

Applications of Homotopy Perturbation Transform Method for Solving Time-Dependent Functional Differential Equations

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Abstract: In this paper, we apply homotopy perturbation transform method for solving linear and nonlinear functional differential equations. This method is a combined form of the Laplace transform method with the homotopy perturbation method. The nonlinear terms can be easily handled by the use of He's polynomials. This technique finds the solutions without any discretization or restrictive assumptions and free from round-off errors and therefore reduces the numerical computations to a great extent. The results are also given to demonstrate the validity and applicability of the present technique. Also the results reveal that the homotopy perturbation transform method is very efficient, simple and can be applied to other nonlinear problems. The fact that this technique solves nonlinear problems without using Adomain's polynomials can be considered as a clear advantage of this algorithm over the decomposition method.

Keywords: Laplace transform method, homotopy perturbation method, Klein-Gordon equations, Emden-Fowler equations, Evolution equations, Cauchy reaction-diffusion equations, He's polynomials.

1 Introduction

The study of nonlinear problems is of crucial importance in all areas of physics and engineering, as well as in other discipline. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one to a given nonlinear problem. There are several methods used to find approximate solutions to nonlinear problems, such as Bäcklund transformation [1], Hirota's bilinear method [2, 3], δ -expansion method [4], homotopy perturbation method [5-7], variational iteration method (VIM) [8-15], Adomain's decomposition method (ADM) [16-18], Laplace decomposition method [19] and variational iteration decomposition method [20]. The linear and nonlinear functional equations as Cauchy reaction-diffusion equation, Evolution equations, Emden-Fowler type equations and Klein-Gordon equations are considerable significance for gauge theory, wave theory, mathematical physics, theoretical physics, chemical physics and applied sciences. Several techniques including the Adomain decomposition method, the variational iteration method, the weighted finite differences method, the Laplace decomposition method, the homotopy perturbation method and the homotopy analysis method have been used to find the solutions of linear and nonlinear functional equations [21-36]. Most of these methods such as Adomain decomposition method, variational iteration method, weighted finite difference method, and Laplace decomposition method have their inbuilt deficiencies like the calculation of Adomain's polynomials, the Lagrange's multipliers, divergent results and huge computational work. It is well known that the homotopy perturbation method (HPM), first proposed by He [38-44] has successfully been applied to solve many types of linear and nonlinear differential equations. This method, which is a combination of homotopy in topology and classical perturbation techniques, provides us with a convenient way to obtain analytic and approximate solutions for a wide variety of problems arising in different fields. The Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to

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deal with these nonlinearities such as the Adomain decomposition method (ADM) [45] and the Laplace decomposition method [46-50]. Furthermore, the homotopy perturbation method is also combined with well known Laplace transform method [51] to produce a highly effective technique for handling many nonlinear problems.

Motivated and inspired by the ongoing research in these areas, we consider a new method, which is called the homotopy perturbation transform method (HPTM) [54]. The suggested HPTM provides the solution in a rapid convergent series which may leads the solution in a closed form. The advantage of this method is its capability of combining of two powerful methods for obtaining exact solution for nonlinear equations. The use of He's polynomials in the nonlinear term was first introduced by Ghorbani [52, 53]. It is worth mentioning that the HPTM is applied without any discretization or restrictive assumptions or transformations and free from round-off errors. Unlike the method of separation of variables that require initial or boundary conditions, the HPTM provides an analytical solution by using the initial conditions only. The boundary conditions can be used only to justify the obtained results. This method work efficiently and the results so far are very encouraging and reliable. We would like to emphasize that the HPTM may be considered as an important and significant refinement of the previously developed techniques and can be viewed as an alternative to the recently developed methods such as Adomain's decomposition method (ADM), variational iteration method (VIM) and Homotopy perturbation method (HPM). In this paper we have considered the effectiveness of the homotopy perturbation transform method (HPTM) for solving linear and nonlinear functional differential equations of various types. Several examples are given to verify the reliability and efficiency of the homotopy perturbation transform method.

