ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.16(2013) No.1,pp.92-96

Radius Problem for Certain Class of Analytic Functions

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(Received 28 February 2012 , accepted 17 October 2012)

Abstract: Let $\mathcal{G}(\beta_1, \beta_2, \beta_3; \lambda)$ be a subclass of analytic functions $f(z)$ satisfies $\frac{f(z)}{z} \neq 0 \;\; (z \in \mathcal{U})$ and

$$\left| \beta_1 z^2 \left(\frac{1}{f(z)} - \frac{1}{z} \right)' + \beta_2 z^3 \left(\frac{1}{f(z)} - \frac{1}{z} \right)'' + \beta_3 z^4 \left(\frac{1}{f(z)} - \frac{1}{z} \right)''' \right| \leq \lambda \qquad (z \in \mathcal{U}),$$

for some complex numbers $\beta_1, \beta_2, \beta_3$ and for some real $\lambda > 0$.The object of the present paper is to obtain radius problem of $\frac{1}{\delta} f(\delta z) \in \mathcal{G}(\beta_1, \beta_2, \beta_3; \lambda)$ where $f(z)$ satisfies the condition $Re\left\{ \frac{z^2 f'(z)}{f^2(z)} \right\} > \alpha$.

Keywords: Analytic, univalent functions, Cauchy-Schwarz inequality, radius problems.

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{B}(\alpha)$ if and only if

$$Re\left\{ \frac{z^2 f'(z)}{f^2(z)} \right\} > \alpha \qquad (z \in \mathcal{U}), \tag{2}$$

for some $\alpha (0 \leq \alpha < 1)$. Frasin and Darus [2] (see also, [1]) have defined the class $\mathcal{B}(\alpha)$ and investigated some interesting properties for this class.

For the class $\mathcal{B}(\alpha)$, we introduce the subclass $\mathcal{B}(\theta, \alpha)$ of $\mathcal{B}(\alpha)$ by

$$\mathcal{B}(\theta, \alpha) = \left\{ f(z) \in \mathcal{B}(\alpha) : \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, b_n = |b_n| e^{in\theta} \right\}.$$

For a function $f(z)$ belonging to $\mathcal{A}$ is said to be in the class $\mathcal{G}(\beta_1, \beta_2, \beta_3; \lambda)$ if it satisfies $\frac{f(z)}{z} \neq 0 \;\; (z \in \mathcal{U})$ and

$$\left| \beta_1 z^2 \left(\frac{1}{f(z)} - \frac{1}{z} \right)' + \beta_2 z^3 \left(\frac{1}{f(z)} - \frac{1}{z} \right)'' + \beta_3 z^4 \left(\frac{1}{f(z)} - \frac{1}{z} \right)''' \right| \leq \lambda \qquad (z \in \mathcal{U}), \tag{3}$$

for some complex numbers $\beta_1, \beta_2, \beta_3$ and for some real $\lambda > 0$. Very recently, Shimoda et al. [3] have studied the class $\mathcal{G}(0, 0, 1; \lambda)$ defined by

$$\left| z^4 \left(\frac{1}{f(z)} - \frac{1}{z} \right)''' \right| \leq \lambda \qquad (z \in \mathcal{U}).$$

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IJNS.2013.08.15/748

Let us consider the function $f_\gamma(z)$ given by $f_\gamma(z)=\frac{z}{(1-z)^\gamma}$ $(\gamma\in\mathbb{R})$. Then, we observe that

$$f_\gamma(z)=\frac{z}{1+\sum\limits_{n=1}^{\infty}(-1)^n\binom{\gamma}{n}z^n}.$$

It follows that

$$\begin{aligned}&\left|\beta_1 z^2\left(\frac{1}{f_\gamma(z)}-\frac{1}{z}\right)'+\beta_2 z^3\left(\frac{1}{f_\gamma(z)}-\frac{1}{z}\right)''+\beta_3 z^4\left(\frac{1}{f_\gamma(z)}-\frac{1}{z}\right)'''\right|\\ \leq\ &\sum_{n=2}^{\infty}(n-1)\binom{\gamma}{n}[|\beta_1|+|\beta_2|(n-2)+|\beta_3|(n-2)(n-3)]\,|z|^n\\ <\ &\sum_{n=2}^{\infty}(n-1)\binom{\gamma}{n}[|\beta_1|+|\beta_2|(n-2)+|\beta_3|(n-2)(n-3)].\end{aligned}$$

