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Stability and Existence Results for a Class of (p, q)-Laplacian Dirichlet Systems Involving Indefinite Weight Functions

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Abstract: This study concerns the existence and stability properties of positive weak solutions to classes of nonlinear systems involving the (p, q)-Laplacian and indefinite weight functions. First by using the method of sub-super solution we establish our existence results. Next we study the stability properties of positive weak solutions under certain conditions.

Keywords: (p, q)-Laplacian; Sub-super solution; Linearized stability

1 Introduction

In this paper we are interested in discussing the existence and stability properties of positive weak solutions to the nonlinear elliptic system

$$-\Delta_p u = \lambda a(x)(v^{\gamma-1} - v^{p-1}) - c \qquad \text{in } \Omega, \tag{1}$$

$$-\Delta_q v = \lambda b(x)(u^{\beta-1} - u^{q-1}) - c \qquad \text{in } \Omega, \tag{2}$$

$$u = v = 0 \qquad \text{on } \partial\Omega, \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with C^2-boundary $\partial\Omega$, $\Delta_s u := \text{div}(|\nabla u|^{s-2}\nabla u)$ is the s-Laplacian operator, $p, q > 1$, λ, c, γ and β are positive parameters and the weight functions $a(x)$, $b(x)$ satisfy $a, b \in L^\infty(\Omega)$ and $a(x) \geq a_0 > 0$, $b(x) \geq b_0 > 0$ for all $x \in \Omega$.

Boundary value problems involving the p-Laplacian arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (for example, see [12]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids.

Equations of the form

$$\begin{cases} -\Delta_p u = \lambda f(u) - c & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

arise in several context in biology and engineering (see [9]). When $c = 0$, it is easy to establish the existence of positive solutions for a large $\lambda \geq 0$. It provides a simple model to describe, for instance, the cooperative interaction of two diffusing biological species u and v represent the densities of two species. See [11] for details on the physical models involving more general reaction-diffusion system. We refer to [1–3, 6–8, 10] for additional results in nonlinear elliptic systems.

The purpose of this paper is to extend this study to the (p, q)-Laplacian case. First by using the method of sub-super solution we study the existence of positive weak solutions. Second we discuss the stability properties of positive solutions directly by analyzing the linearized system. Let $W_0^{1,s}(\Omega)$, $s > 1$, denotes the usual Sobolev space, and $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. We give the definition of sub-super solution of (1)-(3) as follows.

Definition 1 *A pair of non-negative functions (ψ_1, ψ_2) (resp. (z_1, z_2)) in X is called a positive weak subsolution (resp. supersolution) of (1)-(3), if ψ_i $(i = 1, 2)$ satisfies:*

$$\int_\Omega |\nabla\psi_1(x)|^{p-2}\nabla\psi_1(x)\nabla w_1(x)dx \leq \lambda \int_\Omega \Big(a(x)\big(\psi_2(x)^{\gamma-1} - \psi_2(x)^{p-1}\big) - c\Big)w_1(x)dx, \tag{5}$$

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$$\int_\Omega |\nabla \psi_2(x)|^{q-2}\nabla\psi_2(x)\nabla w_2(x)dx \le \lambda \int_\Omega \Big(b(x)\big(\psi_1(x)^{\beta-1} - \psi_1(x)^{q-1}\big) - c\Big) w_2(x)dx \tag{6}$$

$\Bigg($*resp.* z_i $(i = 1, 2)$ *satisfies:*

$$\int_\Omega |\nabla z_1(x)|^{p-2}\nabla z_1(x)\nabla w_1(x)dx \ge \lambda \int_\Omega \Big(a(x)\big(z_2(x)^{\gamma-1} - z_2(x)^{p-1}\big) - c\Big) w_1(x)dx, \tag{7}$$

$$\int_\Omega |\nabla z_2(x)|^{q-2}\nabla z_2(x)\nabla w_2(x)dx \ge \lambda \int_\Omega \Big(b(x)\big(z_1(x)^{\beta-1} - z_1(x)^{q-1}\big) - c\Big) w_2(x)dx\Bigg) \tag{8}$$

for all non-negative test functions $(w_1, w_2) \in X$.

