A Nonlinear Integral Formulation for a Stream-Aquifer Interaction Flow Problem

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Abstract: Computations of phreatic flow in a horizontal aquifer are carried out with a Green element formulation that incorporates the steady and unsteady fundamental solutions. The resulting integral equations are implemented in a manner akin to the finite element method with linear interpolation in time and space for the scalar field. Three models based on the diffusion differential operator of an auxiliary equation are constructed for this purpose. While two of them rely on the steady state fundamental solutions the third incorporates the time-dependent Greens function. The resulting non linear discretized equations are solved by the Newton-Raphson and Picard algorithms with good convergence being achieved for all the test problems. The ability of the models to produce faithful results was tested for the case of recharge and dewatering of an unconfined homogeneous aquifer for a fully penetrating trench. It is noted that in as much as comparable accuracy was obtained for all the models, the ability of the third model to produce reliable results was found to rely heavily on the magnitude of the temporal increment.

Keywords: Green element method; Nonlinear; Newton Raphson; Picard; Unconfined aquifer; unsteady fundamental solution; integral equations

1 Introduction

Surface water and ground water are hydrologically linked and interactions between them plays a huge role in water resources management. A good illustration is the movement of water in and out of an unconfined aquifer either due to a rise or fall of water in a nearby canal or ditch. Seepage loss from an unlined canal may cause the water level to rise in the vicinity of the root zone in an adjacent land with attendant effects on the soil salinity balance, crop yield, or drainage problems. Hence prediction of stream-aquifer interactions is very important in understanding and controlling water table and groundwater recharge rate under stream-aquifer interaction (Hantush M.M. [1]). In arid and semi arid regions the interaction between ground and often scanty surface water is often utilized as a primary source of water for agriculture (Zhenghue Xie and Xing Yuan [2], Bauer P et al. [3], Chen X.H. [4], Chen X and Chen X.H. [5], Chen Y et al. [6], Dages C et al. [7], Hu et al. [8]). Wu R. S. et al. [9]. With it comes the realization that the effective sustainability of limited water yield demands a quantification of surface water-ground water interaction (Courbis et al.[10], Ma et al.[11]).

Until the use of computers became widespread, analytical solutions were the only tools for modeling stream-aquifer interaction. Even though they are limited to simple and idealized cases yet they provide benchmarks for checking numerical solutions. In addition they aid the understanding of the physics underlying the interaction between surface-groundwater system. Such solutions include those of Schmid and Luthin[12], Guitjens and Luthin[13], earlier work of Edelman[14], Verigin[15], Polubarinova-Kochina[16], Carslaw and Jaeger[17] as well as Singh and Rai[18]. Childs[19] later modified the Boussinesq equation to account for the inherent errors arising from the Dupuit-Forchheimer assumptions especially for the case of a sloping bed. No analytical solutions were able to handle these modifications except for some simplified cases studied by Wooding and Chapman[20] and Towner[21]. For a comprehensive review of literature in this area we refer to the work of McWhorter and Sunada[22], Pinder and Gray[23].

One of the most accurate solutions analytical results is reported by Serrano and Workman [24]. They used the Adomian decomposition to analytically solve the Boussinesq equation [25] for the case of steam-aquifer interaction. Other related
work can be found in Ostfeld et al. [26] and Workman et al. [27], Spanoudaki et al. [28], Lockington [29], Upadhyaya and Chauhan [30] presented the type of solutions which we have found to be very useful in this work.

Numerical treatment of stream-aquifer flow problem can be found in finite difference based commercial packages like MODFLOW (McDonald and Harbaugh [31], Rodriguez et al. [32], Fox [33]). One of the finite element work in this field can be found in Huyarkin and Pinder [34].

The purpose of this work is twofold namely: to formulate and apply a hybrid numerical scheme known as the Green element method (GEM) to a stream-aquifer interaction problem. The Green element method (Taigbenu [35], Onyejekwe et al. [36], Onyejekwe et al. [37]) is an integral-based numerical technique which hybridizes the finite and boundary element techniques. It exploits both the integral formulation of the boundary element method (BEM) which avails it of its second order accuracy as well as the finite element discretization which guarantees not only a slender coefficient matrix, but an efficiency in dealing with media heterogeneity and nonlinearity.

