

# Exact Solution of Some Linear Fractional Differential Equations by Laplace Transform

Saeed Kazem \*

Department of Mathematics, Imam Khomeini International University, Ghazvin 34149-16818, Iran

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**Abstract:** This paper aims to apply the Laplace transform for solving linear fractional-order differential equation. The Laplace transform is a powerful tool in applied mathematics and engineering. It will allow us to transform fractional differential equations into algebraic equations and then by solving this algebraic equations, we can obtain the unknown function by using the Inverse Laplace Transform. Illustrative examples are included to demonstrate the validity and applicability of the presented technique.

**Keywords:** Fractional-order differential equations; Laplace Transform; Inverse Laplace Transform.

## 1 Introduction

Although fractional derivatives have a long mathematical history, for many years they were not used in physics. One possible explanation of such unpopularity could be that there are multiple nonequivalent definitions of fractional derivatives [1]. Another difficulty is that fractional derivatives have no evident geometrical interpretation because of their nonlocal character [2]. However, during the last 10 years fractional calculus starts to attract much more attention of physicists and mathematicians. It was found that various, especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [3], and the fluid-dynamic traffic model with fractional derivatives [4] can eliminate the deficiency arising from the assumption of continuum traffic flow. Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested in [5], and differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena [6]. A review of some applications of fractional derivatives in continuum and statistical mechanics is given by Mainardi [7]. The analytic results on the existence and uniqueness of solutions to the fractional differential equations have been investigated by many authors [1, 8]. During the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic systems containing fractional derivatives, such as Adomian's decomposition method [9–13], He's variational iteration method [14–16], homotopy perturbation method [17–19], homotopy analysis method [20], spectral methods [21–24], and other methods [25–27].

The remainder of this paper is organized as follows: We begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory. In section 3, the Laplace transform and the inverse Laplace transform for some functions is demonstrated. In section 4, the proposed method is applied to several examples. Also a conclusion is given in the last section.

## 2 Preliminaries and notations

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this article.

**Definition 1** A real function  $f(x)$ ,  $x > 0$  is said to be in space  $C_\mu$ ,  $\mu \in \mathbb{R}$  if there exists a real number  $p > \mu$ , such that  $f(t) = t^p f_1(t)$ , where  $f_1(t) \in C(0, \infty)$ , and it is said to be in the space  $C_\mu^n$  if and only if  $f^n \in C_\mu$ ,  $n \in \mathbb{N}$ .

\*Corresponding author. E-mail address: saeedkazem@gmail.com

**Definition 2** The Riemann-Liouville fractional integral operator of order  $\alpha > 0$ , of a function  $f \in C_\mu$ ,  $\mu \geq -1$ , is defined as

$$j^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0 \quad (1)$$

$$j^0 f(t) = f(t).$$

Some properties of the operator  $j^\alpha$ , which are needed here, are as follows:

For  $f \in C_\mu$ ,  $\mu \geq -1$ ,  $\alpha, \beta \geq 0$  and  $\gamma \geq -1$ :

$$(1) j^\alpha j^\beta f(t) = j^{\alpha+\beta} f(t)$$

$$(2) j^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}. \quad (2)$$

**Definition 3** The fractional derivative of  $f(t)$  in the Caputo sense is defined as

$$D^\alpha f(t) = j^{m-\alpha} D^m f(t), \quad (3)$$

for  $m-1 < \alpha \leq m$ ,  $m \in \mathbb{N}$ ,  $t > 0$  and  $f \in C_{-1}^m$ .

Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative.

Similar to the integer-order integration, the Riemann-Liouville fractional integral operator is a linear operation:

$$j^\alpha \left( \sum_{i=1}^n c_i f_i(t) \right) = \sum_{i=1}^n c_i j^\alpha f_i(t), \quad (4)$$

where  $\{c_i\}_{i=1}^n$  are constants.