2 Homotopy perturbation transform method (HPTM)

This method is introduced by Khan and Wu [54] by combining the Homotopy perturbation method and Laplace transform method for solving various types of linear and nonlinear systems of partial differential equations. To illustrate the basic idea of HPTM, we consider a general nonlinear partial differential equation with the initial conditions of the form [54].

$$D u(x, t) + R u(x, t) + N u(x, t) = g(x, t), \quad (1)$$

$$u(x, 0) = h(x), u_t(x, 0) = f(x),$$

where D is the second order linear differential operator $D = \partial^2 / \partial t^2$, R is the linear differential operator of less order than D , N represents the general nonlinear differential operator and $g(x, t)$ is the source term. Taking the Laplace transform (denoted in this paper by L) on both sides of Eq. (1):

$$L [D u(x, t)] + L [R u(x, t)] + L [N u(x, t)] = L [g(x, t)]. \quad (2)$$

Using the differentiation property of the Laplace transform, we have

$$L [u(x, t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^2} L [R u(x, t)] + \frac{1}{s^2} L [g(x, t)] - \frac{1}{s^2} L [N u(x, t)]. \quad (3)$$

Operating with the Laplace inverse on both sides of Eq. (3) gives

$$u(x, t) = G(x, t) - L^{-1} \left[\frac{1}{s^2} L [R u(x, t) + N u(x, t)] \right], \quad (4)$$

where $G(x, t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the homotopy perturbation method

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) \quad (5)$$

and the nonlinear term can be decomposed as

$$N u(x, t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (6)$$

for some He's polynomials $H_n(u)$ (see [52, 53]) that are given by

$$H_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, \quad n = 0, 1, 2, 3, \dots \quad (7)$$

Substituting Eq. (5) and Eq. (6) in Eq. (4), we get

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = G(x, t) - p \left(L^{-1} \left[\frac{1}{s^2} L \left[R \sum_{n=0}^{\infty} p^n u_n(x, t) + \sum_{n=0}^{\infty} p^n H_n(u) \right] \right] \right), \tag{8}$$

which is the coupling of the Laplace transform and the homotopy perturbation method using He’s polynomials. Comparing the coefficient of like powers of p , the following approximations are obtained.

$$\begin{aligned} p^0 : u_0(x, t) &= G(x, t), \\ p^1 : u_1(x, t) &= -L^{-1} \left[\frac{1}{s^2} L [R u_0(x, t) + H_0(u)] \right], \\ p^2 : u_2(x, t) &= -L^{-1} \left[\frac{1}{s^2} L [R u_1(x, t) + H_1(u)] \right], \\ p^3 : u_3(x, t) &= -L^{-1} \left[\frac{1}{s^2} L [R u_2(x, t) + H_2(u)] \right], \\ &\vdots \end{aligned} \tag{9}$$

3 Applications

In order to obtain a good approximation, different choices of an arbitrary operator L and initial guesses ν_0 have been used by Chowdhury and Hashim [34] and Babolian et al. [35]. Here we present a new technique without using any arbitrary constants and initial guesses and compare our approximations with solutions obtained in [34, 35]. Here we present two comparative examples to review this classic view by HPTM.

Example 3.1 Consider the time dependent Emden-Fowler equation

$$y_{xx} + \frac{2}{x}y_x - (6 + 4x^2 - \cos(t))y = y_t, \tag{10}$$

with the initial condition

$$y(x, 0) = e^{x^2},$$

and the boundary conditions

$$y(0, t) = e^{\sin(t)}, \quad y_x(x, 0) = 0. \tag{11}$$

Taking Laplace transform both of sides, subject to the initial condition we have

$$L[y(x, t)] = \frac{e^{x^2}}{s} + \frac{1}{s}L \left[y_{xx} + \frac{2}{x}y_x - (6 + 4x^2 - \cos t)y \right]. \tag{12}$$

Taking Inverse Laplace transform we get

$$y(x, t) = e^{x^2} + L^{-1} \left[\frac{1}{s}L \left(y_{xx} + \frac{2}{x}y_x - (6 + 4x^2 - \cos t)y \right) \right]. \tag{13}$$

