ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.16(2013) No.1,pp.12-16

Majorization for Certain Classes of Analytic Functions Defined by Generalized Fractional Calculus Operators

Pranay Goswami[1] * ,Bhavna Shrama[2]
[1]Department of Mathematics , AMITY University Rajasthan, Jaipur-303002, India
[2]Department of Mathematics, Balaji Institute of Engineering and Technology,Jaipur-302022, India
(Received 23 December 2011, accepted 26 April 2013)

Abstract: In this paper, we investigate a majorization problem involving starlike multivalent function of complex order belonging to a certain subclasses of multivalent function defined by generalized fractional calculus operator. Moreover, we point out some new or known consequences of our main result.

Keywords: Analytic functions; Multivalent functions; Starlike functions; Subordination; Fractional calculus operators; Majorization property
2000 MSC: Primary 30C45; Secondary 26A33.

1 Introduction

Let f and g be analytic in the open unit disk

$$\Delta = \{z : z \in \mathbb{C} \ \ , |z| < 1\} . \tag{1.1}$$

We say that f is majorized by g in Δ (see [1]) and write

$$f(z) \ll g(z) \qquad (z \in \Delta), \tag{1.2}$$

if there exists a function φ, analytic in Δ such that

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in \Delta). \tag{1.3}$$

It may be noted here that (1.2) is closely related to the concept of quasi-subordination between analytic functions.

For two functions f and g, analytic in Δ, we say that the function f is subordinate to g in Δ, and we write

$$f(z) \prec g(z),$$

if there exists a Schwarz function ω, which is analytic in Δ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \Delta)$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \Delta).$$

Furthermore, if the function g is univalent in Δ, then we have the following equivalence,

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

*Corresponding author. *E-mail address*: pranaygoswami83@gmail.com,vallabhi_2007@yahoo.com

IJNS.2013.08.15/736

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} := \{1, 2, 3, ...\}), \tag{1.4}$$

which are analytic in the open unit disk Δ. For simplicity, we write $\mathcal{A}_1 =: \mathcal{A}$.

In the following, we recall the Definition of generalized fractional derivative.

Definition 1.1 ([14], see also [11],[13]). *Let* $0 \leq \lambda < 1$ *and* $\mu, \eta \in \mathbb{R}$. *Then*

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)}\int_0^z (z-t)^{-\lambda}\ {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{t}{z}\right) f(t)dt\right), \tag{1.5}$$

where the function is analytic in a simply connected region of the z-plane containing the origin, with the order $f(z) = O(|z|^{\varepsilon})(z \to 0)$, *and* $\varepsilon > \max\{0, \mu-\eta\} - 1$.

It is understood that $(z-t)^{-\lambda}$ denotes the principal value for $0 \leq \arg(z-t) < 2\pi$. The function occurring in the right-hand side of (1.5) is the familiar Gaussian hypergeometric function.

Definition 1.2 ([12]). *Under the hypothesis of Definition 1.1, a fractional calculus operator* $J_{0,z}^{\lambda+m,\mu+m,\eta+m}$ *is defined by,*

$$J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z)\ (z \in \Delta; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.6}$$

We observe that

$$D_z^{\lambda} f(z) = J_{0,z}^{\lambda,\lambda,\eta} f(z)(0 \leq \lambda < 1), \tag{1.7}$$

and

$$D_z^{\lambda+m} f(z) = J_{0,z}^{\lambda+m,\eta+m,\eta+m} f(z)(0 \leq \lambda < 1; \mathrm{m} \in \mathbb{N}_0), \tag{1.8}$$

where $D_z^{\lambda+m} f(z)$ *is the well known fractional derivative operator (see [14] and many others).* Furthermore, in terms of Gamma functions Definition 1.1, readily yields

Lemma 1.1. (Srivastava et al. [15]). If $0 \leq \lambda < 1$; $\mu, \eta \in R$ and $k > max\{0, \mu-\eta\} - 1$, then

$$J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)} z^{k-\mu}.$$

Definition 1.3. *A function* $f \in A_p$ *is said to be in the class of* $M_{p,m}^{\lambda,\mu,\eta}[A, B, \gamma]$, *of p-valent function of complex order* $\gamma \neq 0$ *in* Δ *if and only if*

$$1 + \frac{1}{\gamma}\left(\frac{zJ_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} f(z)}{J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z)} - (p-\mu-m)\right) \prec \frac{1+Az}{1+Bz} \tag{1.9}$$

$(z \in \Delta, p \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \gamma \in \mathbb{C}\backslash\{0\}, 0 \leq \lambda < 1, \mu < 1$ and $\eta > \max(\lambda, \mu) - p - 1)$

Clearly we have the following relationship.