In the present work, the fractional derivatives are considered in the Caputo sense. The reason for adopting the Caputo definition, as pointed by [10], is as follows: to solve differential equations (both classical and fractional), we need to specify additional conditions in order to produce a unique solution. For the case of the Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are akin to those of classical differential equations, and are therefore familiar to us. In contrast, for the Riemann-Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives (and/or integrals) of the unknown solution at the initial point  $x = 0$ , which are functions of  $x$ . These initial conditions are not physical; furthermore, it is not clear how such quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. For more details see [2].

### 3 Laplace operation

The Laplace transform is a powerful tool in applied mathematics and engineering. Virtually every beginning course in differential equations at the undergraduate level introduces this technique for solving linear differential equations. The Laplace transform is indispensable in certain areas of control theory.

#### 3.1 Laplace transform

Given a function  $f(x)$  defined for  $0 < x < \infty$ , the Laplace transform  $F(s)$  is defined as

$$F(s) = L[f(x)] = \int_0^\infty f(x) e^{-sx} dx, \quad (5)$$

at least for those  $s$  for which the integral converges.

Let  $f(x)$  be a continuous function on the interval  $[0, \infty)$  which is of exponential order, that is, for some  $c \in \mathbb{R}$  and  $x > 0$

$$\sup \frac{|f(x)|}{e^{cx}} < \infty.$$

In this case the Laplace transform (5) exists for all  $s > c$  [27].

Some of the useful Laplace transforms which are applied in this paper, are as follows:

For  $L[f(x)] = F(s)$  and  $L[g(x)] = G(s)$

$$L[f(x) + g(x)] = F(s) + G(s),$$

$$L[x^\beta] = \frac{\Gamma(\beta + 1)}{s^{\beta+1}}, \quad \beta > -1, \tag{6}$$

$$L[f^{(n)}(x)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0), \tag{7}$$

$$L[x^n f(x)] = (-1)^n F^{(n)}(s),$$

$$L\left[\int_0^x f(t)dt\right] = \frac{F(s)}{s},$$

$$L\left[\int_0^x f(x-t)g(t)dt\right] = F(s)G(s).$$

**Lemma 1** The Laplace transform of Riemann-Liouville fractional integral operator of order  $\alpha > 0$  can be obtained in the form of:

$$L\left[j^\alpha f(x)\right] = \frac{F(s)}{s^\alpha}.$$

**Proof.** The Laplace transform of Riemann-Liouville fractional integral operator of order  $\alpha > 0$  is :

$$L\left[j^\alpha f(x)\right] = L\left[\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt\right] = \frac{1}{\Gamma(\alpha)} F(s)G(s),$$

where

$$G(s) = L[x^{\alpha-1}] = \frac{\Gamma(\alpha)}{s^\alpha}$$

■

**Lemma 2** The Laplace transform of Caputo fractional derivative for  $m - 1 < \alpha \leq m, m \in \mathbb{N}$ , can be obtained in the form of:

$$L\left[D^\alpha f(x)\right] = \frac{s^m F(s) - s^{m-1} f(0) - s^{m-2} f'(0) - \dots - f^{(m-1)}(0)}{s^{m-\alpha}}.$$

**Proof.** The Laplace transform of Caputo fractional derivative of order  $\alpha > 0$  is :

$$L\left[D^\alpha f(x)\right] = L\left[j^{m-\alpha} f^{(m)}(x)\right] = \frac{L\left[f^{(m)}(x)\right]}{s^{m-\alpha}}.$$

■

Using equation 7. Now, we can transform fractional differential equations into algebraic equations and then by solving this algebraic equations, we can obtain the unknown Laplace function  $F(s)$ .

### 3.2 Inverse Laplace transform

The function  $f(x)$  in (5) is called the inverse Laplace transform of  $F(s)$  and will be denoted by  $f(x) = L^{-1}[F(s)]$  in the paper. In practice when one uses the Laplace transform to, for example, solve a differential equation, one has to at some point invert the Laplace transform by finding the function  $f(x)$  which corresponds to some specified  $F(s)$ .