Now, if there exists a subsolution and a supersolution (ψ_1, ψ_2) and (z_1, z_2), respectively, such that $0 \le \psi_i(x) \le z_i(x)$ $(i = 1, 2)$ for all $x \in \Omega$, then (1)-(3) has a positive solution $(u, v) \in X$ such that $\psi_1(x) \le u(x) \le z_1(x)$ and $\psi_2(x) \le v(x) \le z_2(x)$ for all $x \in \Omega$. We shall obtain the existence of positive weak solution to the system (1)-(3) by constructing a positive subsolution (ψ_1, ψ_2) and a positive supersolution (z_1, z_2).

2 Existence results

In this section we prove that if $\gamma < p$ and $\beta < q$, then there exist positive constants c_o and λ^* such that the system (1)-(3) has a positive solution when $c \le c_o$ and $\lambda \ge \lambda^*$. We shall obtain the existence of positive weak solution to the system (1)-(3) by constructing a positive subsolution (ψ_1, ψ_2) and a positive supersolution (z_1, z_2).

To precisely state our theorem, we first consider the eigenvalue problems

$$\begin{cases} -\Delta_p\phi_1 = \lambda_1|\phi_1|^{p-2}\phi_1 & \text{in } \Omega, \\ \phi_1 = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta_q\phi_2 = \lambda_2|\phi_2|^{q-2}\phi_2 & \text{in } \Omega, \\ \phi_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Let λ_1 and λ_2 be the respective first eigenvalues of Δ_p and Δ_q with Dirichlet boundary conditions and ϕ_1 and ϕ_2 the corresponding eigenfunctions with $\phi_1, \phi_2 > 0$ and $\|\phi_1\|_\infty = \|\phi_2\|_\infty = 1$. Hence there exist $\eta \ge 0$, $\mu \in (0, 1]$ and $k > 0$ such that

$$\begin{cases} |\nabla\phi_1|^p - \lambda_1\phi_1^p \ge k & \text{in } \overline{\Omega}_\eta, \\ \phi_1 \ge \mu & \text{in } \Omega \setminus \overline{\Omega}_\eta, \end{cases}$$

and

$$\begin{cases} |\nabla\phi_2|^q - \lambda_2\phi_2^q \ge k & \text{in } \overline{\Omega}_\eta, \\ \phi_2 \ge \mu & \text{in } \Omega \setminus \overline{\Omega}_\eta, \end{cases}$$

where $\overline{\Omega}_\eta = \{x \in \Omega | d(x, \partial\Omega) \le \eta\}$. We will also consider the unique solutions $\zeta_1, \zeta_2 \in C^1(\overline{\Omega})$ of the boundary value problems

$$\begin{cases} -\Delta_p\zeta_1 = 1 & \text{in } \Omega, \\ \zeta_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta_q\zeta_2 = 1 & \text{in } \Omega, \\ \zeta_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

To discuss our existence result, it is known that $\zeta_1, \zeta_2 > 0$ in Ω and $\frac{\partial\zeta_1}{\partial n}, \frac{\partial\zeta_2}{\partial n} < 0$ on $\partial\Omega$, where n denotes the outward unit normal to $\partial\Omega$. Our existence result is formulated as follows.

Theorem 1 *Let* $\gamma < p$ *and* $\beta < q$. *Then there exist positive constants* c_o *and* λ^* *such that the system (1)-(3) has a positive solution when* $c \le c_o$ *and* $\lambda \ge \lambda^*$.

Proof. We construct a positive subsolution (ψ_1,ψ_2) and a positive supersolution (z_1,z_2) to obtain existence of positive solution for the system (1)-(3). We shall verify that $(\psi_1,\psi_2)=(\frac{p-1}{p}\,\phi_1^{\frac{p}{p-1}},\frac{q-1}{q}\,\phi_2^{\frac{q}{q-1}})$ is a subsolution of (1)-(3). Note that $\|\psi_1\|_\infty,\|\psi_2\|_\infty\le 1$. For ψ_1 we have

$$\int_\Omega |\nabla\psi_1(x)|^{p-2}\nabla\psi_1(x)\nabla w_1(x)dx=\int_\Omega \phi_1(x)|\nabla\phi_1(x)|^{p-2}\nabla\phi_1(x)\nabla w_1(x)dx$$
$$=\int_\Omega |\nabla\phi_1(x)|^{p-2}\nabla\phi_1(x)\nabla(\phi_1(x)w_1(x))dx-\int_\Omega|\nabla\phi_1(x)|^p w_1(x)dx$$
$$=\int_\Omega(\lambda_1\phi_1(x)^p-|\nabla\phi_1(x)|^p)w_1(x)dx.$$