1. $M_{p,m}^{\lambda,\lambda,\eta}[A, B; \gamma] \equiv M_{p,m}^{\lambda,\eta}[A, B; \gamma]$ where $M_{p,m}^{\lambda,\eta}[A, B; \gamma]$ represents the class of functions $f \in A_p$ *satisfying the condition*
$$1 + \frac{1}{\gamma}\left(\frac{zD_z^{\lambda+1} f(z)}{D_z^{\lambda} f(z)} - (p-\mu-m)\right) \prec \frac{1+Az}{1+Bz}$$
2. $M_{1,0}^{0,0,\eta}[A, B; \gamma] \equiv S^*[A, B; \gamma]$
3. $M_{1,0}^{1,1,\eta}[1, -1; \gamma] \equiv K(\gamma)$
4. $M_{1,0}^{0,0,\eta}[1, -1; \gamma] \equiv S(\gamma)$
5. $M_{1,0}^{0,0,\eta}[1, -1; 1-\alpha] \equiv S^*(\alpha)$

The class $S^*[A, B; \gamma]$ is studied by Polatoglu [10], which is a well known class of starlike function. The classes $S(\gamma)$ and $K(\gamma)$ are classes of starlike and convex of complex order $\gamma \neq 0$ in Δ which were considered by Naser and Aouf [8], and Wiatrowski [16]. The class of starlike functions of order $S^*(\alpha)$ in Δ.

A majorization problem for the class $S(\gamma)$ has recently been investigated by Altinas *et al.* [1]. Also, majorization problems for the class $S^* = S^*(0)$ have been investigated by MacGregor [7]. Further, majorization problem for different different classes have been studied by Goyal and Goswami [5], Goyal at el. [6], Goswami and Aouf [2], Goswami and Wang [3] and Goswami et al.[4]. In the present paper, we investigate a majorization problem for the class $M_{p,m}^{\lambda,\mu,\eta}[A, B, \gamma]$.

2 Majorization problem for the class $M_{p,m}^{\lambda,\mu,\eta}[A, B, \gamma]$.

We begin by proving

Theorem 2.1. *Let the function $f \in A_p$ and suppose that $g \in M_{p,m}^{\lambda,\mu,\eta}[A, B, \gamma]$.If $J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z)$ is majorized by $J_{0,z}^{\lambda+m,\mu+m,\eta+m} g(z)$ in Δ , then*

$$\left|J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} f(z)\right| \leq \left|J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} g(z)\right| \tag{2.1}$$

for $|z| = r_1$, where r_1 is the smallest positive root of the following equation

$$\begin{aligned} &|\gamma A - B(\gamma - p + \mu + m)|\, r^3 - (|p - \mu - m| + 2B)\, r^2 \\ &-(|\gamma A - B(\gamma - p + \mu + m)| + 2)r + |p - \mu - m| = 0 \end{aligned} \tag{2.2}$$

where $(z \in \Delta, p \in \mathbb{N}, m, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \gamma \in \mathbb{C}\backslash\{0\}, 0 \leq \lambda < 1, \mu < 1\ , |p - \mu - m| \geq |\gamma A - B(\gamma - p + \mu + m)|$ and $\eta > \max(\lambda, \mu) - p - 1)$

Proof. Since $g \in M_{p,m}^{\lambda,\mu,\eta}[A, B, \gamma]$, we have

$$1 + \frac{1}{\gamma}\left(\frac{zJ_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} g(z)}{J_{0,z}^{\lambda+m,\mu+m,\eta+m} g(z)} - (p - \mu - m)\right) = \frac{1 + Aw(z)}{1 + Bw(z)} \tag{2.3}$$

We note that $w(z) = c_1 z + c_2 z^2 + ... \in \mathcal{P}$,where $\mathcal{P}$ denotes the well-known class of bounded analytic function in Δ (see [9]) and satisfies the conditions

$$w(0) = 0 \ \text{ and } \ |\mathrm{w(z)}| \leq |\mathrm{z}| \ \ (\mathrm{z} \in \Delta)$$