The Inverse Laplace Transform of  $F(s)$  is defined as:

$$f(x) = L^{-1}[F(s)] = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\sigma-iT}^{\sigma+iT} e^{sx} F(s) ds,$$

where  $\sigma$  is large enough that  $F(s)$  is defined for the real part of  $s \geq \sigma$ . Surprisingly, this formula isn't really useful. Therefore, in this section some useful function  $f(x)$  is obtained from their Laplace transform. In the first we define the most important special functions used in fractional calculus the Mittag-Leffler functions and the generalized Mittag-Leffler functions

For  $\alpha, \beta > 0$  and  $z \in \mathbb{C}$

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)},$$

$$E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}.$$

Now, we prove some Lemmas which are useful for finding the function  $f(x)$  from its Laplace transform.

**Lemma 3.** For  $\alpha, \beta > 0$ ,  $a \in \mathbb{R}$  and  $s^{\alpha} > |a|$  we have the following inverse Laplace transform formula

$$L^{-1} \left[ \frac{s^{\alpha-\beta}}{s^{\alpha} + a} \right] = x^{\beta-1} E_{\alpha, \beta}(-ax^{\alpha}).$$

**Proof.**  $s^{\alpha-\beta}/(s^{\alpha} + a)$  by using the series expansion can be rewritten as

$$\frac{s^{\alpha-\beta}}{s^{\alpha} + a} = \frac{1}{s^{\beta}} \frac{1}{1 + \frac{a}{s^{\alpha}}} = \frac{1}{s^{\beta}} \sum_{n=0}^{\infty} \left( \frac{-a}{s^{\alpha}} \right)^n = \sum_{n=0}^{\infty} \frac{(-a)^n}{s^{n\alpha + \beta}}.$$

The inverse Laplace transform of above function is

$$\sum_{n=0}^{\infty} \frac{(-a)^n x^{n\alpha + \beta - 1}}{\Gamma(n\alpha + \beta)} = x^{\beta-1} \sum_{n=0}^{\infty} \frac{(-ax^{\alpha})^n}{\Gamma(n\alpha + \beta)} = x^{\beta-1} E_{\alpha, \beta}(-ax^{\alpha}).$$

■

**Lemma 4.** For  $\alpha \geq \beta > 0$ ,  $a \in \mathbb{R}$  and  $s^{\alpha-\beta} > |a|$  we have

$$L^{-1} \left[ \frac{1}{(s^{\alpha} + as^{\beta})^{n+1}} \right] = x^{\alpha(n+1)-1} \sum_{k=0}^{\infty} \frac{(-a)^k \binom{n+k}{k}}{\Gamma(k(\alpha - \beta) + (n+1)\alpha)} x^{k(\alpha-\beta)}.$$

**Proof.** Using the series expansion of  $(1+x)^{-n-1}$  of the form

$$\frac{1}{(1+x)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k} (-x)^k$$

we have:

$$\frac{1}{(s^{\alpha} + as^{\beta})^{n+1}} = \frac{1}{(s^{\alpha})^{n+1}} \frac{1}{\left(1 + \frac{a}{s^{\alpha-\beta}}\right)^{n+1}} = \frac{1}{(s^{\alpha})^{n+1}} \sum_{k=0}^{\infty} \binom{n+k}{k} \left( \frac{-a}{s^{\alpha-\beta}} \right)^k$$

Giving the inverse Laplace transform of above function can prove the Lemma. ■

**Lemma 5.** For  $\alpha \geq \beta$ ,  $\alpha > \gamma$ ,  $a \in \mathbb{R}$ ,  $s^{\alpha-\beta} > |a|$  and  $|s^{\alpha} + as^{\beta}| > |b|$  we have

$$L^{-1} \left[ \frac{s^{\gamma}}{s^{\alpha} + as^{\beta} + b} \right] = x^{\alpha-\gamma-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-b)^n (-a)^k \binom{n+k}{k}}{\Gamma(k(\alpha - \beta) + (n+1)\alpha - \gamma)} x^{k(\alpha-\beta) + n\alpha}.$$

**Proof.**  $s^{\gamma}/(s^{\alpha} + as^{\beta} + b)$  by using the series expansion can be rewritten as

$$\frac{s^{\gamma}}{s^{\alpha} + as^{\beta} + b} = \frac{s^{\gamma}}{s^{\alpha} + as^{\beta}} \frac{1}{1 + \frac{b}{s^{\alpha} + as^{\beta}}} = \sum_{n=0}^{\infty} \frac{s^{\gamma} (-b)^n}{(s^{\alpha} + as^{\beta})^{n+1}}$$

Now by using Lemma 4 the Lemma can be proved. ■

## 4 Illustrative examples

This section is applied the method presented in the paper and give an exact solution of some linear fractional differential equations.