First, the case when $x\in\overline{\Omega}_\eta$, we have $\lambda_1\,\phi_1^p-|\nabla\phi_1|^p\le -k$. Therefore if $c\le c_0=k$, then

$$\begin{aligned}\lambda_1\phi_1^p-|\nabla\phi_1|^p &\le -k\le -c\\ &\le \lambda a(x)(\psi_2^{\gamma-1}-\psi_2^{p-1})-c,\end{aligned}$$

since $\lambda a(x)(\psi_2^{\gamma-1}-\psi_2^{p-1})\ge 0$. Next, consider the case when $x\in\Omega\setminus\overline{\Omega}_\eta$, we note that $\phi_1,\phi_2\ge\mu>0$. Now if

$$\lambda\ge\lambda_1^*:=\frac{\lambda_1+c}{\alpha_1 a_0},$$

where

$$\alpha_1:=\inf\left\{f(t):(\frac{p-1}{p})\mu^{\frac{p}{p-1}}\le t\le\frac{p-1}{p}\right\}\qquad\text{with}\qquad f(t)=t^{\gamma-1}-t^{p-1},$$

then we have

$$\begin{aligned}\lambda_1\,\phi_1^p-|\nabla\phi_1|^p &\le \lambda_1\\ &\le \lambda\alpha_1 a_0-c\\ &\le \lambda a(x)(\psi_2^{\gamma-1}-\psi_2^{p-1})-c.\end{aligned}$$

Thus we proved (5). Similarly, for ψ_2 we can prove (6) with

$$\lambda\ge\lambda_2^*:=\frac{\lambda_2+c}{\alpha_2 b_0}$$

where

$$\alpha_2:=\inf\left\{g(t):(\frac{q-1}{q})\mu^{\frac{q}{q-1}}\le t\le\frac{q-1}{q}\right\}\qquad\text{with}\qquad g(t)=t^{\beta-1}-t^{q-1}.$$

Now if $c\le c_0$ and $\lambda\ge\lambda^*:=\max\{\lambda_1^*,\lambda_2^*\}$, then (ψ_1,ψ_2) is a subsolution. Next, we construct a supersolution (z_1,z_2) of (1)-(3). We denote

$$z_1(x)=A\zeta_1(x),\qquad z_2(x)=B\zeta_2(x),$$

where the constants $A,B>0$ are large and to be chosen later. We shall verify that (z_1,z_2) is a supersolution of (1)-(3). To this end, let $w_1\in W_0^{1,p}(\Omega)$ and $w_2\in W_0^{1,q}(\Omega)$ with $w_1,w_2\ge 0$. Then we obtain

$$\int_\Omega|\nabla z_1(x)|^{p-2}\nabla z_1(x)\nabla w_1(x)dx=A^{p-1}\int_\Omega w_1(x)dx,$$
$$\int_\Omega|\nabla z_2(x)|^{q-2}\nabla z_2(x)\nabla w_2(x)dx=B^{q-1}\int_\Omega w_2(x)dx.$$

There exist positive large constants A,B such that

$$A\ge(A_1)^{\frac{1}{P-1}}\qquad\text{where}\qquad A_1:=\lambda\|a\|_\infty\sup_{t\in[0,1]}f(t)$$

and

$$B \geq (B_1)^{\frac{1}{q-1}} \qquad \text{where} \qquad B_1 := \lambda \|b\|_\infty \sup_{t\in[0,1]} g(t).$$

These imply that

$$A^{p-1} \geq \lambda\, a(x)\,(z_1^{\gamma-1}(x) - \, z_1^{p-1}(x)) \geq \lambda\, a(x)\,(z_1^{\gamma-1}(x) - \, z_1^{p-1}(x)) - c$$

and

$$B^{q-1} \geq \lambda\, b(x)\,(z_2^{\beta-1}(x) - \, z_2^{q-1}(x)) \geq \lambda\, b(x)\,(z_2^{\beta-1}(x) - \, z_2^{q-1}(x)) - c.$$