Therefore, if $\gamma=2$, then

$$\left|\beta_1 z^2\left(\frac{1}{f_2(z)}-\frac{1}{z}\right)'+\beta_2 z^3\left(\frac{1}{f_2(z)}-\frac{1}{z}\right)''+\beta_3 z^4\left(\frac{1}{f_2(z)}-\frac{1}{z}\right)'''\right|<|\beta_1|.$$

This implies that $f_2(z)\in\mathcal{G}(\beta_1,\beta_2,\beta_3;\lambda)$ for $\lambda>|\beta_1|$. If $\gamma=3$, then we have

$$\left|\beta_1 z^2\left(\frac{1}{f_3(z)}-\frac{1}{z}\right)'+\beta_2 z^3\left(\frac{1}{f_3(z)}-\frac{1}{z}\right)''+\beta_3 z^4\left(\frac{1}{f_3(z)}-\frac{1}{z}\right)'''\right|<5|\beta_1|+2|\beta_2|.$$

Thus, $f_3(z)\in\mathcal{G}(\beta_1,\beta_2,\beta_3;\lambda)$ for $\lambda>5|\beta_1|+2|\beta_2|$. Further, if $\gamma=4$, then we have

$$\begin{aligned}&\left|\beta_1 z^2\left(\frac{1}{f_4(z)}-\frac{1}{z}\right)'+\beta_2 z^3\left(\frac{1}{f_4(z)}-\frac{1}{z}\right)''+\beta_3 z^4\left(\frac{1}{f_4(z)}-\frac{1}{z}\right)'''\right|\\ <\ &17|\beta_1|+14|\beta_2|+6|\beta_3|.\end{aligned}$$

Therefore, $f_4(z)\in\mathcal{G}(\beta_1,\beta_2,\beta_3;\lambda)$ for $\lambda>17|\beta_1|+14|\beta_2|+6|\beta_3|$.

2 Radius problem for the class $\mathcal{G}(\beta_1,\beta_2,\beta_3;\lambda)$

To obtain the radius problem for the class $\mathcal{P}(\beta_1,\beta_2,\beta_3;\lambda)$, we need the following lemmas.

Lemma 1 *Let $f(z)\in\mathcal{A}$ and $\frac{z}{f(z)}=1+\sum\limits_{n=1}^{\infty}b_n z^n\neq 0$ $(z\in\mathcal{U})$. If $f(z)$ satisfies*

$$\sum_{n=2}^{\infty}(n-1)(|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|)|b_n|\leq\lambda, \tag{4}$$

for some complex numbers β_1,β_2,β_3 and for some real $\lambda>0$, then $f(z)\in\mathcal{G}(\beta_1,\beta_2,\beta_3;\lambda)$.

Proof. We observe that

$$\begin{aligned}&\left|\beta_1 z^2\left(\frac{1}{f(z)}-\frac{1}{z}\right)'+\beta_2 z^3\left(\frac{1}{f(z)}-\frac{1}{z}\right)''+\beta_3 z^4\left(\frac{1}{f(z)}-\frac{1}{z}\right)'''\right|\\ =\ &\left|\beta_1\sum_{n=2}^{\infty}(n-1)b_n z^n+\beta_2\sum_{n=3}^{\infty}(n-1)(n-2)b_n z^n\right.\\ &\left.+\beta_3\sum_{n=4}^{\infty}(n-1)(n-2)(n-3)b_n z^n\right|\\ \leq\ &\sum_{n=2}^{\infty}(n-1)(|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|)|b_n|.\end{aligned}$$

Therefore, if $f(z)$ satisfies the inequality (4), then $f(z) \in \mathcal{G}(\beta_1, \beta_2, \beta_3; \lambda)$. ■