From (2.3), we get

$$\frac{zJ_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} g(z)}{J_{0,z}^{\lambda+m,\mu+m,\eta+m} g(z)} = \frac{w(z)\{\gamma A - B(\gamma - p + \mu + m)\} + (p - \mu - m)}{1 + Bw(z)}, \tag{2.4}$$

which yields

$$\begin{aligned} &\left|J_{0,z}^{\lambda+m,\mu+m,\eta+m} g(z)\right| \\ &\leq \tfrac{|z|(1+B|z|)}{[|(p-\mu-m)|-|z||(\gamma A-B(\gamma-p+\mu+m))|]} \left|J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} g(z)\right|. \end{aligned} \tag{2.5}$$

Next, since $J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z)$ is majorized by $J_{0,z}^{\lambda+m,\mu+m,\eta+m} g(z)$ i.e.

$$J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \phi(z) J_{0,z}^{\lambda+m,\mu+m,\eta+m} g(z)$$

Differentiating the above equation with respect to 'z', we get

$$J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} f(z) = \phi'(z) J_{0,z}^{\lambda+m,\mu+m,\eta+m} g(z) + \phi(z) J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} g(z). \tag{2.6}$$

Since $\phi \in \mathcal{P}$ satisfies the inequality (see eg. Nehari [9])

$$|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}, (\mathrm{z} \in \Delta),$$

by using it in (2.6), we easily get

$$\begin{aligned}\left|J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1}f(z)\right| &\leq \frac{1-|\phi(z)|^2}{1-|z|^2}\left|J_{0,z}^{\lambda+m,\mu+m,\eta+m}g(z)\right| \\ &+|\phi(z)|\left|J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1}g(z)\right|.\end{aligned}$$

Using (2.5) in above equation, we obtain

$$\begin{gathered}\left|J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1}f(z)\right| \leq [\tfrac{1-|\phi(z)|^2}{1-|z|^2}. \\ \tfrac{|z|(1+B|z|)}{(|(p-\mu-m)|-|z||(\gamma A-B(\gamma-p+\mu+m))|)}]+|\phi(z)|\left|J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1}g(z)\right|.\end{gathered} \tag{2.7}$$

Let $|\phi(z)|=\rho$ $(0\leq\rho\leq 1)$ and $|z|=r$, then

$$\begin{gathered}\left|J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1}f(z)\right| \\ \leq \tfrac{\psi(r,\rho)}{(1-r^2)(|(p-\mu-m)|-r|(\gamma A-B(\gamma-p+\mu+m))|)}\left|J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1}g(z)\right|\end{gathered} \tag{2.8}$$

where

$$\begin{aligned}\psi(r,\rho) &= -\rho^2 r(1+Br)+\rho[(1-r^2)\{|(p-\mu-m)| \\ &\quad -r\,|(\gamma A-B(\gamma-p+\mu+m))|\}]+r(1+Br).\end{aligned}$$

Now we have to prove

$$\left|J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1}f(z)\right| \leq \left|J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1}g(z)\right|$$

To prove it, it is sufficient to show that

$$\frac{\psi(\rho)}{(1-r^2)[|(p-\mu-m)|-r\,|(\gamma A-B(\gamma-p+\mu+m))|]}\leq 1,$$

which is equivalent

$$\begin{gathered}(1-\rho)[-(1+\rho)r(1+Br)+(1-r^2)\{|(p-\mu-m)| \\ -r\,|(\gamma A-B(\gamma-p+\mu+m))|\}\geq 0,\end{gathered}$$

this implies

$$\begin{gathered}u(r,\rho)=[(1-r^2)\{|(p-\mu-m)|-r\,|(\gamma A-B(\gamma-p+\mu+m))|\} \\ -(1+\rho)r(1+Br)]\geq 0.\end{gathered}$$

while the function $u(r,\rho)$ takes its minimum values at $\rho=1$,i.e.

$$\min\{u(r,\rho):\rho\in[0,1]\}=u(r,1)=v(r),$$

where

$$\begin{aligned}v(r) &= |\gamma A-B(\gamma-p+\mu+m)|\,r^3-(|p-\mu-m|+2B)\,r^2 \\ &\quad -(|\gamma A-B(\gamma-p+\mu+m)|+2)\,r+|p-\mu-m|\end{aligned}$$