**Example 1.** As the first example, we consider the following initial value problem in the case of the inhomogeneous Bagley-Torvik equation [22, 28]

$$\begin{aligned} D^2 y(x) + D^{\frac{3}{2}} y(x) + y(x) &= 1 + x, \\ y(0) = y'(0) &= 1. \end{aligned} \quad (8)$$

This equation by using Laplace transform is converted to

$$\begin{aligned} s^2 F(s) - sy(0) - y'(0) + \frac{s^2 F(s) - sy(0) - y'(0)}{s^{\frac{1}{2}}} + F(s) &= \frac{1}{s} + \frac{1}{s^2}, \\ s^2 F(s) - s - 1 + \frac{s^2 F(s) - s - 1}{s^{\frac{1}{2}}} + F(s) &= \frac{1}{s} + \frac{1}{s^2}, \\ F(s)(s^2 + s^{\frac{3}{2}} + 1) &= \left(\frac{1}{s} + \frac{1}{s^2}\right)(s^2 + s^{\frac{3}{2}} + 1), \\ F(s) &= \frac{1}{s} + \frac{1}{s^2}. \end{aligned}$$

Using the inverse Laplace transform the exact solution of this problem  $y(x) = 1 + x$  can be obtained.

**Example 2.** Our second example covers the inhomogeneous linear equation [29]

$$\begin{aligned} D^\alpha y(x) + y(x) &= \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + x^2 - x, \\ y(0) &= 0, \quad 0 < \alpha \leq 1. \end{aligned} \quad (9)$$

Using the Laplace transform  $F(s)$  is obtained as follows

$$\begin{aligned} \frac{sF(s) - y(0)}{s^{1-\alpha}} &= \frac{2}{s^{3-\alpha}} - \frac{1}{s^{2-\alpha}} - F(s) + \frac{2}{s^3} - \frac{1}{s^2}, \\ F(s)(s^\alpha + 1) &= 2\frac{s^\alpha + 1}{s^3} - \frac{s^\alpha + 1}{s^2}, \\ F(s) &= \frac{2}{s^3} - \frac{1}{s^2}. \end{aligned}$$

Then  $y(x) = x^2 - x$  is obtained by using the inverse Laplace transform.

**Example 3.** Consider the following linear initial value problem [20, 22, 24]

$$\begin{aligned} D^\alpha y(x) + y(x) &= 0, \\ y(0) = 1 \quad y'(0) &= 0. \end{aligned} \quad (10)$$

The second initial condition is for  $\alpha > 1$  only.

In two cases of  $\alpha$ ,  $L\left[D^\alpha f(x)\right]$  is obtained as

1. For  $\alpha < 1$

$$L\left[D^\alpha f(x)\right] = \frac{s^2 F(s) - s}{s^{2-\alpha}} = \frac{sF(s) - 1}{s^{1-\alpha}},$$

2. For  $\alpha > 1$

$$L\left[D^\alpha f(x)\right] = \frac{sF(s) - 1}{s^{1-\alpha}},$$

which are the same. Now the Laplace transform,  $F(s)$  is obtained as

$$\frac{sF(s) - 1}{s^{1-\alpha}} + F(s) = 0 ,$$

$$F(s) = \frac{s^{\alpha-1}}{1 + s^{\alpha}}$$

Using the lemma 3, the exact solution of this problem can be obtained as:

$$y(x) = E_{\alpha}(-x^{\alpha})$$

**Example 4.** Consider the following linear initial value problem [30]

$$D^{\alpha}y(x) = y(x) + 1 , \quad 0 < \alpha \leq 1$$

$$y(0) = 0. \quad (11)$$

Using the Laplace transform  $F(s)$  is obtained as follows

$$\frac{sF(s)}{s^{1-\alpha}} = F(s) + \frac{1}{s} ,$$

$$F(s) = \frac{s^{-1}}{s^{\alpha} - 1} .$$

Using the Lemma 3 the exact solution of this problem can be obtained as:

$$y(x) = x^{\alpha} E_{\alpha, \alpha+1}(x^{\alpha})$$

**Example 5.** Consider the composite fractional oscillation equation [30]

$$y''(x) - aD^{\alpha}y(x) - by(x) = 8 , \quad 1 < \alpha \leq 2$$

$$y(0) = y'(0) = 0.$$

Using the Laplace transform,  $F(s)$  is obtained as follows

$$s^2F(s) - a \frac{s^2F(s)}{s^{2-\alpha}} - bF(s) = \frac{8}{s} ,$$

$$F(s) = \frac{8s^{-1}}{s^2 - as^{\alpha} - b} .$$

Using the lemma 5 the exact solution of this problem can be obtained as:

$$y(x) = 8x^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{b^n a^k \binom{n+k}{k} x^{(2-\alpha)k+2n}}{\Gamma\left((2-\alpha)k + 2(n+1) + 1\right)}$$

The graph of  $y(x)$  for  $a = b = 1$  and various values of  $\alpha$  is shown in Fig. (1).

**Example 6.** Consider the following system of fractional algebraic-differential equations [31]

$$D^{\alpha}x(t) - ty'(t) + x(t) - (1+t)y(t) = 0 , \quad 0 < \alpha \leq 1$$

$$y(t) - \sin t = 0 ,$$

subject to the initial conditions

$$x(0) = 1 , \quad y(0) = 0.$$

Using the Laplace transform  $F(s) = L^{-1}[y(t)]$  and  $G(s) = L^{-1}[x(t)]$  is obtained as follows

$$\begin{aligned} \frac{sG(s) - 1}{s^{1-\alpha}} + F(s) + sF'(s) + G(s) - F(s) + F'(s) &= 0, \\ F(s) &= \frac{1}{s^2 + 1}, \quad F'(s) = \frac{-2s}{(s^2 + 1)^2} \\ G(s)(s^\alpha + 1) &= \frac{2s(s + 1)}{(1 + s^2)^2} + \frac{1}{s^{1-\alpha}}, \\ G(s) &= \frac{2s}{s^\alpha + 1} \frac{s + 1}{(1 + s^2)^2} + \frac{s^{\alpha-1}}{s^\alpha + 1}. \end{aligned}$$

The exact solution for  $\alpha = 1$  is  $x(t) = t \sin t + e^{-t}$ . Using the Lemma 3 and 4 the exact solution for  $0 < \alpha \leq 1$  can be obtained as:

$$\begin{aligned} x(t) &= 2t^{\alpha+1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{n+k} (k+1) t^{2k} \\ &\quad \left( \frac{t^{n\alpha+1}}{\Gamma((n+1)\alpha + 2k + 3)} + \frac{t^{n\alpha}}{\Gamma((n+1)\alpha + 2k + 2)} \right) + E_\alpha(-t^\alpha) \\ &= 2t^{\alpha+1} \sum_{k=0}^{\infty} (-1)^{\lfloor \frac{k}{2} \rfloor} \left( \left\lfloor \frac{k}{2} \right\rfloor + 1 \right) t^k E_{\alpha, \alpha+k+2}(-t^\alpha) + E_\alpha(-t^\alpha) \\ y(t) &= \sin t, \end{aligned}$$

where  $\lfloor \cdot \rfloor$  denotes the integer part of the number. The graph of  $y(x)$  for various values of  $\alpha$  is shown in Fig. (2).

