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Solving a Class of Volterra Integral Equation Systems by the Differential Transform Method

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Abstract: In this paper, Numerical solution linear and non-linear systems of Volterra integral equations of the first kind and second kind is considered by differential transform method. We applied these method to three example. this powerful method catches the exact solution. This examples are prepared to show the efficiency and simplicity of the method.

Keywords: Volterra integral equations system; differential transform method; taylor series expansion

1 Introduction

The solution of volterra integral equations have a major role in the fields of science and engineering.When a physical system is modeled under the differential sense; There are various techniques for solving a system of Volterra integral equations, e.g. Adomian decomposition method (ADM) [1,2]. DTM is a semi analytical-numerical technique that depends on Taylor series. It was first introduced by Zhou in a study about electrical circuits [3], difference equations [4], differential difference equations [5], fractional differential equations [6], pantograph equations [7] by using this method. This paper outlines the application of DTM to the systems of volterra integral equations. Two problems for Volterra integral equation systems of the first kind and one problem for Volterra integral equation systems of the second kind are solved to make clear the application of the transform. Differential transform method is based on taylor series expansion.

2 Differential transform method

Differential transform of function $y(x)$is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}, \tag{1}$$

In equation (1), $y(x)$ is the original function and $Y(k)$ is the transformed function, which is called the T-function. Differential inverse transform of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k), \tag{2}$$

from equation (1) and (2), we obtain

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}, \tag{3}$$

Equation (3) implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which are

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described by the transformed equations of the original functions. In this study we use the lower case letter to represent the original function and upper case letter represent the transformed function. From the definitions of equations (1) and (2), it is easily proven that the transformed functions comply with the basic mathematics operations shown in Table 1. In actual applications, the function $y(x)$ is expressed by a finite series and equation (2) can be written as

$$y(x) = \sum_{k=0}^{m} x^k Y(k), \tag{4}$$

Equation (3) implies that$y(x) = \sum_{k=m+1}^{\infty} x^k Y(k)$ is negligibly small. In fact, m is decided by the convergence of natural frequency in this study.

Table 1: The fundamental operations of one-dimensional differential transform method

Original function	Transformed function
$y(x) = u(x) + v(x)$	$Y(k) = U(k) + V(k)$
$y(x) = cw(x)$	$Y(k) = cW(k)$
$y(x) = \frac{dw(x)}{dx}$	$Y(k) = (k+1)W(k+1)$
$y(x) = \frac{d^j w(x)}{dx^j}$	$Y(k) = (k+1)(k+2)...(k+j)W(k+j)$
$y(x) = u(x)v(x)$	$Y(k) = \sum_{r=0}^{k} U(r)V(k-r)$
$y(x) = x^j$	$Y(k) = \delta(k-j) = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$

Theorem 1 *if* $y(x) = \int_{x_0}^{x} u(t)dt$ *then* $Y(k) = \frac{1}{k}U(k-1)$, *where* $k \geq 1$.

Theorem 2 *if* $y(x) = \int_{x_0}^{x} u_1(t)u_2(t)dt$ *then* $Y(k) = \frac{1}{k}\sum_{r=1}^{k-1} U_1(r)U_2(k-r-1)$, *where* $k \geq 1$.

Theorem 3 *if* $y(x) = \int_{x_0}^{x} u_1(t)u_2(t)...u_{n-1}(t)u_n(t)dt$ *then* $Y(k) = \frac{1}{k}\sum_{k_{n-1}=0}^{k-1}\sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} U_1(k_1)$ $U_2(k_2-k_1)...U_{n-1}(k_{n-1}-k_{n-2})U_n(k-k_{n-1}-1)$, *where* $k \geq 1$.

3 Numerical examples

Example 1. Consider the following system of integral equations of the first kind with the exact solutions $u(x) = x^2$ and $v(x) = x$.

$$\int_0^x (1 - x^2 + t^2)(u(t) + v^3(t))dt = \frac{-1}{12}x^6 - \frac{2}{15}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3,$$
$$\int_0^x (5 + x - t)(u^3(t) + v(t))dt = \frac{-5}{2}x^2 - \frac{1}{6}x^3 + \frac{5}{7}x^7 + \frac{1}{56}x^8, \tag{5}$$

