Three-Unit Repairable System with Priority and Single Vacation

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Abstract: Three-unit repairable system with preemptive priority of part 1 and single vacation of a repairman is studied in this paper. It is assumed that the system has a shut down rule, parts can be repaired as good as new, part 1 has the preemptive priority right to be repaired, the other parts are "first bad first repair", the life-length of units has exponential distribution, the system are obtained by using the method of supplement variables.

Keywords: the repairable system; single vacation; priority; availability; reliability

1 Introduction

Three-unit repairable system (Figure 1) in recent years has become more popular. Li Wei et al [1] studied the repairable system with two repairmen, and got the system reliability and availability. Kovalenko [2] discussed the repairable system with one repairman, and obtained the system average operating time and failure frequency. Hu Lin-min, etc. [3] [4], Cheng Jiang et al [5], considering the repairman can leave the system in part time for vacation, studied the Three-unit repairable system with multi-vacations, part 1 has non-preemptive repair priority, Part 2 and part 3 are first bad first repaired, and gave some important results of reliability. For priority, it is mentioned little in the Three-unit repairable system, especially in preemptive priority right to be repaired. Consider the actual maintenance of the system, such as the operation maintenance of a high performance system, in which some components of the system require to be served at first. In this paper, it is assumed that the system has a certain shut down rule, parts can be repaired as good as new, part 1 has the preemptive priority right to be repaired, the other parts are first bad first repaired, by using the method of supplement variables, vector Markov process method and the tool of LT (Laplace transform), some availability and reliability indices of the system are obtained.

2 Model description

Figure 1 shows the system, and we assuming that: (1) system consists of three dissimilar components, i.e. part 1, part 2 and part 3; (2) the system has a shut down rule: If part 1 fails, the remained parts will be shut down (that means the units will be

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temporarily halted and the life-length of the units will be accumulated when the system reoperates), or if parts 2, 3 fail, part 1 will be shut down; (3) the system has one repairman, parts can be repaired as good as new. Part 1 has the preemptive repair priority, that is, when part 2 or part 3 is in repair, if part 1 fails, the repairman stops his service at once and changes to serve part 1 until the service finished, then continues the interrupted repairs; Part 2 and part 3 are first bad first repaired, that is, if both units 2 and 3 fail, the one that fails later has to wait for repair until the repair of the unit which failed earlier is completed; (4) the life-length of parts, the vacation time, and the repair-time are independent. (5) initially all parts are good, the system is in operation and the repairman is on vacation. Further assuming: Part 1 follows exponential distribution with a variable failure rate (the normal failure rate of part 1 is $\lambda_1$; when one of the other parts faults, its failure rate will be changed to $\lambda_2$ if $\lambda_2 < \lambda_1$); part 2, part 3 follow exponential distribution with the failure rate $\lambda_2, \lambda_3$ respectively, then the lifetime distribution functions are $F_i(t) = 1 - e^{-\lambda_i t}$ ($\lambda_i > 0$) for $i=1,2,3,4$; parts repair times follow the same general continuous distribution with different means, their distribution functions are $G_i(t) = \int_0^t g_i(t)dt = 1 - e^{-\int_0^t \lambda_i(t)dt}$, and the average repair time are $1/\mu_i, \mu_i > 0$ for $i = 1, 2, 3$ respectively. Repairman has a single vacation, the vacation time follows a general continuous distribution with distribution function $H(t) = \int_0^t h(t)dt = 1 - e^{-\int_0^t \gamma(t)dt}$.

3 The state differential equations and solutions

State is defined as:

State 0: three components are good, the system is in operation, repairman is on vacation.

State 1: Part 1 is waiting for repairing, parts 2, 3 are shut down, the system is interrupted, repairman is on vacation.

State 2: Part 1 is running, Part 2 is interrupted, Part 3 is shut down, the system is operating, repairman is on vacation.

State 3: Part 1 is running, Parts 2 is shut down, the system is operating, repairman is on vacation.

State 4: Part 2 is repaired, repairman is on vacation.