By homotopy perturbation method, we get

$$y(x, t) = \sum_{n=0}^{\infty} p^n y_n(x, t). \tag{14}$$

Substitute Eq. (14) in Eq. (13), we get

$$\begin{aligned} \sum_{n=0}^{\infty} p^n y_n(x, t) &= e^{x^2} + pL^{-1} \left[\frac{1}{s}L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) \right]_{xx} + pL^{-1} \left[\frac{2}{xs}L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) \right]_x \\ &\quad - pL^{-1} \left[L \left(\frac{(6 + 4x^2 - \cos t)}{s} \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) \right) \right]. \end{aligned} \tag{15}$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned} p^0 : y_0(x, t) &= e^{x^2}, \\ p^1 : y_1(x, t) &= e^{x^2} \frac{\sin t}{1!}, \\ p^2 : y_2(x, t) &= e^{x^2} \frac{\sin^2 t}{2!}, \end{aligned} \quad (16)$$

proceeding in a similar manner, we get

$$\begin{aligned} p^3 : y_3(x, t) &= e^{x^2} \frac{\sin^3 t}{3!}, \\ p^4 : y_4(x, t) &= e^{x^2} \frac{\sin^4 t}{4!}, \\ &\vdots \end{aligned} \quad (17)$$

Then the approximate solution in a series form is given by

$$y(x, t) = e^{x^2} \left[1 + \frac{\sin t}{1!} + \frac{\sin^2 t}{2!} + \frac{\sin^3 t}{3!} + \frac{\sin^4 t}{4!} + \dots \right] = e^{x^2 + \sin t}, \quad (18)$$

which is an exact solution and is same as obtained in [34, 35].

Example 3.2 Consider the Cauchy reaction-diffusion equation

$$y_t = y_{xx} - y, \quad (19)$$

with the initial condition

$$y(x, 0) = e^{-x} + x. \quad (20)$$

By applying aforesaid method, we have

$$\sum_{n=0}^{\infty} p^n y_n(x, t) = e^{-x} + x + pL^{-1} \left[\frac{1}{s} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)_{xx} - \frac{1}{s} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) \right]. \quad (21)$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned} p^0 : y_0(x, t) &= e^{-x} + x, \\ p^1 : y_1(x, t) &= -xt, \\ p^2 : y_2(x, t) &= x \frac{t^2}{2!}, \end{aligned} \quad (22)$$

proceeding in a similar manner, we get

$$\begin{aligned} p^3 : y_3(x, t) &= -x \frac{t^3}{3!}, \\ p^4 : y_4(x, t) &= x \frac{t^4}{4!}, \\ &\vdots \end{aligned} \quad (23)$$

Then the approximate solution is given by

$$y(x, t) = e^{-x} + x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \dots \right) = e^{-x} + xe^{-t}, \quad (24)$$

which is an exact solution and is same as obtained in [35].

4 Examples

4.1 Evolution equation

Babolian et al. [35] and Ganji et al. [36] have been solved the given problem by HPM using $L\phi = \frac{\partial\phi}{\partial t}$ and taking the initial condition $v_0(x, t) = 0$ as initial guesses. We have present here this example by using HPTM without choosing an arbitrary initial guesses and compare our approximations with solutions obtained in [35, 36].

Example 4.1.1 Consider the equation

$$y_t - y_{xxxx} = 0, \tag{25}$$

with the initial condition

$$y(x, 0) = \sin x.$$

By applying aforesaid method we have

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = \sin x + pL^{-1} \left[\frac{1}{s} L \left(\sum_{n=0}^{\infty} p^n u_n(x, t) \right)_{xxxx} \right]. \tag{26}$$

Comparing the coefficients of various powers of p , we have

$$\begin{aligned} p^0 : \quad u_0(x, t) &= \sin x, \\ p^1 : \quad u_1(x, t) &= \sin x \, t, \\ p^2 : \quad u_2(x, t) &= \sin x \frac{t^2}{2!}, \end{aligned} \tag{27}$$

proceeding in a similar manner, we get

$$\begin{aligned} p^3 : \quad u_3(x, t) &= \sin x \frac{t^3}{3!}, \\ p^4 : \quad u_4(x, t) &= \sin x \frac{t^4}{4!}, \end{aligned} \tag{28}$$

Then the approximate solution is given by

$$y(x, t) = \sin x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \right) = \sin x e^t, \tag{29}$$

which is an exact solution and is same as obtained by HPM [35, 36].