By differentiation we have:

$$u(x) + v^3(x) - 2x\int_0^x (u(t) + v^3(t))dt = \frac{-1}{2}x^5 - \frac{2}{3}x^4 + x^3 + x^2,$$
$$u^3(x) - v(x) + \frac{1}{5}\int_0^x (u^3(t) - v(t))dt = -x - \frac{1}{10}x^2 + x^6 + \frac{1}{35}x^7, \tag{6}$$

Then

$$u(x) = \frac{-1}{2}x^5 - \frac{2}{3}x^4 + x^3 + x^2 - v^3(x) + 2x\int_0^x (u(t) + v^3(t))dt,$$
$$v(x) = x + \frac{1}{10}x^2 - x^6 - \frac{1}{35}x^7 + u^3(x) + \frac{1}{5}\int_0^x (u^3(t) - v(t))dt, \tag{7}$$

Eq. (7) is transformed by using the fundamental operations of one-dimensional differential transform method in Table 1 and Theorems 2 and 3 as follows:

$$\begin{aligned} U(k) &= \frac{-1}{2}\delta(k-5) - \frac{2}{3}\delta(k-4) + \delta(k-3) + \delta(k-2) \\ &- \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} V(k_1)V(k_2-k_1)V(k-k_2) + \sum_{l=1}^{k}\frac{1}{l}2\delta(k-l-1)U(l-1) \\ &+ \sum_{l=1}^{k}\left(\frac{1}{l}2\delta(k-l-1)\sum_{k_2=0}^{l-1}\sum_{k_1=0}^{k_2} V(k_1)V(k_2-k)V(l-k_2-1)\right), \end{aligned} \tag{8}$$

$$\begin{aligned} V(k) &= \delta(k-1) + \frac{1}{10}\delta(k-2) - \delta(k-6) \\ &- \frac{1}{35}\delta(k-7) + \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} U(k_1)U(k_2-k_1)U(k-k_2) \\ &+ \frac{1}{5k}\sum_{k_2=0}^{k-1}\sum_{k_1=0}^{k_2} U(k_1)U(k_2-k_1)U(k-1-k_2) - \frac{1}{5k}V(k-1), \end{aligned} \tag{9}$$

Note that the transformations of integral terms are considered for $k \geq 1$ according to Theorems 2 and 3 The valuse $U(k)$ and $V(k)$ in $k = 0, 1, 2, ...$ of Eqs.(8) and (9) can be evaluated as follows:

$$\begin{aligned} &U(0) = 0, U(1) = 0, U(2) = 1, U(i+1) = 0, i = 2, 3, 4, ..., \\ &V(0) = 0, V(1) = 1, V(j+1) = 0, j = 1, 2, 3, ... \end{aligned} \tag{10}$$

by using the inverse transformation rule in Eq. (2), the following solution can be obtained:

$$u(x) = x^2, v(x) = x. \tag{11}$$

Example 2. Consider the following system of Volterra integral equations

$$\begin{aligned} cos(x) - \frac{1}{2}sin^2(x) + \int_0^x f_1(s)f_2(s)ds = f_1(x), \\ sin(x) - x + \int_0^x f_1^2(s)ds + \int_0^x f_2^2(s)ds = f_1(x), \end{aligned} \tag{12}$$

where the exact solutions are $f_1(x) = cos(x)$ and $f_2(x) = sin(x)$.

By using the fundamental operations of one-dimensional differential transform method in Table 1, Eq. (1) transforms to the following recurrence relations.

$$\frac{1}{k!}cos(\frac{k\pi}{2}) - \frac{1}{2}\sum_{l=0}^{k}\frac{1}{l!}sin(\frac{l\pi}{2})\frac{1}{(k-l)!}sin(\frac{(k-l)\pi}{2}) + \frac{1}{k}\sum_{l=0}^{k-1}F_1(l)F_2(k-l-1) = F_1(k), \tag{13}$$

$$\frac{1}{k!}sin(\frac{k\pi}{2}) - \delta(k-1) + \frac{1}{k}\sum_{l=0}^{k-1}F_1(l)F_1(k-l-1) + \frac{1}{k}\sum_{l=0}^{k-1}F_2(l)F_2(k-l-1) = F_2(k). \tag{14}$$