Secondly to compare the numerical results obtained herein with existing benchmark analytical results for the simulation of water table profiles resulting from an impulsive rise or drop in boundary head in an unconfined homogeneous aquifer which interacts with an adjacent stream.

2 Green element formulation

Relying on the Dupuit-Forcheimer assumptions which hold that streamlines are horizontal and the horizontal gradient is equal to the slope of the water table; the unsteady state unconfined groundwater flow can be considered as approximately one-dimensional and is described by the nonlinear Boussinesq equation:

\[ \Phi \frac{\partial}{\partial x} \left( \Omega(h) \frac{\partial h}{\partial x} \right) = \frac{\partial h}{\partial t} + f(x,t) \tag{1} \]

where, for the Boussinesq equation \( \Omega(h) = h \), \( h[L] \) is the piezometric head, \( x[L] \) is the horizontal coordinate, \( t[T] \) is the time, \( \Phi = K/S \), \( K[L/T] \) is the hydraulic conductivity, and \( S \) is the specific yield, and \( f(x,t) \) represents external point and distributed head sources into the medium. The unconfined aquifer is assumed to be homogeneous, isotropic and semi-infinite. The canal banks as well as the aquifer are also assumed to have the same hydraulic properties.

To obtain unique solutions to equation (1), data for the head \( h \) and/or its spatial derivative \( \partial h/\partial x \) is specified at the end points of the computational domain at all times and the data for \( h \) throughout the flow region at the initial time \( t_0 \). Boundary conditions can be specified in a way that admits the first, second and third types and is represented as:

\[ \vartheta_1 + \vartheta_2 \Omega \frac{\partial h}{\partial x} = \varphi(t) \tag{2} \]

where \( \vartheta_1, \vartheta_2 \) are constants. The Green element method (GEM) is the adopted numerical technique for this work. GEM implements the singular boundary integral theory in a way that is similar to the finite element method (FEM). This element-by-element approach as opposed to its boundary-driven variety guarantees a more efficient handling of domain non-uniformity resulting from heterogeneity as well cases which admit nonlinearity. As a result, a valid case has been established for GEM as an efficient numerical approach for implementing the boundary integral theory (Taigbenu and Onyejekwe [38]).

Along the lines of GEM implementation, the initial task is to seek for the one-dimensional integral replication of equation (1) within a generic element. The first two models (mod1 and mod2) applied in this study achieve this objective by using the 1-D Laplacian as an auxiliary equation: \( \delta^2 G / \delta x^2 = \delta (x - x_i) \), \( x \in (-\infty, \infty) \) with a solution or free space Green function given as: \( G(x, x_i) = (|x - x_i| + k)/2 \) where \( k \) is an arbitrary constant, and \( x - x_i \) is the distance between the point of application of the instantaneous unit input \( x_i \) and any other point.

The third model, referred to (hereafter) as mod3 adopts the transient heat equation as the auxiliary equation. This model though similar in concept to the first two models acquires a more rigorous formulation because of the extra demand introduced by the time scale of the auxiliary differential equation. This is given by: \( \delta^2 G / \delta x^2 - 1/D \delta h / \delta t = \delta (x - x_i) \delta (t - \tau), -\infty \leq \tau \leq 0 \), whose free space Green function is given as:

\[ G(x, t; x_i, \tau) = \frac{-H(t - \tau)}{\sqrt{4\pi D(t - \tau)}} \exp \left[ -\frac{(x - x_i)^2}{4D(t - \tau)} \right] \tag{3} \]

where \( \Omega(h) = D(h) \).
2.1 Model 1

We convert the governing partial differential equation (equation 1) into its integral form using both the auxiliary equation and the Greens second identity. Before we do this, we need to convert equation 1 into its Poisson form:

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{D} \left[ -\Phi \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} + f(x, t, h) \right] \quad (4)$$