4.2 Cauchy reaction-diffusion equation

Cauchy reaction-diffusion equations have been solved by various methods such as HAM [31] and HPM [35]. Here we present the reliability of the HPTM and compare our approximations with solutions obtained via [31, 35].

Example 4.2.1 Consider the equation

$$y_t = y_{xx} + 2ty, \tag{30}$$

with the initial condition

$$y(x, 0) = e^x.$$

By applying aforesaid method we have

$$\sum_{n=0}^{\infty} p^n y_n(x, t) = e^x + pL^{-1} \left[\frac{1}{s} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)_{xx} + \frac{2}{s} L \left(t \cdot \sum_{n=0}^{\infty} p^n y_n(x, t) \right) \right]. \tag{31}$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned} p^0 : \quad y_0(x, t) &= e^x, \\ p^1 : \quad y_1(x, t) &= e^x(t + t^2), \end{aligned} \tag{32}$$

$$\begin{aligned}
 p^2 : \quad y_2(x, t) &= e^x \frac{(t+t^2)^2}{2!}, \\
 p^3 : \quad y_3(x, t) &= e^x \frac{(t+t^2)^3}{3!}, \\
 &\vdots
 \end{aligned} \tag{33}$$

Then the approximate solution is given by

$$y(x, t) = e^x \left(1 + (t+t^2) + \frac{(t+t^2)^2}{2!} + \frac{(t+t^2)^3}{3!} + \dots \right) = e^{x+t+t^2}, \tag{34}$$

which is an exact solution and is same as obtained via HAM [31] and HPM [35].

Example 4.2.2 We consider the equation

$$y_t = y_{xx} - (1 + 4x^2)y, \tag{35}$$

with the initial condition

$$y(x, 0) = e^{x^2}.$$

By applying aforesaid method, we have

$$\sum_{n=0}^{\infty} p^n y_n(x, t) = e^{x^2} + pL^{-1} \left[\frac{1}{s} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)_{xx} - \frac{(1 + 4x^2)}{s} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) \right]. \tag{36}$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned}
 p^0 : \quad y_0(x, t) &= e^{x^2}, \\
 p^1 : \quad y_1(x, t) &= e^{x^2} t, \\
 p^2 : \quad y_2(x, t) &= e^{x^2} \frac{t^2}{2!}, \\
 p^3 : \quad y_3(x, t) &= e^{x^2} \frac{t^3}{3!}, \\
 &\vdots
 \end{aligned} \tag{37}$$

Then the approximate solution is given by

$$y(x, t) = e^{x^2} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = e^{x^2+t}, \tag{38}$$

which is an exact solution and is same as obtained via HAM [31] and HPM [35].

Example 4.2.3 Consider the equation

$$y_t = y_{xx} - (4x^2 - 2t + 2)y, \tag{39}$$

with the initial condition

$$y(x, 0) = e^{x^2}.$$

By applying aforesaid method, we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} p^n y_n(x, t) &= e^{x^2} + pL^{-1} \left[\frac{1}{s} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)_{xx} - \frac{4x^2}{s} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) \right] \\
 &+ pL^{-1} \left[\frac{2}{s} L \left(t \sum_{n=0}^{\infty} p^n y_n(x, t) \right) - \frac{2}{s} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) \right].
 \end{aligned} \tag{40}$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned}
 p^0 : \quad y_0(x, t) &= e^{x^2}, \\
 p^1 : \quad y_1(x, t) &= e^{x^2} t^2, \\
 p^2 : \quad y_2(x, t) &= e^{x^2} \frac{t^4}{2!}, \\
 p^3 : \quad y_3(x, t) &= e^{x^2} \frac{t^6}{3!}, \\
 &\vdots
 \end{aligned}
 \tag{41}$$

Then the approximate solution is given by

$$y(x, t) = e^{x^2} \left(1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \dots \right) = e^{x^2+t^2}, \tag{42}$$

which is an exact solution and is same as obtained by HAM [31] and HPM [35].

4.3 Emden-Fowler equations

Chowdhury and Hashim have been solved the time dependent Emden-Fowler type equations by HPM [34] and Sami Bataineh solved this type of equations by HAM [32]. Here we solved this type of problem using HPTM and compared our approximations with solutions obtained in [32, 34].