Note that the transformations of integral terms are considered for $k \geq 1$ according to Theorems 2 and 3. The valuse $F_1(k)$ and $F_2(k)$ in $k = 0, 1, 2, ...$ of Eqs. (13) and (14) can be evaluated as follows:

$$\begin{aligned} &F_1(0) = 1, F_1(1) = 0, F_1(2) = \frac{-1}{2}, F_1(3) = 0, F_1(4) = \frac{1}{24}, \\ &F_1(5) = 0, F_1(6) = \frac{-1}{720}, F_1(7) = 0, F_1(8) = \frac{1}{40320}, ..., \\ &F_2(0) = 0, F_2(1) = 1, F_2(2) = 0, F_2(3) = \frac{-1}{6}, F_2(4) = 0, \\ &F_2(5) = \frac{1}{120}, F_2(6) = 0, F_2(7) = \frac{-1}{5040}, F_2(8) = 0, F_2(9) = \frac{1}{362880}, ... \end{aligned} \tag{15}$$

and then, by using the inverse transformation rule in Eq. (2), the following solution can be obtained:

$$f_1(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - ... = cosx,$$
$$f_2(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - ... = sinx, \tag{16}$$

Example 3. Consider the following system of linear integral equations with the exact solutions: $f(x) = x^2$ and $g(x) = x$.

$$\int_0^x ((1 - x^2 + t^2)f(t) - (2x - t)g(t))dt = \frac{-1}{3}x^3 - \frac{2}{15}x^5,$$
$$\int_0^x ((x + t^2)f(t) - (2x - t)g(t))dt = x^2 - \frac{1}{6}x^3 + \frac{1}{3}x^4 - \frac{1}{5}x^5. \tag{17}$$

By differentiation with respect to x we have:

$$f(x) - xg(x) - 2\int_0^x (xf(t) + g(t))dt = -x^2 - \frac{2}{3}x^4,$$
$$(x^2 + x)f(x) - 2g(x) + \int_0^x (f(t) - g(t))dt = 2x - \frac{x^2}{2} + \frac{4}{3}x^3 + x^4, \tag{18}$$

and then:

$$f(x) = -x^2 - \frac{2}{3}x^4 + xg(x) + 2\int_0^x (xf(t) + g(t))dt,$$
$$g(x) = x - \frac{x^2}{4} + \frac{2}{3}x^3 + \frac{1}{2}x^4 - \frac{1}{2}(x^2 + x)f(x) - \frac{1}{2}\int_0^x (f(t) - g(t))dt, \tag{19}$$

Equation system (19) is transformed by using the fundamental operations of one-dimensional differential transform method in Table 1 and Theorems 1 and Theorems 2 to obtain the following recurrence relations:

$$F(k) = -\delta(k-2) - \frac{2}{3}\delta(k-4) + \sum_{l=1}^{k}\delta(l-1)G(k-l)$$
$$+ 2\sum_{l=1}^{k}\frac{1}{l}\delta(k-l-1)F(l-l) + \frac{2}{k}G(k-l), \tag{20}$$
$$G(k) = \delta(k-2) - \frac{1}{4}\delta(k-2) + \frac{2}{3}\delta(k-3) + \frac{1}{2}\delta(k-4)$$
$$- \frac{1}{2}\sum_{l=0}^{k}\delta(l-1)F(k-l) - \frac{1}{2}\sum_{l=0}^{k}\delta(l-2)F(k-l) - \frac{2}{k}F(k-l) + \frac{2}{k}G(k-l). \tag{21}$$

Note that the transformations of integral terms are considered for $k \geq 1$ according to Theorems 2 and 3. The value $F(k)$ and $G(k)$ in $k = 0, 1, 2, ...$ of Eqs. (19) and (20) can be evaluated as follows:

$$F(0) = 0, F(1) = 0, F(2) = 1, F(i+1) = 0, i = 2, 3, 4, ...,$$
$$G(0) = 0, G(1) = 1, G(j+1) = 0, j = 1, 2, 3, ... \tag{22}$$

by using the inverse transformation rule in Eq. (2), the following solution can be obtained:

$$f(x) = x^2, g(x) = x. \tag{23}$$

4 Conclusion

In this study, the differential transform method for the solution of Volterra integral equation systems is successfully expanded. In the first two examples Volterra integral equation systems of the first kind and the last problem for Volterra integral equation systems of the second kind are considered. It is observed that the method is robust and is applicable to various types of integral equation systems. The method gives rapidly converging series solutions. The accuracy of the obtained solution can be improved by taking more terms in the solution.

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