State 5: Part 1 is repaired, Part 2 is shut down, the system is operating, repairman is on vacation.

State 6: Part 1 is repaired, Part 2 is running, Part 3 is shut down, the system is operating, repairman is on vacation.

State 7: Part 1 is shut down, Part 2 is running, Part 3 is shut down, the system is operating, repairman is on vacation.

State 8: Part 1 is waiting for repairing, Parts 2, 3 are shut down, the system is interrupted, repairman is on vacation.

State 9: Part 1 is waiting for repairing, Parts 2 is shut down, the system is interrupted.

State 10: Part 1 is waiting for repairing, Part 3 is shut down, the system is interrupted.

State 11: Part 1 is waiting for repairing, Part 3 is running, the system is operating.

State 12: Part 1 is running, Part 2 is interrupted repair, Part 3 is shut down, the system is interrupted.

State 13: Part 1 is running, Part 2 is running, Part 3 is shut down, the system is interrupted.

State 14: Part 1 is running, Part 2 is running, Part 3 is waiting for repairing, the system is interrupted.

State 15: Part 1 is shut down, Part 2 is shut down, the system is interrupted, repairman is on vacation.

State 16: Part 1 is shut down, Part 2 is shut down, Part 3 is shut down, the system is interrupted.

Let $S(t)$ be the state of the system at time which is random process with the state space

$J = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$, obviously it is not a Markov process. Introduction of additional variables as follows:

Let: $X(t)$ be the past vacation time of repairman at time $t$; $Y(t)$ the past repair time of part 1 at time $t$; $Z(t)$ the past repair time of part 2 at time $t$; $L(t)$ the past repair time of part 3 at time $t$; thus $\{S(t), X(t), Y(t), Z(t), L(t)|t \geq 0\}$ is a vector Markov process.

State probability is defined as:

$$P_1(t, x)dx = P\{S(t) = i, x < X(t) < x + dx\}, i = 1, \ldots, 7$$

$$P_8(t) = P\{S(t) = 8\}$$

$$P_9(t, y)dy = P\{S(t) = 9, y < Y(t) < y + dy\},$$

$$P_j(t, z)dz = P\{S(t) = j, z < Z(t) < z + dz\}, \quad j = 10, 11$$

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Boundary conditions

By a standard probabilistic argument, we can derive the following differential equations:

\[ P_0(t,0) = P\{ S(t) = k, l \leq L(t) < l + dl \}, \quad k = \Pi, s \]

\[ P(t,y,z)dydz = P\{ S(t) = 12, y \leq Y(t) < y + dy, z \leq Z(t) < z + dz \} \]

\[ P(t,y,l)dydl = P\{ S(t) = 13, y \leq Y(t) < y + dy, l \leq L(t) < l + dl \} \]

By a standard probabilistic argument, we can derive the following differential equations:

\[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_1 + \lambda_2 + \lambda_3 + \gamma(x)P_0(t,x) = 0 \]

\[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \gamma(x)P_1(t,x) = \lambda_1P_0(t,x) \]

\[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_3 + \lambda_4 + \gamma(x)P_2(t,x) = \lambda_2P_1(t,x) \]

\[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_2 + \lambda_4 + \gamma(x)P_3(t,x) = \lambda_3P_2(t,x) \]

\[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \gamma(x)P_4(t,x) = \lambda_4P_3(t,x) \]

\[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_3 + \lambda_4 + \gamma(x)P_5(t,x) = \lambda_5P_4(t,x) \]

\[ \frac{\partial}{\partial t} + \lambda_1 + \lambda_2 + \lambda_3P_6(t) = \int_{0}^{\infty} \gamma(x)P_0(t,x)dx \]

\[ \frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu_1(y)P_5(t,y) = 0 \]

\[ \frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \lambda_3 + \lambda_4 + \mu_2(z)P_{10}(t,z) = \int_{0}^{\infty} \mu_1(y)P_{12}(t,y,z)dy \]

\[ \frac{\partial}{\partial t} + \lambda_2 + \lambda_4 + \mu_3(l)P_{11}(t,l) = \int_{0}^{\infty} \mu_1(y)P_{13}(t,y,l)dy \]

\[ \frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu_1(y)P_{12}(t,y,z) = 0 \]

\[ \frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu_1(y)P_{13}(t,y,l) = 0 \]

\[ \frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \mu_2(z)P_{14}(t,z) = \lambda_3P_{10}(t,z) \]

\[ \frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu_3(l)P_{15}(t,l) = \lambda_2P_{11}(t,l) \]