Applying the Greens second identity, the integral form can now be written as:

$$-\lambda h(x_1, t) + G^*(x_2, x_1)h(x_2, t) - G^*(x_1, x_1)h(x_1, t)$$

$$-G(x_2, x_1)\varphi(x_2, t) + G(x_1, x_1)\varphi(x_1, t)$$

$$+ \int_{x_1}^{x_2} G(x, x_1) \left[ -\frac{\partial h}{\partial x} \varphi(x, t) + \frac{1}{D} \left( \frac{\partial h}{\partial t} + f(x, t, h) \right) \right] dx = 0 \quad (5)$$

Equation (5) is the so called element equation that categorizes the Green element method (GEM) as a hybrid approach between the classical BEM formulation and the Finite element method (FEM). Keeping faith with FEM methodology, we simplify the line integral of equation (5) by approximating all the functional quantities by linear interpolation in space for example $h(x, t) = \xi(\gamma)h_j(t)$, $\lambda_\Omega = \Theta(x, t)\approx \xi(\gamma)\Theta_j(t)$ where $\xi(\gamma)$ is the interpolation function, and the local coordinate $\xi$ is given by $\xi = (x - x_1)/l$, and the $l$, the length of a generic element is given by $l = x_2 - x_1$. Substituting these approximations into equation(5) and evaluating the line integral results in the system of element equations:

$$R_{ij}h_j + \left( L_{ij} - V_{i\nu j}\Theta_n \right)\varphi_j + U_{ijkl}\Psi_j \left( \frac{dh_l}{dt} + f_l \right) = 0 \quad (6)$$

where $1/D(h) = \Psi$.

Next we approximate the temporal derivative by a modified fully implicit time discretization to yield:

$$\frac{dh_l}{dt} |_{h^{m+1}} = \omega/\Delta t \left( h^{m+1}_l - h^m_l \right) + (1 - \omega) \left( \frac{dh_l}{dt} \right) |_{h^m} \quad (7)$$

$$1 \leq \omega \leq 2$$

where $\omega$ is a time approximation factor, $m$ and $m + 1$ represent the previous and the current times and $\Delta t$ is the time step. Substituting equation(7) into equation(6) and applying the Picard algorithm gives the model 1 equation:

$$\sum_{c=1}^{M} \left( R_{ij}^c h^{m+1,s+1}_j + \left[ L_{ij}^c - V_{i\nu j}^c \Theta^{m+1,s+1}_n \right] \varphi_j^{m+1,s+1} \right) = $$

$$\sum_{c=1}^{M} U_{ijkl}^c \Psi_j^{m+1,k} \left( \frac{\omega}{\Delta t} \{ h^m_k \} - \{ 1 - \omega \} \left[ \frac{dh_k}{dt} \right]^m \right) \quad (8)$$

where $e$ and $s$ are element and iteration counters respectively.

2.2 Model 2

We achieve a substantial simplification of equation(5) by assuming that the piezometric head function is constant within an element. In other words we average this functional term within a generic element as done in Taigbenu and Onyejekwe[38]:

$$\Omega(h) \approx \tilde{\Omega} = \Omega(\tilde{h}) = \Omega \left[ \alpha\tilde{h}^{(m+1)} + \beta \tilde{h}^m \right], \quad 0 \leq \alpha \leq 1, \beta = 1 - \alpha \quad (9)$$

where

$$\tilde{h}^{(m+1)} = \left( h_1^{(m+1)} + h_2^{(m+1)} \right)/2, \tilde{h}^m = \left( h_1^m + h_2^m \right)/2$$

where $m + 1$ and $m$ indicates the current and previous times. The implication of these approximations is that the spatial derivative of the piezometric head function remains invariant within an element and as a result equation (6) becomes:

$$R_{ij}h_j + (L_{ij})\varphi_j + T_{ij}\Psi \left( \frac{dh_j}{dt} + f_j \right) = 0 \quad (10)$$