## 5 Conclusions

The Laplace transform is a powerful tool in applied mathematics and engineering and have been applied for solving linear differential equations. In this paper, the application of Laplace transform is investigated to obtain an exact solution of some linear fractional differential equations. The fractional derivatives are described in the Caputo sense which obtained by Riemann-Liouville fractional integral operator. Solving some problems show that the Laplace transform is a powerful and efficient techniques for obtaining analytic solution of linear fractional differential equations.

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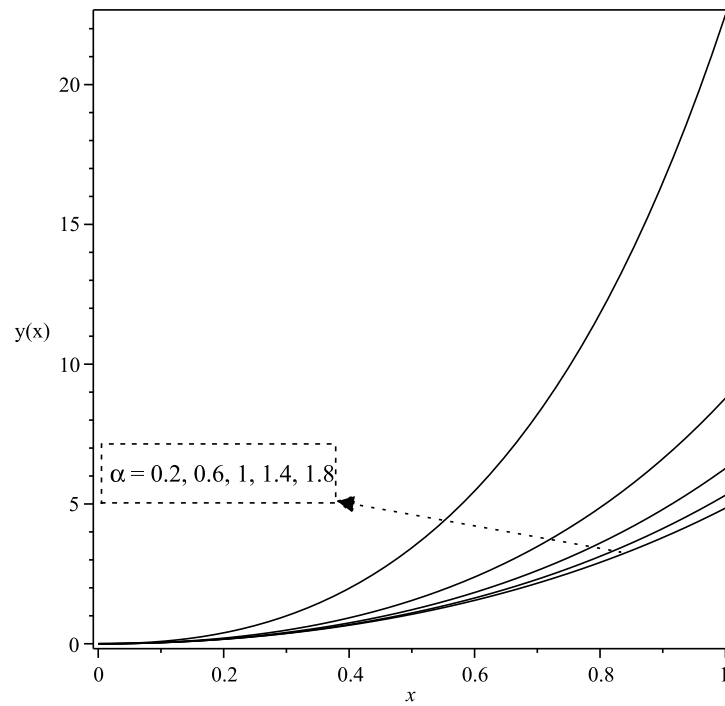


Figure 1: The graph of  $y(x)$  for  $a = b = 1$  and  $\alpha = 0.2, 0.6, 1, 1.4$  and  $1.8$ .

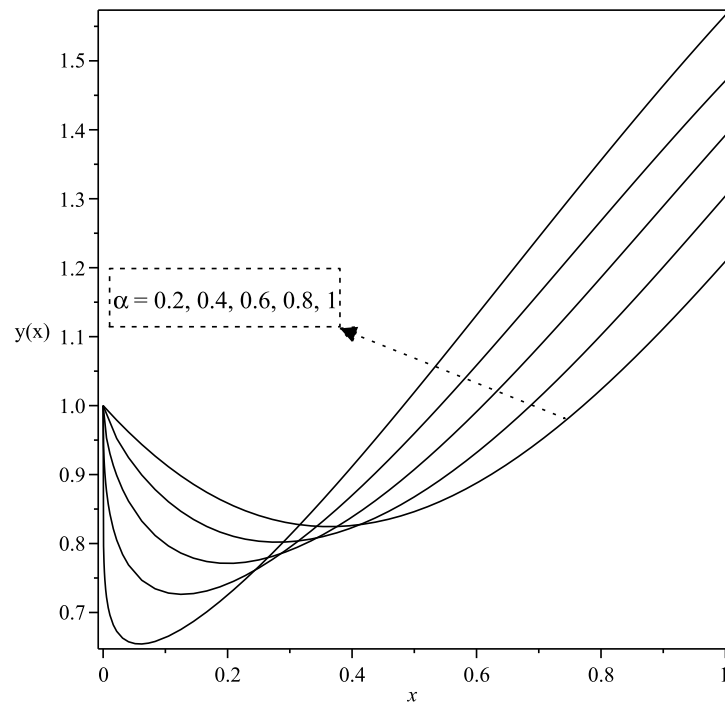


Figure 2: The graph of  $y(x)$  for  $\alpha = 0.2, 0.4, 0.6, 0.8$  and  $1$ .