Boundary conditions

\[ P_0(t,0) = \int_{0}^{\infty} \mu_1(y)P_0(t,y)dy + \int_{0}^{\infty} \mu_2(z)P_{10}(t,z)dz + \int_{0}^{\infty} \mu_3(l)P_{11}(t,l)dl + \delta(t) \]
\[ P_0(t, 0) = \lambda_1 P_0(t) + \int_{0}^{\infty} \gamma(x) P_1(t, x) \, dx \]
\[ P_{10}(t, 0) = \lambda_2 P_0(t) + \int_{0}^{\infty} \gamma(x) P_2(t, x) \, dx + \int_{0}^{\infty} \mu_1(y) P_{12}(t, y, 0) \, dy + \int_{0}^{\infty} \mu_3(l) P_{15}(t, l) \, dl \]
\[ P_{11}(t, 0) = \lambda_3 P_0(t) + \int_{0}^{\infty} \gamma(x) P_3(t, x) \, dx + \int_{0}^{\infty} \mu_1(y) P_{13}(t, y, 0) \, dy + \int_{0}^{\infty} \mu_3(z) P_{14}(t, z) \, dz \]
\[ P_{12}(t, 0, z) = \lambda_4 P_{10}(t, z); \quad P_{12}(t, 0, 0) = \int_{0}^{\infty} \gamma(x) P_4(t, x) \, dx \]
\[ P_{13}(t, y, l) = \lambda_4 P_{11}(t, l); \quad P_{13}(t, 0, 0) = \int_{0}^{\infty} \gamma(x) P_5(t, x) \, dx \]
\[ P_{14}(t, 0) = \int_{0}^{\infty} \gamma(x) P_6(t, x) \, dx \]
\[ P_{15}(t, 0) = \int_{0}^{\infty} \gamma(x) P_7(t, x) \, dx \]
\[ P_{1}(t, 0) = 0, i = 1, 7 \]

Initial conditions
\[ P_0(0, x) = \delta(x) \text{ the rest is zero.} \]

4 Solution of equations

Note that
\[ H^*(s) = \int_{0}^{\infty} H(t) e^{-st} \, dt, \quad H(x) = 1 - H(x) \]