Hence (7) and (8) are valid. Since $\zeta_1, \zeta_2 > 0$ and $\partial\zeta_1/\partial n < 0,\ \partial\zeta_2/\partial n < 0$ on $\partial\Omega$, we can choose A, B large enough so that $\psi_i \leq z_i$ is also satisfied $(i = 1, 2)$. Thus, by comparison principle, there exists a solution (u, v) of (1)-(3) with $\psi_1 \leq u \leq z_1$, $\psi_2 \leq v \leq z_2$. This completes the proof of the theorem. ■

3 Stability results

In this section we shall prove the instability (stability) of a positive solution (u, v) to the system (1)-(3) directly by analyzing the linearized equation.

We recall that, if (u, v) is any positive solution to elliptic system of the form

$$\begin{cases} -\Delta_p u = \lambda f(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = \lambda g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

then the linearized equation about (u, v) is

$$-(p-1)\text{div}(|\nabla u|^{p-2}\nabla w) - \lambda f_u(x,u,v)w - \lambda f_v(x,u,v)z = \eta w \qquad \text{in } \Omega, \tag{9}$$

$$-(q-1)\text{div}(|\nabla v|^{q-2}\nabla z) - \lambda g_u(x,u,v)w - \lambda g_v(x,u,v)z = \eta z \qquad \text{in } \Omega, \tag{10}$$

$$w = z = 0 \qquad \text{on } \partial\Omega, \tag{11}$$

where f_t denotes the partial derivative of f with respect to t. Eqs. (9)-(11) obtained from the formal derivative of the operators Δ_p and Δ_q.

Definition 2 *Let η_1 denote the first eigenvalue of (9)-(11). We say that (u, v) is linearly stable if all eigenvalues of (9)-(11) are strictly positive, which can be inferred if the principal eigenvalue $\eta_1 > 0$. Otherwise (u, v) is linearly unstable.*

Let (u, v) be any positive solution of (1)-(3). Then from (9)-(11) the linearized equation about (u, v) is

$$-(p-1)\text{div}(|\nabla u|^{p-2}\nabla w) - \lambda a(x)\big((\gamma-1)v^{\gamma-2} - (p-1)v^{p-2}\big)z = \eta w \qquad \text{in } \Omega, \tag{12}$$

$$-(q-1)\text{div}(|\nabla v|^{q-2}\nabla z) - \lambda b(x)\big((\beta-1)u^{\beta-2} - (q-1)u^{q-2}\big)w = \eta z \qquad \text{in } \Omega, \tag{13}$$

$$w = z = 0 \qquad \text{on } \partial\Omega. \tag{14}$$

Let η_1 be the principal eigenvalue and (ϕ, ψ) be the corresponding eigenfunction. We make take ϕ, ψ such that $\phi, \psi > 0$ in Ω and $\|\phi\|_\infty = \|\psi\|_\infty = 1$ (see [4]). Now we state our stability result.

Theorem 2 *Suppose that $c \leq c_0$ and $\lambda \geq \lambda^*$. Let (u, v) be the solution of the system (1)-(3) obtained in Theorem 1. Moreover, let γ, β, c and λ be such that*

$$c\big[(p-1)\phi + (q-1)\psi\big] + \lambda\big[a(x)(\gamma-1)v^{\gamma-2} - (p-1)v^{p-2})\phi u + b(x)(\beta-1)u^{\beta-2} - (q-1)u^{q-2})\psi v\big]$$

$$< \lambda\big[\phi(p-1)a(x)(v^{\gamma-1} - v^{p-1}) + (q-1)b(x)(u^{\beta-1} - u^{q-1})\psi\big]$$

for all $x \in \Omega$. Then (u, v) is linearly stable.