Lemma 2 *If $f(z) \in \mathcal{B}(\theta, \alpha)$; $0 \le \alpha < 1$ and $\frac{z}{f(z)} = 1 + \sum\limits_{n=1}^{\infty} b_n z^n$, $b_n = |b_n| e^{in\theta}$, then*

$$\sum_{n=2}^{\infty}(n-1)|b_n| \le 1 - \alpha. \tag{5}$$

Proof. *Let $f(z) \in \mathcal{B}(\theta, \alpha)$ and $\frac{z}{f(z)} = 1 + \sum\limits_{n=1}^{\infty} b_n z^n$, $b_n = |b_n| e^{in\theta}(n = 1, 2, \cdots)$. Then,*

$$\begin{aligned} Re\left(\frac{z^2 f'(z)}{f^2(z)}\right) &= Re\left(\frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)'\right) \\ &= Re\left(1 - \sum_{n=1}^{\infty}(n-1)b_n z^n\right) \\ &= Re\left(1 - \sum_{n=1}^{\infty}(n-1)|b_n| e^{in\theta} z^n\right) > \alpha, \quad (z \in \mathcal{U}). \end{aligned}$$

If we consider a point $z = |z| e^{-i\theta}$, then we have

$$1 - \sum_{n=2}^{\infty}(n-1)|b_n| |z|^n > \alpha.$$

Letting $|z| \to 1^-$, we obtain the inequality (5). ■

Remark 3 *If $f(z) \in \mathcal{B}(\theta, \alpha)$; $0 \le \alpha < 1$, then the inequality*

$$\sum_{n=2}^{\infty}(n-1)|b_n| \le 1 - \alpha$$

implies that

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 \le 1 - \alpha.$$

Applying the above lemma, we derive the following theorem.

Theorem 4 *If $f(z) \in \mathcal{B}(\theta, \alpha)$; $0 \le \alpha < 1$ and $\delta \in \mathbb{C}$ $(0 < |\delta| < 1)$. Then the function $\frac{1}{\delta} f(\delta z) \in \mathcal{G}(\beta_1, \beta_2, \beta_3; \lambda)$ for $0 < |\delta| \le |\delta_0(\lambda)|$, where $|\delta_0(\lambda)|$ is the smallest positive root of the equation*

$$\begin{aligned} &|\beta_1| \frac{|\delta|^2 \sqrt{1-\alpha}}{1-|\delta|^2} + |\beta_2| \frac{|\delta|^3 \sqrt{2(1+2|\delta|^2)(1-\alpha-|b_2|^2)}}{(1-|\delta|^2)^2} \\ &+ |\beta_3| \frac{|\delta|^4 \sqrt{12(1+6|\delta|^2+3|\delta|^4)(1-\alpha-|b_2|^2-2|b_3|^2)}}{\left(1-|\delta|^2\right)^3} \\ = \ &\lambda \end{aligned}$$

in $0 < |\delta| < 1$.

Proof. Since $\frac{z}{f(z)} = 1 + \sum\limits_{n=1}^{\infty} b_n z^n \ne 0$ $(z \in \mathcal{U})$ for $f(z) \in \mathcal{B}(\theta, \alpha)$, we see that $\frac{z}{\frac{1}{\delta} f(\delta z)} = 1 + \sum\limits_{n=1}^{\infty} \delta^n b_n z^n$ for $0 < |\delta| < 1$. Thus, to show that $\frac{1}{\delta} f(\delta z) \in \mathcal{G}(\beta_1, \beta_2, \beta_3; \lambda)$, from Lemma 1, it is sufficient to prove that

$$\sum_{n=2}^{\infty}(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|\delta|^n |b_n| \le \lambda.$$

Applying Cauchy-Schwarz inequality, we note that

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n-1)(|\beta_1|+(n-2)\,|\beta_2|+(n-2)(n-3)\,|\beta_3|)\,|\delta|^n\,|b_n| \\
\leq\ & |\beta_1|\left(\sum_{n=2}^{\infty}(n-1)\,|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}(n-1)\,|b_n|^2\right)^{\frac{1}{2}} \\
&+|\beta_2|\left(\sum_{n=3}^{\infty}(n-1)(n-2)^2\,|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=3}^{\infty}(n-1)\,|b_n|^2\right)^{\frac{1}{2}} \\
&+|\beta_3|\left(\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2\,|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=4}^{\infty}(n-1)\,|b_n|^2\right)^{\frac{1}{2}} \\
\leq\ & |\beta_1|\left(\sum_{n=2}^{\infty}(n-1)\,|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha} \\
&+|\beta_2|\left(\sum_{n=3}^{\infty}(n-1)(n-2)^2\,|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha-|b_2|^2} \\
&+|\beta_3|\left(\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2\,|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha-|b_2|^2-2\,|b_3|^2}.
\end{aligned} \tag{6}$$