Solve above equations:
\[ P_{0}^*(s, x) = P_{0}^*(s, t) \hat{H}(x) e^{-(s+\lambda_1+\lambda_2+\lambda_3)x}, \]
\[ P_{1}^*(s, x) = P_{0}^*(s, 0) \lambda_1 e^{-sx} \hat{H}(x) \left[ 1 - e^{-(s+\lambda_1+\lambda_2+\lambda_3)x} \right]/(\lambda_1 + \lambda_2 + \lambda_3) \]
\[ P_{2}^*(s, x) = P_{0}^*(s, 0) \lambda_2 \hat{H}(x) e^{-(s+\lambda_3+\lambda_4)x} \left[ 1 - e^{-(\lambda_1+\lambda_2-\lambda_4)x} \right]/(\lambda_1 + \lambda_2 - \lambda_4) \]
\[ P_{3}^*(s, x) = P_{0}^*(s, 0) \lambda_3 \hat{H}(x) e^{-(s+\lambda_2+\lambda_3)x} \left[ 1 - e^{-(\lambda_1+\lambda_3-\lambda_4)x} \right]/(\lambda_1 + \lambda_3 - \lambda_4) \]
\[ P_{4}^*(s, x) = P_{0}^*(s, 0) \frac{\lambda_4 \lambda_2}{\lambda_1 + \lambda_2 - \lambda_4} \hat{H}(x) e^{-sx} \left[ 1 - e^{-(\lambda_3+\lambda_4)x} \right]/\lambda_3 + \lambda_4 - 1 - e^{-(\lambda_1+\lambda_2+\lambda_3)x}/\lambda_1 + \lambda_2 + \lambda_3 \]
\[ P_{5}^*(s, x) = P_{0}^*(s, 0) \frac{\lambda_4 \lambda_3}{\lambda_1 + \lambda_3 - \lambda_4} \hat{H}(x) e^{-sx} \left[ 1 - e^{-(\lambda_2+\lambda_4)x} \right]/\lambda_2 + \lambda_4 - 1 - e^{-(\lambda_1+\lambda_2+\lambda_3)x}/\lambda_1 + \lambda_2 + \lambda_3 \]
\[ P_{6}^*(s, x) = P_{0}^*(s, 0) \frac{\lambda_2 \lambda_3}{\lambda_1 + \lambda_2 - \lambda_4} \hat{H}(x) e^{-sx} \left[ 1 - e^{-(\lambda_3+\lambda_4)x} \right]/\lambda_2 + \lambda_4 - 1 - e^{-(\lambda_1+\lambda_2+\lambda_3)x}/\lambda_1 + \lambda_2 + \lambda_3 \]
\[ P_{7}^*(s, x) = P_{0}^*(s, 0) \frac{\lambda_3 \lambda_2}{\lambda_1 + \lambda_3 - \lambda_4} \hat{H}(x) e^{-sx} \left[ 1 - e^{-(\lambda_2+\lambda_4)x} \right]/\lambda_2 + \lambda_4 - 1 - e^{-(\lambda_1+\lambda_2+\lambda_3)x}/\lambda_1 + \lambda_2 + \lambda_3 \]

\[ P_{8}^*(s) = P_{0}^*(s, 0) h^*(s + \lambda_1 + \lambda_2 + \lambda_3)/(s + \lambda_1 + \lambda_2 + \lambda_3) \]

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\[ P^*_9(s,y) = P^*_0(s,0)\lambda_1G_1'(y)e^{-sy}[h^*(s + \lambda_1 + \lambda_2 + \lambda_3) \frac{h^*(s) - h^*(s + \lambda_1 + \lambda_2 + \lambda_3)}{(s + \lambda_1 + \lambda_2 + \lambda_3)}] \]

\[ P^*_0(s,z) = P^*_0(s,0)\lambda_1G_2'(z)e^{-(s+\lambda_3+\lambda_4-\lambda_4g_1'(s))z} \]

\[ P^*_1(s,l) = P^*_{11}(s,0)G_3(l)e^{-(s+\lambda_2+\lambda_4-\lambda_4g_1'(s))l} \]

\[ P^*_2(s, y, z) = P^*_0(s,0)\lambda_4G_1'(y)G_2(z)e^{-sy}e^{-(s+\lambda_3+\lambda_4-\lambda_4g_1'(s))z} \]

\[ P^*_3(s, y, l) = P^*_0(s,0)\lambda_4G_1'(y)G_3(l)e^{-sy}e^{-(s+\lambda_2+\lambda_4-\lambda_4g_1'(s))l} \]

\[ P^*_4(s, z) = \{ P^*_{10}(s,0)\lambda_3 \frac{1-e^{-(\lambda_3+\lambda_4-\lambda_4g_1'(s))z}}{\lambda_3+\lambda_4-\lambda_4g_1'(s)} \begin{pmatrix} h^*(s) - h^*(s + \lambda_3 + \lambda_4) \\ h^*(s) - h^*(s + \lambda_3 + \lambda_4) \end{pmatrix} \} G_2(z)e^{-sz} \]