Example 4.3.1 Consider the equation

$$y_{xx} + \frac{2}{x}y_x - (5 + 4x^2)y = y_t + (6 - 5x^2 - 4x^4), \tag{43}$$

subject to the initial condition

$$y(x, 0) = x^2 + e^{x^2}.$$

By applying aforesaid method, we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} p^n y_n(x, t) &= (x^2 + e^{x^2}) + pL^{-1} \left[\frac{1}{s} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)_{xx} + \frac{2}{xs} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)_x \right] \\
 &- pL^{-1} \left[\frac{(5 + 4x^2)}{s} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) + \frac{(6 - 5x^2 - 4x^4)}{s} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)^0 \right]. \tag{44}
 \end{aligned}$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned}
 p^0 : \quad y_0(x, t) &= x^2 + e^{x^2}, \\
 p^1 : \quad y_1(x, t) &= e^{x^2} t, \\
 p^2 : \quad y_2(x, t) &= e^{x^2} \frac{t^2}{2!}, \\
 p^3 : \quad y_3(x, t) &= e^{x^2} \frac{t^3}{3!}, \\
 &\vdots
 \end{aligned}
 \tag{45}$$

Then the approximate solution is given by

$$y(x, t) = x^2 + e^{x^2} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = x^2 + e^{x^2+t}, \tag{46}$$

which is an exact solution and is same as obtained by HAM [32] and HPM [34].

Example 4.3.2 Consider the following equation

$$y_{xx} + \frac{2}{x}y_x - (5 + 4x^2)y = y_{tt} + (12x - 5x^3 - 4x^5),$$

subject to the initial condition

$$y(x, 0) = x^3 + e^{x^2}, y_t(x, 0) = -e^{x^2}, \quad (47)$$

By applying aforesaid method, we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n y_n(x, t) &= x^3 + (1-t)e^{x^2} + pL^{-1} \left[\frac{1}{s^2} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)_{xx} \right] \\ &+ pL^{-1} \left[\frac{2}{xs^2} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)_x - \frac{1}{s^2} (5 + 4x^2) L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) \right] \\ &- pL^{-1} \left[\frac{1}{s^2} (12x - 5x^3 - 4x^5) L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)^0 \right]. \end{aligned} \quad (48)$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned} p^0 : \quad y_0(x, t) &= x^3 + (1-t)e^{x^2}, \\ p^1 : \quad y_1(x, t) &= e^{x^2} \left(\frac{t^2}{2!} - \frac{t^3}{3!} \right), \\ p^2 : \quad y_2(x, t) &= e^{x^2} \left(\frac{t^4}{4!} - \frac{t^5}{5!} \right), \\ p^3 : \quad y_3(x, t) &= e^{x^2} \left(\frac{t^6}{6!} - \frac{t^7}{7!} \right), \\ &\vdots \end{aligned} \quad (49)$$

Then the approximate solution is given by

$$y(x, t) = x^3 + e^{x^2} \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \dots \right) = x^3 + e^{x^2-t}, \quad (50)$$

which is an exact solution and is same as obtained by HAM [32] and HPM [34].

Example 4.3.3 Consider the equation

$$y_{tt} = y_{xx} + \frac{4}{x}y_x - (18x + 9x^4)y + 2 + (18x + 9x^4)t^2,$$

subject to the initial condition

$$y(x, 0) = e^{x^3}, y_t(x, 0) = 0. \quad (51)$$

By applying aforesaid method, we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n y_n(x, t) &= e^{x^3} + pL^{-1} \left[\frac{1}{s^2} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)_{xx} + \frac{4}{xs^2} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)_x \right] \\ &+ pL^{-1} \left[-\frac{(18x + 9x^4)}{s^2} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) + \frac{2}{s^2} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)^0 \right] \\ &+ pL^{-1} \left[\frac{(18x + 9x^4)}{s^2} L \left(t^2 \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right)^0 \right) \right]. \end{aligned} \quad (52)$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned} p^0 : \quad y_0(x, t) &= e^{x^3}, \\ p^1 : \quad y_1(x, t) &= t^2, \\ p^2 : \quad y_2(x, t) &= 0, \\ &\vdots \end{aligned} \tag{53}$$

In general $y_n(x, t) = 0, \quad n \geq 2.$

$$y(x, t) = e^{x^3} + t^2, \tag{54}$$

which is an exact solution and is same as obtained by HAM [32] and HPM [34].