We note that

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}, \qquad (|x|<1),$$

thus, we have

$$\sum_{n=2}^{\infty}(n-1)x^n=\frac{x^2}{(1-x)^2}. \tag{7}$$

Since

$$\sum_{n=3}^{\infty}(n-2)x^{n-1}=x^2\left(\sum_{n=3}^{\infty}(n-2)x^{n-3}\right)=x^2\left(\sum_{n=3}^{\infty}x^{n-2}\right)'=\frac{x^2}{(1-x)^2},$$

we see that

$$\sum_{n=3}^{\infty}(n-1)(n-2)^2x^n=x^3\left(\frac{x^2}{(1-x)^2}\right)''=\frac{2x^3+4x^4}{(1-x)^4}. \tag{8}$$

Furthermore, we have

$$\begin{aligned}
\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2x^n &= x^4\left(\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2x^{n-4}\right) \\
&= x^4\left(\sum_{n=4}^{\infty}(n-2)(n-3)x^{n-1}\right)''',
\end{aligned}$$

but

$$\sum_{n=4}^{\infty}(n-2)(n-3)x^{n-1}=x^3\left(\sum_{n=4}^{\infty}(n-2)(n-3)x^{n-4}\right)=\frac{2x^3}{(1-x)^3}$$

thus, we have

$$\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2x^n=\frac{12x^4+72x^5+36x^6}{(1-x)^6}. \tag{9}$$

Therefore, from (6)- (9) with $|\delta|^2 = x$, we obtain

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|\delta|^n|b_n| \\
\leq\ & |\beta_1| \frac{|\delta|^2\sqrt{1-\alpha}}{1-|\delta|^2} + |\beta_2| \frac{|\delta|^3\sqrt{2(1+2|\delta|^2)(1-\alpha-|b_2|^2)}}{(1-|\delta|^2)^2} \\
&+ |\beta_3| \frac{|\delta|^4\sqrt{12(1+6|\delta|^2+3|\delta|^4)(1-\alpha-|b_2|^2-2|b_3|^2)}}{\left(1-|\delta|^2\right)^3}.
\end{aligned}$$

Now, let us consider the complex number δ $(0 < |\delta| < 1)$ such that

$$\begin{aligned}
& |\beta_1| \frac{|\delta|^2\sqrt{1-\alpha}}{1-|\delta|^2} + |\beta_2| \frac{|\delta|^3\sqrt{2(1+2|\delta|^2)(1-\alpha-|b_2|^2)}}{(1-|\delta|^2)^2} \\
& + |\beta_3| \frac{|\delta|^4\sqrt{12(1+6|\delta|^2+3|\delta|^4)(1-\alpha-|b_2|^2-2|b_3|^2)}}{\left(1-|\delta|^2\right)^3} \\
=\ & \lambda.
\end{aligned}$$

If we define the function $h(|\delta|)$ by

$$\begin{aligned}
h(|\delta|) =\ & |\beta_1||\delta|^2(1-|\delta|^2)^2\sqrt{1-\alpha} \\
& + |\beta_2||\delta|^3\left(1-|\delta|^2\right)\sqrt{2(1+2|\delta|^2)(1-\alpha-|b_2|^2)} \\
& + |\beta_3||\delta|^4\sqrt{12(1+6|\delta|^2+3|\delta|^4)(1-\alpha-|b_2|^2-2|b_3|^2)} \\
& -\lambda(1-|\delta|^2)^3,
\end{aligned}$$

then we have $h(0) = -\lambda < 0$ and $h(1) = |\beta_3|\sqrt{120(1-\alpha-|b_2|^2-2|b_3|^2)} > 0$. This means that there exists some δ_0 such that $h(|\delta_0|) = 0$ $(0 < |\delta_0| < 1)$. This completes the proof of the theorem. ∎

Acknowledgments

The author would like to thank the referees for valuable comments and suggestions.

References

[1] B.A. Frasin.A note on certain analytic univalent functions,*Southeast Asian Bull. Math*, 8(2004):1-8.

[2] B. A. Frasin, and M. Darus.On certain analytic univalent functions.*Internat. J. Math. and Math. Sci*,25(5)(2001). 305-310.

[3] Y. Shimoda, T. Hayami, Y. Hashidume and S. Owa.Radius properties of certain analytic functions.*Internat. J. Open Problems Complex Anal*, 1(2009):30-34.