\[ P^*_5(s, l) = \{ P^*_{11}(s,0)\lambda_2 \frac{1-e^{-(\lambda_2+\lambda_4-\lambda_4g_1'(s))z}}{\lambda_2+\lambda_4-\lambda_4g_1'(s)} \begin{pmatrix} h^*(s) - h^*(s + \lambda_2 + \lambda_4) \\ h^*(s) - h^*(s + \lambda_2 + \lambda_4) \end{pmatrix} \} G_3(l)e^{-sl} \]

\[ P^*_6(s, 0) = \frac{\varphi_0(s)\varphi_1(s) - \varphi_0(s)\psi_1(s)}{(1-\lambda_4g_1'(s))}\varphi_1(s) \varphi_0(s) P^*_0(s,0) \]

\[ P^*_7(s, 0) = \frac{\varphi_0(s) - \psi_0(s)}{\varphi_1(s) - \varphi_0(s)} P^*_0(s,0) \]

where

\[ \varphi_0(s) = \lambda_2 \frac{g_2'(s) - g_2'(s + \lambda_2 + \lambda_4 - \lambda_4g_1'(s))}{\lambda_2 + \lambda_4 - \lambda_4g_1'(s)} \]

\[ \varphi_1(s) = \lambda_3 \frac{g_2'(s) - g_2'(s + \lambda_3 + \lambda_4 - \lambda_4g_1'(s))}{\lambda_3 + \lambda_4 - \lambda_4g_1'(s)} \]

\[ \psi_0(s) = \lambda_2 \frac{h^*(s + \lambda_1 + \lambda_2 + \lambda_3) + h^*(s + \lambda_3 + \lambda_4) - h^*(s + \lambda_1 + \lambda_2 + \lambda_3)}{\lambda_2 + \lambda_4 - \lambda_4g_1'(s)} \]

\[ \psi_1(s) = \lambda_3 \frac{h^*(s + \lambda_1 + \lambda_2 + \lambda_3) + h^*(s + \lambda_3 + \lambda_4) - h^*(s + \lambda_1 + \lambda_2 + \lambda_3)}{\lambda_2 + \lambda_4 - \lambda_4g_1'(s)} \]

\[ P^*_0(s,0) = \frac{B_1(s)}{s[M_1(s)B_1(s) + M_2(s)B_2(s) + M_3(s)B_3(s)]} \]

where

\[ B_1(s) = (1 - \lambda_4g_1'(s))((\varphi_1(s) - \varphi_0(s)) \varphi_1(s) - \varphi_0(s) \psi_1(s)) \]

\[ B_2(s) = \psi_0(s)\varphi_1(s) - \varphi_0(s) \psi_1(s) \]

\[ B_3(s) = (1 - \lambda_4g_1'(s))(\varphi_0(s) - \psi_1(s)) \]

\[ M_2(s) = \lambda_3 \frac{g_2'(s) - g_2'(s + \lambda_2 + \lambda_4 - \lambda_4g_1'(s))}{\lambda_2 + \lambda_4 - \lambda_4g_1'(s)} + (1 + \lambda_4g_1'(s))G_2'(s + \lambda_3 + \lambda_4 - \lambda_4g_1'(s)) \]

\[ M_3(s) = \lambda_3 \frac{g_2'(s) - g_2'(s + \lambda_2 + \lambda_4 - \lambda_4g_1'(s))}{\lambda_2 + \lambda_4 - \lambda_4g_1'(s)} + (1 + \lambda_4g_1'(s))G_3'(s + \lambda_2 + \lambda_4 - \lambda_4g_1'(s)) \]
\[ M_1(s) = \lambda_1 B^*(s) - B^*(s + \lambda_1 + \lambda_2 + \lambda_3) + \lambda_2 B^*(s + \lambda_3 + \lambda_4) - B^*(s + \lambda_1 + \lambda_2 + \lambda_3) \]

\[ H^*(s + \lambda_1 + \lambda_2 + \lambda_3) + \lambda_3 B^*(s + \lambda_3 + \lambda_4) - B^*(s + \lambda_1 + \lambda_2 + \lambda_3) \]