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Equation (7) is adopted for the discrete approximation of the time derivative. We however hasten to comment that equation (10) is still nonlinear and needs one of the linearization techniques for its solution. After the substitution of the temporal approximation to equation (10); we adopt the Newton Raphsons algorithm, for its linearization. The Jacobian matrix is given as:

\[
\left. \frac{\partial g_i}{\partial h_j} \right|_{\theta_j = \theta_j^{m+1}} = \left\{ R_{ij} + \omega T_{ij}/\Delta t \right\} + \left\{ \omega T_{ij}/\Delta t \theta_j \right\} \frac{d\theta_j}{dt} + \frac{df}{d\theta_j} + T_{ij} \frac{df}{d\theta_j} + T_{ij} \frac{df}{d\theta_j} \]

Starting from a statement of Greens identity integrated within the time domain

\[
\int_0^t \int_{x_1}^{x_2} \left\{ \frac{\partial^2 G}{\partial x^2} - \frac{\partial h}{\partial x} \right\} dx dt = \int_0^t \int_{x_1}^{x_2} \left\{ \frac{\partial G}{\partial t} \right\} dx dt
\]

we obtain:

\[
G(x, t, x_1, t) = \frac{H(t - \tau)}{d(t - \tau) exp} \left[ -\frac{(x - x_1)^2}{4D(t - \tau)} \right]
\]

It can be shown that:

\[
\int_0^t \int_{x_1}^{x_2} h(x, t) \delta(x - x_1) \delta(t - \tau) dx dt = \lambda h(x_1, t)
\]

The integral analog of equation (1) within a spatial \([x_1, x_2]\) and temporal \([t_m, t_{m+1}]\) elements now becomes:

\[
-\lambda h^{m+1}_1 + \int_{t_m}^{t_{m+1}} [h(x_2, t)G(x_2, t, x_1, \tau) - h(x_1, t)G(x_1, t, x_1, \tau)] d\tau
\]

\[
\int_{t_m}^{t_{m+1}} [\Psi(\tau)G(x_2, t_{m+1}, x_1, \tau) - \Psi(\tau)G(x_1, t_{m+1}, x_1, \tau)] d\tau = \frac{1}{D} \int_{x_1}^{x_2} G(x_2, t_{m+1}, x_1, \tau) h(x, t_m) dx
\]

\[
+ \frac{1}{D} \int_{x_1}^{x_2} G(x, t_{m+1}, x_1, \tau) f(\tau) d\tau = 0
\]

where

\[
\frac{\partial G}{\partial x} = G^*(x, t_{m+1}; x_1, \tau) = \frac{(x - x_1)}{2D(t_{m+1} - \tau)} G(x, t_{m+1}; x_1, \tau)
\]
Newton Raphson’s method. Applying the Picard’s algorithm to equation (27) yields:

\[ h(x, t) = \xi_j(x)\xi^{(m)}(\tau)h_j^{m,j} j = 1, 2, m = 1, 2 \]  \hfill (20)

As a result of linear interpolations in space and time, we have the following relationships.

\[ \int_{t_m}^{t_{m+1}} [h(x_2, \tau)G^*(x_2, t; x_i, \tau)] d\tau \]

\[ = \Delta t \int_0^1 [\xi_1(\tau)h_2^{m} + \xi_2(\tau)h_2^{m+1}] G^*(x_2, \Delta t(1 - \tau); x_i, 0) d\tau \]  \hfill (21)

\[ \int_{t_m}^{t_{m+1}} [h(x_1, \tau)G^*(x_1, t; x_i, \tau)] d\tau \]

\[ = \Delta t \int_0^1 [\xi_1(\tau)h_1^{m} + \xi_2(\tau)h_1^{m+1}] G^*(x_1, \Delta t(1 - \tau); x_i, 0) d\tau \]  \hfill (22)

\[ \int_{t_m}^{t_{m+1}} [\Psi(x_2, \tau)G(x_2, t; x_i, \tau)] d\tau \]

\[ = \Delta t \int_0^1 [\xi_1(\tau)\Psi_2^{m} + \xi_2(\tau)\Psi_2^{m+1}] G(x_2, \Delta t(1 - \tau); x_i, 0) d\tau \]  \hfill (23)