It follows that $v(r)\geq 0$ for all $r\in[0,r_1]$ *, where r_1 is the smallest positive root of the* equation given by (2.2). ■
Upon setting $\lambda=\mu$, we get
Corollary 2.1. *Let the function $f\in\mathcal{A}_p$ and suppose that $g\in M_p^{\lambda}[A,B;\gamma]$.If $D_{0,z}^{\lambda+m}g(z)$ is majorized by $D_{0,z}^{\lambda+m}g(z)$ in the unit disk Δ , then*

$$\left|D_{0,z}^{\lambda+m+1}f(z)\right|\leq\left|D_{0,z}^{\lambda+m+1}g(z)\right| \quad \text{for } |z|\leq r_2$$

where r_2 is the smallest positive root of the following equation,

$$|\gamma A-B(\gamma-p+\lambda+m)|\,r^3-(|p-\lambda-m|+2B)\,r^2$$

$$-\left(|\gamma A-B(\gamma-p+\lambda+m)|+2\right)r+|p-\mu-m|=0$$

$$\begin{aligned}(z &\in \Delta, p\in, j\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, \gamma\in\mathbb{C}\setminus\{0\}, 0\le\lambda<1,\\ \textit{and}\,\eta &> \max\lambda-p-1).\end{aligned}$$

Putting $\lambda=\mu=m=0$ and $p=1$, we have
Corollary 2.2. *$f\in\mathcal{A}_p$ and suppose that $g\in S^*[A,B;\gamma]$. If $f(z)$ is majorized by $g(z)$*, then

$$|f'(z)|\le|g'(z)| \quad \text{for } |z|\le r_3,$$

where r_3 is the smallest positive root of the following equation,

$$|\gamma A-B(\gamma-1)|\,r^3-(1+2B)\,r^2-\left(|\gamma A-B(\gamma-1)|+2\right)r+1=0.$$

Remarks :
(i) Putting $\lambda=\mu=m=p=1$, and $A=1, B=-1$, we have a known result obtain by Altinas et al. [1],
(ii) Putting $\lambda=m=0, \mu=p=1$ and $A=1, B=-1$, we have the known result obtained by Mac-Gregor [7].

References

[1] O. Altinas, O. Ozkan and H. M. Srivastava, Majorization by starlike functions of complex order, *Complex var*, 46(2001):207-218.
[2] P. Goswami and M. K. Aouf, Majorization properties for certain classes of analytic functions using the Salagean operator. *Appl. Math. Lett*, 23(11)(2010):1351-1354
[3] P. Goswami and Z.-G. Wang, Majorization for certain classes of analytic functions, *Acta Univ. Apulensis Math. Inform*, (21)(2010):97-104.
[4] P. Goswami, B. Sharma and T. Bulboaca, Majorization for certain classes of analytic functions using multiplier transformation. *Appl. Math. Lett*, 23(5)(2010):633-637.
[5] S. P. Goyal and P. Goswami, Majorization for certain classes of analytic functions defined by fractional derivatives, *Appl. Math. Lett*, 22(12)(2009):1855-1858.
[6] S. P. Goyal, S. K. Bansal and P. Goswami, Majorization for the subclass of analytic functions defined by linear operators using differential subordination,*J. Appl. Math. Stat. Informatics (JAMSI)*, 6(2)(2010):45-50.
[7] T. H. MacGreogor, Majorization by univalent functions, *Duke Math. J*, 34(1967):95-102.
[8] M.A. Naser and M.K. Aouf, Starlike function of complex order, *J. Natur. Sci. Math*, 25(1985):1-12.
[9] Z. Nehari, *Conformal mapping ,MacGra-Hill Book Company, New York, Toronto and London*, (1955).
[10] Y.Polatoglu, M.Bolcal and A. Sen, The radius of starlikeness for convex functions of complex order, *Int. J. Math. Math. Sci* , (45)(2004):2423-2428.
[11] R. K. Raina and I. B. Bapna, Inequalities defining certain subclasses of analytic and multivalent functions involving fractional calculus operator, *J. Ineq. Pure Appl. Math*, 5(2)(2005):28.
[12] R. K. Raina and C. H. Choi, Some results connected with a subclass of analytic functions involving certain fractional calculus operators, *J. Fracl. Cal* , 23(2003):19-25.
[13] R.K. Raina and H. M. Srivastava, A certain subclass of analytic functions associated with operators of fractional calculus, *Comput. Math. Appl*, 32(1996):13-19.
[14] H.M. Srivastava, S. Owa (Eds.), Univalent Functions, Fractional Calculus, and Their Applications, *Halsted Press (Ellis Horwood Limited, Chichester, John Wiley and Sons, New York*, 1989.
[15] H.M. Srivastava, M.Saigo and S. Owa, A class of distortion theorems involing certain operators of fractional calculus, *J. Math. Anal. Appl*, 131(1988):412-420.
[16] P. Wiatrowski, On the coefficients of some family of holomorphic functions, *Zeszyry Nauk. Uniw. Lddz. Nauk. Mat.-Przyrod*, 39(1970):75-85.