\[ + \lambda_2 (\lambda_1 + \lambda_4) \left( B^*(s) - B^*(s + \lambda_1 + \lambda_2 + \lambda_3) \right) - \lambda_1 (\lambda_1 + \lambda_2 + \lambda_3) \]

\[ + \lambda_2 (\lambda_1 + \lambda_4) \left( B^*(s - \lambda_1 - \lambda_2 - \lambda_3) \right) - \lambda_1 (\lambda_1 + \lambda_2 + \lambda_3) \]

\[ + (1 + \lambda_1 G_1^*(s)) h^*(s + \lambda_1 + \lambda_2 + \lambda_3) + \lambda_1 G_1^*(s) h^*(s - h^*(s + \lambda_1 + \lambda_2 + \lambda_3)) \]

\[ + \lambda_1 G_1^*(s) h^*(s - h^*(s + \lambda_1 + \lambda_2 + \lambda_3)) \]

\[ + \lambda_1 G_1^*(s) h^*(s - h^*(s + \lambda_2 + \lambda_3)) \]

\[ + \lambda_1 G_1^*(s) h^*(s - h^*(s + \lambda_2 + \lambda_3)) \]

\[ + \lambda_1 G_1^*(s) h^*(s - h^*(s + \lambda_2 + \lambda_3)) \]

\[ + \lambda_1 G_1^*(s) h^*(s - h^*(s + \lambda_2 + \lambda_3)) \]

5 The system reliability indices

According to the probability analysis of the system in Section 4, we can obtain some indices of the system. Denote: the transient availability as \( A(t) \), the System Reliability as \( R(t) \).

**Theorem 1** The L- transform of the transient availability is

\[ A^*(s) = \frac{B_1(s) N(s)}{s[M_1(s)B_1(s) + M_2(s)B_2(s) + M_3(s)B_3(s)]} \]

where

\[ N(s) = H^*(s + \lambda_1 + \lambda_2 + \lambda_3) - \lambda_1 \left( \frac{B^*(s + \lambda_3 + \lambda_4) - B^*(s + \lambda_1 + \lambda_2 + \lambda_3)}{\lambda_1 + \lambda_2 + \lambda_3} \right) \]

\[ + \lambda_2 \left( \frac{B^*(s + \lambda_3 + \lambda_4) - B^*(s + \lambda_1 + \lambda_2 + \lambda_3)}{\lambda_1 + \lambda_2 + \lambda_3} \right) + h^*(s - h^*(s + \lambda_1 + \lambda_2 + \lambda_3)) \]

\[ + \frac{B_2(s)}{B_1(s)} G_2^*(s + \lambda_3 + \lambda_1 - \lambda_4 G_1^*(s)) + \frac{B_3(s)}{B_1(s)} G_3^*(s + \lambda_2 + \lambda_1 - \lambda_4 G_1^*(s)) \]

**Proof.** Based on the definition of the availability of the system and the fact that the system is in operation, if and only if the stochastics process \( S(t) \) is in state 0,2,3,8,10,11, we know:

\[ A(t) = \int_0^t P_0(t,x)dx + \int_0^t P_2(t,x)dx + \int_0^t P_3(t,x)dx + \int_0^t P_8(t) + \int_0^t P_{10}(t,z)dz + \int_0^t P_{11}(t,l)dl \]

Its L- transformation is

\[ A^*(s) = \int_0^\infty P_0^*(s) dx + \int_0^\infty P_2^*(s) dx + \int_0^\infty P_3^*(s) dx + \int_0^\infty P_8^*(s) + \int_0^\infty P_{10}^*(s,z)dz + \int_0^\infty P_{11}^*(s,l)dl \]

Thus, form the solution of the equations, we can get the result. ■

**Note:** when we consider the states 1,4,5,6,7,9,12,13,14,15 as absorbing states, \( \{ S(t), X(t), Y(t), Z(t), L(t) | t \geq 0 \} \) is a generalized vector Markov process with absorbing states, and we can get the Laplace- transform of the System Reliability \( R(t) \).

**References**


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