\[ \int_{t_m}^{t_{m+1}} [\Psi(x_1, \tau)G(x_1, t; x_i, \tau)] d\tau \]

\[ = \Delta t \int_0^1 [\xi_1(\tau)\Psi_1^{m} + \xi_2(\tau)\Psi_1^{m+1}] G(x_1, \Delta t(1 - \tau); x_i, 0) d\tau \]  \hfill (24)

\[ \frac{1}{D} \int_{x_1}^{x_2} G(x, t_{m+1}; x_i, t_m)h(x, t_m)dx = \frac{1}{D} \int_0^1 \xi_j(\varsigma)h_j^{m}G(l(\varsigma - \varsigma_i), \Delta t, 0, 0) d\varsigma \]  \hfill (25)

\[ \frac{1}{D} \int_{t_m}^{t_{m+1}} \int_{x_1}^{x_2} G(x, t_{m+1}; x_i, \tau)f(x, \tau)dxd\tau = \]  \hfill (26)

\[ \frac{1}{D} \int_0^1 \int_0^1 G(l(\varsigma - \varsigma_i), \Delta t(1 - \tau), 0, 0) [\xi_1(\tau)\xi_j(\varsigma)\xi_j^{m+1}(\tau)\xi_j(\varsigma)\xi_j^{m+1}] d\varsigma d\tau \]

The system of nonlinear discrete equations obtained after interpolation is given by:

\[ \bar{A}_{ij} h_j^{(m+1)} + \bar{A}_{ij}^* h_j^{(m)} + B_{ij} \Psi_j^{(m+1)} + \bar{B}_{ij}^* \Psi_j^{(m)} + C_{ij} f_j^{(m+1)} + \bar{C}_{ij}^* f_j^{(m)} = 0 \]  \hfill (27)

where all the element matrices have been put under a bar to indicate their dependence on the averaged piezometric head function. For completeness their integral evaluations are given in the appendix.

Equation (27) is solved by the Picard method to obviate the task needed for the computation of the Jacobian of the Newton Raphson’s method. Applying the Picard’s algorithm to equation (27) yields:

\[ \bar{A}_{ij} h_j^{(m+1,k+1)} + B_{ij} \Psi_j^{(m+1,k+1)} + C_{ij} f_j^{(m+1,k+1)} = -\bar{A}_{ij} h_j^{(m+1,k)} \]

\[ -B_{ij} \Psi_j^{(m+1,k)} - \bar{C}_{ij}^* f_j^{(m+1,k)} \]  \hfill (28)

Equation(21) is model 3.
3 Numerical results and discussions

The validity of GEM formulation is tested for the case of prediction of water table profiles in a semi-infinite flow region due to an abrupt rise or drop in a canal or drain water level for both recharging and discharging scenarios at various times. GEM numerical results are compared with existing analytical solutions based on $L_2$, $L_\infty$ norms.

Polubarinova-Kochina[16] used a linearization method to determine the water table profile for a stream aquifer configuration (Fig.1) involving an instantaneous lowering of the water level in a ditch. The same problem, but different methods of linearization were solved by both Verigin[15], and Edelman[14].

Table 1 shows a comparison of the three Green element models with those of Polubarinova-Kochina[16] whose results have been shown to be superior to the other two researchers [30]. Both space and time increments; $\Delta x$ and $\Delta t$ are taken to be 0.0025days and 2.0m respectively, while the hydraulic conductivity and specific yield are 20.0m/day and 0.27. The aquifer is considered to be underlain by a a horizontal impermeable base. For this test case, the water level in the trench is initially kept at 2.0m, and the water level in an adjoining trench is impulsively raised to 3.0m to induce a recharging aquifer. The performance of the 3 models; mod1, mod2 and mod3, shows that based on the magnitude of the error norms, mod1 was found to give the closest results to those of [16], followed by mod2 and lastly by mod3. It may be expected that the level of numerical rigor involved in the formulation of mod3 would guarantee a better performance. Two important numerical considerations are to be noted, namely: scalar interpolation was carried out in space and time (see equation 20) and the smaller the time step, the more the Greens function adopted for its formulation is forced to behave like a dirac-delta function (equation 15). As a result, the accompanying sharp gradient is not easy to resolve numerically. It may also be mentioned in passing that when the value of $\Delta t$ is increased from .0025days to say 0.1 days, the quality of the solution obtained from mod3 improves considerably while mod1 and mod2 begin to experience some numerical difficulties. The same trend in model performance for the prediction of water table heights can be observed in Table 2 after 5 days of recharging of the aquifer.