Proof. We calculate $(1)(p-1)\,\phi-(12)\,u+(2)(q-1)\,\psi-(13)\,v$ and then integrating over Ω yields

$$\begin{aligned}
&(p-1)\int_\Omega \big[u(x)\text{div}(|\nabla u(x)|^{p-2}\nabla\phi(x))-\phi(x)\text{div}(|\nabla u(x)|^{p-2}\nabla u(x))\big]dx\\
&+(q-1)\int_\Omega \big[v(x)\text{div}(|\nabla v(x)|^{q-2}\nabla\psi(x))-\psi(x)\text{div}(|\nabla v(x)|^{q-2}\nabla v(x))\big]dx\\
&+\lambda\int_\Omega \big[a(x)(\gamma-1)v(x)^{\gamma-2}-(p-1)v(x)^{p-2})\phi(x)u(x)+b(x)(\beta-1)u(x)^{\beta-2}\\
&-(q-1)u(x)^{q-2})\psi(x)v(x)\big]dx \qquad (15)\\
&-\lambda\int_\Omega \big[\phi(x)(p-1)a(x)(v(x)^{\gamma-1}-v(x)^{p-1})+(q-1)b(x)(u(x)^{\beta-1}-u(x)^{q-1})\psi(x)\big]dx\\
&+c\int_\Omega \big[(p-1)\phi(x)+(q-1)\psi(x)\big]dx\\
&=-\eta_1\int_\Omega (u(x)\phi(x)+v\psi(x))dx.
\end{aligned}$$

But by using the Green's first identity we obtain

$$\begin{aligned}
\int_\Omega u(x)\text{div}(|\nabla u(x)|^{p-2}\nabla\phi(x))dx &= -\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x)\nabla\phi(x))dx\\
&+\int_{\partial\Omega} u(x)|\nabla u(x)|^{p-2}(\frac{\partial\phi}{\partial n})ds \qquad (16)\\
&=-\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x)\nabla\phi(x))dx,
\end{aligned}$$

$$\begin{aligned}
\int_\Omega \phi(x)\text{div}(|\nabla u(x)|^{p-2}\nabla u(x))dx &= -\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x)\nabla\phi(x))dx\\
&+\int_{\partial\Omega} \phi(x)|\nabla u(x)|^{p-2}(\frac{\partial u}{\partial n})ds \qquad (17)\\
&=-\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x)\nabla\phi(x))dx,
\end{aligned}$$

$$\begin{aligned}
\int_\Omega v(x)\text{div}(|\nabla v(x)|^{q-2}\nabla\psi(x))dx &= -\int_\Omega |\nabla v(x)|^{q-2}(\nabla v(x)\nabla\psi(x))dx\\
&+\int_{\partial\Omega} v(x)|\nabla v(x)|^{q-2}(\frac{\partial\psi}{\partial n})ds \qquad (18)\\
&=-\int_\Omega |\nabla v(x)|^{q-2}(\nabla v(x)\nabla\psi(x))dx,
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \psi(x)\text{div}(|\nabla v(x)|^{q-2}\nabla v(x))dx &= -\int_\Omega |\nabla v(x)|^{q-2}(\nabla v(x)\nabla\psi(x))dx\\
&+\int_{\partial\Omega} \psi(x)|\nabla v(x)|^{q-2}(\frac{\partial v}{\partial n})ds \qquad (19)\\
&=-\int_\Omega |\nabla v(x)|^{q-2}(\nabla v(x)\nabla\psi(x))dx.
\end{aligned}$$

By using (16)-(19) in (15) we get

$$\begin{aligned}
-\eta_1 \int_\Omega (u(x)\phi(x) + v(x)\psi(x))dx = \lambda \int_\Omega &\big[a(x)(\gamma-1)v(x)^{\gamma-2} - (p-1)v(x)^{p-2})\phi(x)u(x) \\
&+ b(x)(\beta-1)u(x)^{\beta-2} - (q-1)u(x)^{q-2})\psi(x)v(x)\big]dx \\
&- \lambda \int_\Omega \big[\phi(x)(p-1)a(x)(v(x)^{\gamma-1} - v(x)^{p-1}) \\
&+ (q-1)b(x)(u(x)^{\beta-1} - u(x)^{q-1})\psi(x)\big]dx \\
&+ c\int_\Omega \big[(p-1)\phi(x) + (q-1)\psi(x)\big]dx \\
&< 0.
\end{aligned} \tag{20}$$

Hence

$$-\eta_1 \int_\Omega (u(x)\phi(x) + v(x)\psi(x))dx < 0. \tag{21}$$

But $\phi(x), \psi(x) > 0$ for all $x \in \Omega$ and also $u, v > 0$ and hence $\eta_1 > 0$. ■

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