4.4 Klein-Gordon equations

Klein-Gordon equations have been solved using HPM by Chowdhury and Hashim [37], where they used a technical initial guess and obtain a very good approximation. Also Odibat and Momani [33] have solved the same problem by HPM. Here we test the efficiency of HPTM and compare our approximations with solutions obtained by [33, 37].

Example 4.4.1 we consider the linear Klein-Gordon equation

$$y_{tt} - y_{xx} = y, \tag{55}$$

with the initial condition

$$y(x, 0) = 1 + \sin x, y_t(x, 0) = 0.$$

By applying aforesaid method, we have

$$\sum_{n=0}^{\infty} p^n y_n(x, t) = (1 + \sin x) + pL^{-1} \left[\frac{1}{s^2} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) \right]_{xx} + \frac{1}{s^2} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right). \tag{56}$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned} p^0 : \quad y_0(x, t) &= 1 + \sin x, \\ p^1 : \quad y_1(x, t) &= \frac{t^2}{2!}, \\ p^2 : \quad y_2(x, t) &= \frac{t^4}{4!}, \\ p^3 : \quad y_3(x, t) &= \frac{t^6}{6!}, \\ &\vdots \end{aligned} \tag{57}$$

Then the approximate solution is given by

$$y(x, t) = \sin x + \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \dots \right) = \sin x + \cosh t, \tag{58}$$

which is an exact solution and is same as obtained by HPM [33,37].

Example 4.4.2 We consider the nonlinear nonhomogeneous Klein-Gordon equation

$$y_{tt} = y_{xx} + 2y - 2 \sin x \sin t, \tag{59}$$

with the initial condition $y(x, 0) = 0$ and $y_t(x, 0) = \sin x.$

By applying aforesaid method, we have

$$\sum_{n=0}^{\infty} p^n y_n(x, t) = \sin x t - 2 \sin x (t - \sin t) + pL^{-1} \left[\frac{1}{s^2} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) \right]_{xx}$$

$$+pL^{-1} \left[\frac{2}{s^2} L \left(\sum_{n=0}^{\infty} p^n y_n(x, t) \right) \right]. \quad (60)$$

Comparing the coefficients of various powers of p , we get

$$\begin{aligned} p^0 : \quad y_0(x, t) &= t \sin x - 2t \sin x + 2 \sin x \sin t, \\ p^1 : \quad y_1(x, t) &= -\sin x \frac{t^3}{3!} + 2t \sin x - 2 \sin x \sin t, \\ p^2 : \quad y_2(x, t) &= -\sin x \frac{t^5}{5!} + \sin x \frac{t^3}{3!} + 2 \sin x \sin t - 2t \sin x, \\ p^3 : \quad y_3(x, t) &= -\sin x \frac{t^7}{7!} + \sin x \frac{t^5}{60} - 2 \sin x \sin t + 2t \sin x - \sin x \frac{t^3}{3}, \\ &\vdots \end{aligned} \quad (61)$$

It is important to recall here that the noise terms appears between the components $y_0(x, t)$ and $y_1(x, t)$. The noise terms between the components $y_0(x, t)$, $y_1(x, t)$ and higher approximations are cancelled out and the remaining terms will satisfy the equation.

Then an approximate solution is given by

$$y(x, t) = \sin x \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right) = \sin x \sin t, \quad (62)$$

which is an exact solution and is same as obtained by HPM [33,37].

Example 4.4.3 We consider the nonlinear non-homogeneous Klein-Gordon equation

$$y_{tt} - y_{xx} + y^2 = -x \cos t + x^2 \cos^2 t, \quad (63)$$

with the initial condition

$$y(x, 0) = x, y_t(x, 0) = 0,$$

which has the exact solution $y(x, t) = x \cos t$.

Our analytical seems complicated but still converges to the exact solution.