For the aquifer discharging process, the water level in the aquifer was kept at an elevation of 3m and that in the canal was lowered to 2.0m. The water level profiles at times $t=1$ to $t=5$ days together with both the analytical and numerical results can be seen in Table3 and Table 4. The time increment was increased from .025days to 0.3 days. The effect of this time increment is reflected on the quality of the results provided by mod3. Mod3 can be seen to have performed better than the other two models.

It can be seen that small time steps introduce a dirac-delta singularity for model3. It therefore means that model3 requires relatively large time steps to achieve acceptable accuracy. A look at equation (15) shows that the numerator within the exponential sign represents the square of the element size while the denominator reflects a temporal step size. It can be surmised that the Greens function or the fundamental solution is highly influenced by an exponential function for mod3. Improvement in performance of model3 will therefore require relatively small values of the exponential function. This can be realized by selecting a large time step, a small element size or a combination of the two of them. Since the element size within the exponential term is squared, the proper combination of time step with the square of the element size for an efficient model3 performance is still a moot topic. Experience of the author so far on this topic is that a lot depends on the physics of the problem and to what level of interpolation we are willing to apply on the space and time domains without being involved with parasitic round-off errors. For the problems encountered in this study, the following ratio produced fairly accurate results:

$$0.425 \leq \Re \leq 0.675$$ (29)

where $\Re$ is the ratio of the numerator to the denominator for the terms within the exponent in equation (15)

Lockington[29] used a weighted residual approach to obtain closed form solutions for the nonlinear one-dimensional Boussinesq equation. Equations (35) and (36) of his work, yield expressions for the boundary fluxes for the case of dewatering and recharging of a semi-infinite homogeneous aquifer. The ability of GEM to compute fluxes as part of the solution is tested by comparing GEM numerical solutions of boundary fluxes with those of Lockington[29]. After considerable trial error a time step of 0.185 was adopted for this comparison to allow acceptable performance of all the models. Fig. 2 and Fig. 3 show the closeness of the predicted fluxes with the analytical results of Lockington[29]. The profiles are hardly distinguishable except for the minor instability shown by mod1 for the first day of application.

4 Conclusions

Three Green element models have been applied to the solution of a boundary value problem involving flow in a semi-infinite unconfined aquifer. The analytical solutions of Polubarinova-Kochina[16], Edelman[14], Verigin[15], and Lockington[29] were compared with GEM numerical results for their ability to predict water surface profiles and fluxes for the
case of a sudden change in boundary heads. Comparative model performances were assessed by the $L_2$ and $L_\infty$ error norms. Based on these criteria it was found that the performance of model3 is highly dependent on both the time and space increment values. It was found that model3 in particular gave acceptable results as long as the exponential function of the Greens fundamental solution is less than unity. The relatively large time-step incorporated into model3 to produce accurate results for a shorter computational time compensates for its more rigorous formulation.

