = \sum_{k=1}^m\ a_k\ u(\tau_k).$$

Let $-\,y(t)\ =\ u''(t),$ then

$$u(t)\ =\ u(0)\ -\ I^2 y(t) \tag{7}$$

where $y$ is the solution of the fractional-order functional integral equation (5).
Let $t=\tau_k$ in equation (7), we get

$$u(\tau_k) = -\int_0^{\tau_k} (\tau_k - s)\ y(s)\ ds + u(0)$$

and

$$\sum_{k=1}^{m} a_k u(\tau_k) = -\sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\ y(s)\ ds +\ u(0)\sum_{k=1}^{m} a_k.$$

From equation (2), we get

$$-\int_0^1 (1-s)\ y(s)\ ds + u(0) = -\sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\ y(s)\ ds + u(0)\sum_{k=1}^{m} a_k,$$

then

$$u(0)\ = A\left(\int_0^1 (1-s)\ y(s)\ ds\ -\ \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\ y(s)\ ds\right),\ \text{ where } A = (1\ -\ \sum_{k=1}^{m} a_k)^{-1}.$$

and

$$u(t)\ = A\int_0^1 (1-s)\ y(s)\ ds\ -\ A\sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\ y(s)\ ds\ -\ \int_0^t (t-s) y(s)\ ds. \tag{8}$$

This equation can be written as

$$\begin{aligned} u(t) = \{ &\int_0^1 (1-s)\ y(s)\ ds\ +\ A\sum_{k=1}^{m} a_k \tau_k \int_0^1 (1-s) y(s)\ ds \\ &-\ A\sum_{k=1}^{m} a_k\ \int_0^{\tau_k} (\tau_k - s) y(s)\ ds\}\ -\ \int_0^t (t-s) y(s)\ ds. \end{aligned} \tag{9}$$

which, by Theorem 3.1, has at least one solution $u \in C(0,1)$ and

$$u(0) = \lim_{t \to\ 0^+} u(t) = 0$$

$$u(1) = \lim_{t \to\ 1^-} u(t) = A\sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\ y(s)\ ds -\ A\int_0^1 (1-s)\ y(s)\ ds -\ \int_0^1 (1-s) y(s)\ ds$$

from which we deduce that equation (9) has at least one solution $u \in C[0,1]$.
Now,

$$\begin{aligned} \sum_{k=1}^{m} a_k\ \int_0^{\tau_k} (\tau_k - s) y(s)\ ds &< \sum_{k=1}^{m} a_k \tau_k\ \int_0^1 (1 - \frac{s}{\tau_k}) y(s)\ ds \\ &< \sum_{k=1}^{m} a_k \tau_k\ \int_0^1 (1-s) y(s)\ ds \end{aligned}$$

and

$$\int_0^t (t-s) y(s)\ ds < \int_0^1 (1-s) y(s)\ ds$$

then the solution of (9) is positive.
To complete the proof, we prove that the integral equation (9) satisfies the nonlocal problem (1)-(2).

Differentiating (9), we obtain

$$\frac{d^2u}{dt^2} = -\, y(t),$$

$$D^{\alpha+1}\, u(t) = I^{1-\alpha}\, \frac{d^2}{dt^2} u(t) = -\, I^{1-\alpha} y(t)$$

and

$$u''(t) \,+\, f(t,\, D^{\alpha+1}u(t)) = 0.$$

Also

$$\begin{aligned}
\sum_{k=1}^{m} a_k u(\tau_k) &= A \sum_{k=1}^{m} a_k \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds - \, A \sum_{k=1}^{m} a_k \int_0^1 (1-s)\, y(s)\, ds \\
&- \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) y(s)\, ds \\
&= (A \sum_{k=1}^{m} a_k \,-\, 1) \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds - \, A \sum_{k=1}^{m} a_k \int_0^1 (1-s)\, y(s)\, ds \\
&= (A \,+\, 1 \,-\, 1) \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds - \, (A \,+\, 1) \int_0^1 (1-s)\, y(s)\, ds \\
&= A \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds - \, A \int_0^1 (1-s)\, y(s)\, ds - \int_0^1 (1-s)\, y(s)\, ds \,=\, u(1)
\end{aligned}$$

This complete the proof of the equivalent between the nonlocal problem (1)-(2) and the integral equation (9) .
This implies that there exists at least one positive solution $u \in C[0,1]$ of the nonlocal problem (1)-(2). ■

**Corollary 3.1** *Let the assumptions (i) and (ii) are satisfied, then the nonlocal problem*

$$u''(t) \,+\, f(t, u''(t)) = 0$$

$$u'(0) = 0,\; u(1) = \sum_{k=1}^{m} a_k\, u(\tau_k),\; \tau_k \in (a,\, b) \subset \,(0,\, 1),\; 0 < \sum_{k=1}^{m} a_k \,< 1.$$

*has at least one positive solution.*

**Proof.** Letting $\alpha \,\to\, 1$ in Theorem (3.1) we get (see [21] )the result. ■

**Corollary 3.2** *Let the assumptions (i) and (ii) are satisfied, then the nonlocal problem*

$$u''(t) \,+\, f(t, u'(t)) = 0$$

$$u'(0) = 0,\; u(1) = \sum_{k=1}^{m} a_k\, u(\tau_k),\; \tau_k \in (a,\, b) \subset \,(0,\, 1),\; 0 < \sum_{k=1}^{m} a_k \,< 1.$$

*has at least one positive solution.*

**Proof.** Letting $\alpha \,\to\, 0$ in Theorem (3.1) we get (see [21] )the result. ■

# 4 Nonlocal integral condition

Let $u \in C[0,1]$ be the solution of the nonlocal problem (1)-(2).
Let $a_k = t_j - t_{k-1},\;\; \eta_k \in (t_{k-1}, t_j),\; a \,=\, t_0 < t_1 < t_2, ... < t_n = b$ then the nonlocal condition (2) will be

$$u(1) = \sum_{k=1}^{m} \,(t_j - t_{k-1})\, u(\eta_k).$$

From the continuity of the solution $u$ of the nonlocal problem (1)-(2) we deduce that

$$\lim_{m\to\infty}\sum_{k=1}^{m} (t_j - t_{k-1})\, u(\eta_k) \;=\; \int_a^b u(s)\, ds.$$

and the nonlocal condition (2) transformed to the integral one

$$u'(0) = 0,\ u(1) = \int_a^b u(s)\, ds.$$

Now, we have the following Theorem

**Theorem 4.1** *Let the assumptions of Theorem 3.2 are satisfied. Then there exist at least one positive solution* $u \in AC[0,1]$ *of the nonlocal problem with integral condition,*

$$u'(t) \;+\; f(t,\ D^{\alpha+1}x(t)) = 0\ , \alpha \in (0,\ 1),\ a.e.\ t \in (0,\ 1),$$

$$u'(0) = 0,\ u(1) = \int_a^b u(s)\, ds.$$

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