The 4-term approximations is

$$\begin{aligned} y_{app}(x, t) &= y_0(x, t) + y_1(x, t) + y_2(x, t) + y_3(x, t) \\ &= \cos t \left\{ \frac{1}{8} \cos^2 t - \frac{15}{8} t^2 - 4 \cos t + \frac{1}{8} t^4 + \frac{31}{8} \right\} x \\ &\quad + \cos t \left\{ \frac{1}{4} \cos^2 t + \frac{1}{4} t^2 - \frac{1}{12} t^4 - \frac{1}{4} + \cos t + \frac{1}{90} t^6 \right\} x^2 \\ &\quad + \cos t \left\{ -\frac{64}{9} + 2t \sin t + \frac{43}{6} \cos t + 2t^2 - \frac{1}{6} t^4 - \frac{t^2}{2} \cos t - \frac{1}{18} \cos^3 t \right\} x^3 \\ &\quad + \cos t \left\{ \frac{1}{32} t^2 \cos^2 t + \frac{5}{192} t^4 - \frac{33}{256} \cos^2 t - \frac{23}{256} t^2 + \frac{1}{256} \cos^4 t + \frac{1}{8} \right. \\ &\quad \left. - \frac{7}{1440} t^6 - \frac{1}{16} t \cos t \sin t \right\} x^4. \end{aligned} \quad (65)$$

In table-1, we have computed the absolute errors for this approximation at some points which shows efficiency of our choice. It seems that the errors increases by increasing x and t . It is because we have used only 4-terms in our approximations. So it isn't a big problem. For obtaining more accurate results (with smaller error values) one should use more terms in his / her approximation.

Example 4.4.4 Consider the nonlinear non-homogeneous Klein-Gordon equation

$$y_{tt} = y_{xx} - y^2 + 6xt(x^2 - t^2) + x^6 + t^6, \quad (66)$$

Table 1: Absolute errors of a 4-term approximation

x_i	t_i	$ y - y_{app} $
0.1	0.1	3.749×10^{-12}
0.2	0.2	1.128×10^{-9}
0.3	0.3	3.175×10^{-8}
0.4	0.4	3.229×10^{-7}
0.5	0.5	1.781×10^{-6}

Table 2: Absolute errors of a 4-term approximation

x_i	t_i	$ y - y_{app} $
0.1	0.1	1.247×10^{-16}
0.2	0.2	2.044×10^{-12}
0.3	0.3	5.968×10^{-10}
0.4	0.4	3.349×10^{-8}
0.5	0.5	7.615×10^{-7}

with the initial condition $y(x, 0) = 0$ and $y_t(x, 0) = 0$.

$$\begin{aligned}
 y_{app}(x, t) &= y_0(x, t) + y_1(x, t) + y_2(x, t) + y_3(x, t) \\
 &= x^3 t^3 + \frac{53}{4200} x^4 t^{10} - \frac{13}{92400} x^2 t^{12} - \frac{1}{959616} x^{12} t^{18} + \frac{1}{19600} x^7 t^{15} \\
 &\quad - \frac{1}{4368} x^9 t^{13}.
 \end{aligned} \tag{67}$$

Again we can use more terms in our approximation to get more accurate results for larger values of t and x . Moreover we should indicate that our choice in computing the error in point (x, t) with equal values of x and t one can get very close results other (not equal) values. Also it is worth mentioning that the initial guess proposed by Chowdhury and Hashim in [34] may not be an effective choice when used for other types of problems. For example when their initial guess is applied to time dependent Emden-Fowler type equation [34], yields a divergent result.

5 Conclusions

In this paper, we used the homotopy perturbation transform method for solving functional equations. The solution procedure by using He's polynomials [52, 53] is simple, but the calculation of Adomain polynomials is complex. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still mentioning the high level of accuracy of the numerical results; the size reduction amounts to an improvement of the performance of the approach. Also the proposed scheme does not need any arbitrary initial guesses L and v_0 which have been used in [31, 32, 34 and 35]. The fact that the HPTM solves nonlinear problems without using the Adomain's polynomials can be considered as a clear advantage of this algorithm over the decomposition method. It is observed that the proposed scheme exploits full advantage of Homotopy perturbation method [34, 35, 36] and Homotopy analysis method [31]. Finally, we conclude that HPTM can be considered as a nice refinement in existing numerical technique and might find wide applications.

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