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References


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Appendix: A

\[ A_{11} = A_{22} = -0.5 \]

\[ A_{12} = A_{21} = \Delta t \int_0^1 \Omega_2(\tau)G^*(l, \Delta t(1-\tau); 0, 0) d\tau \]

\[ = \frac{\sqrt{\gamma}}{2\sqrt{\pi}} \int_0^1 \frac{\tau}{(1-\tau)^{\frac{3}{2}}} \exp \left[ -\frac{\gamma}{1-\tau} \right] d\tau \]

\[ = \frac{1}{\sqrt{\pi}} \left[ \sqrt{\pi} \left( \frac{1}{2} + \gamma \right) erfc(\sqrt{\gamma}) - \sqrt{\gamma} \exp(-\gamma) \right] \]

\[ B_{11} = -B_{22} = \Delta t \int_0^1 \Omega_2(\tau)G(0, \Delta t(1-\tau); 0, 0) d\tau \]

\[ = \frac{1}{4\sqrt{\pi \gamma}} \int_0^1 \frac{\tau}{(1-\tau)^{\frac{3}{2}}} d\tau = -\frac{1}{3\sqrt{\pi \gamma}} \]

\[ B_{12} = -B_{21} = \Delta t \int_0^1 \Omega_2(\tau)G(l, \Delta t(1-\tau); 0, 0) d\tau \]
\[ A_{11} = A_{22} = -l \int_0^1 \Omega_1(\zeta)G(l\zeta, \Delta; 0, 0) d\zeta \]
\[ A_{21} = \Delta t \int_0^1 \Omega_1(\tau)G^*(l, \Delta t(1-\tau); 0, 0) d\tau - l \int_0^1 \Omega_2(\zeta)G(l\zeta, \Delta t, 0, 0) d\zeta \]

where \[ \zeta = \frac{t^2}{4\Delta t}, \quad erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-s^2) ds, \quad erfc(x) = 1 - erf(x) \]

\[ C_{11} = C_{22} = l \Delta t \int_0^1 \int_0^1 \Omega_2(\tau)\Omega_1(\zeta)G(l\zeta, \Delta t(1-\tau); 0, 0) d\tau d\zeta \]
\[ C_{12} = C_{21} = l \Delta t \int_0^1 \int_0^1 \Omega_2(\tau)\Omega_2(\zeta)G(l\zeta, \Delta t(1-\tau); 0, 0) d\tau d\zeta \]

\[ \text{LINS homepage: http://www.nonlinearscience.org.uk/} \]
\[
= \frac{\Delta t}{2\sqrt{\pi}} \left\{ \sqrt{\pi} \int_0^1 0 \text{erf} \left( \frac{\sqrt{y}}{\sqrt{\pi}} \right) \, dr + \frac{2}{5 \sqrt{\pi}} \left[ \frac{2}{\sqrt{\pi}} + \frac{4}{15} \gamma \exp(-\gamma) + 4 \sqrt{\pi \gamma} \exp \left( \frac{\gamma}{\sqrt{\gamma}} \right) \right] \right\}
\]

\[
C_{12}^* = C_{21}^* = \frac{\Delta t}{\sqrt{\pi}} \int_0^1 \int_0^1 \Omega_1(\tau) \Omega_2(\zeta) G(\zeta, \Delta t(1-\tau); 0, 0) \, d\tau d\zeta
\]

\[
= \frac{\Delta t}{5 \sqrt{\pi \gamma}} \left[ -1 + \frac{1}{3} \left( (3 + 4\gamma^2 - 2\gamma) \exp(-\gamma) - 4 \sqrt{\pi \gamma} \frac{\exp(\gamma)}{\sqrt{\gamma}} \right) \right]
\]

### Tables and Figures

#### Table 1: Comparison of three Green element models with analytic solution for a recharging aquifer at \( t = 1 \) day

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#### Table 2: Comparison of three Green element models with analytic solution for a recharging aquifer at \( t = 5 \) days

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L₂ Norm: 1.3290 e-03 6.7520 e-03

L∞ Norm: 1.8388 e-03 3.4760 e-03

IJNS email for contribution: editor@nonlinearscience.org.uk
Table 3: Comparison of three Green element models with analytic solution for a discharging aquifer at $t = 1$ day

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Table 4: Comparison of three Green element models with analytic solution for a discharging aquifer at $t = 5$ days

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Figure 1: Water level variations in an aquifer-ditch combination.

Figure 2: Comparison of GEM and analytic fluxes for a recharging aquifer.

Figure 3: Comparison of GEM ad analytic fluxes for a discharging aquifer.