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Study for System of Fractional Differential Equations

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Abstract: In this work, we study existence and uniqueness theorems for the initial value problem for the system of fractional differential equations $D^\alpha \bar{x}(t) = A\bar{x}(t)$, $t^{\alpha-1}\bar{x}(t) = \bar{b}$, where D^α denotes standard Riemann-Liouville fractional derivative,$0 < \alpha < 1$, $\bar{x}(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T$, $\bar{b} = [b_1, b_2, \cdots, b_n]^T$ and A is a square matrix. At the same time, power-type estimate for them have been given.

Keywords: Riemann-Liouville fractional derivative; Weighted Cauchy-type problem ; fractional differential equations

1 Introduction

Let M_n denote the $n \times n$ matrix over real fields $\mathbb{R}$ or complex fields $\mathbb{C}$. For $h > 0, C_r^0([0,h])[4] := \{f \in C^0((0,h]) : \lim_{t\to 0^+} t^r f(t)$ exists and is finite$\}$, here, $C^0((0,h])$ is the usual space of continuous functions on $(0,h]$, which is a Banach space with the norm $\|f\|_r := \max_{0\le t\le h} t^r|f(t)|$. The space $C_{1-\alpha}^\alpha([0,h])$ is defined by [4]$C_{1-\alpha}^\alpha([0,h]) = \{f \in C_{1-\alpha}([0,h])$: there exists $c \in \mathbb{R}$ and $f^* \in C_{1-\alpha}^0([0,h])$ s.t. $f(t) = ct^{\alpha-1} + I^\alpha f^*(t)\}$.

The existence and uniqueness of initial value problems for fractional order differential equations have been studied in many literatures such as [2,3,4,5]. In this paper, we present the analysis of the system of fractional differential equations

$$\begin{cases} D^\alpha \bar{x}(t) = A\bar{x}(t), \\ t^{\alpha-1}\bar{x}(t) = \bar{b} \end{cases} \qquad (*)$$

where D^α denotes standard Riemann-Liouville fractional derivative,$\frac{1}{2} < \alpha < 1$. $\bar{x}(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T$, $D^\alpha \bar{x}(t) = [D^\alpha x_1(t), D^\alpha x_2(t), \cdots, D^\alpha x_n(t)]^T$, $\bar{b} = [b_1, b_2, \cdots, b_n]^T$ and A is a square matrix.

Now,we give some definitions.

Definition 1[2] Let f be continuous function defined on $[a,b]$ and $n-1 \le \alpha < n, n \in \mathbb{N}$. Then the expression

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, x > a$$

is called left-sided fractional derivatives of order α.

Definition 2[2]Let f be continuous function defined on $[a,b]$ and $\alpha > 0$. Then the expression

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{-\alpha+1}} dt, x > a$$

is called left-sided fractional integral of order α.

2 Lemmas

To prove the main results, we begin with hypotheses and some lemmas.

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H1: $f(t,x)$ is continuous on $\mathbb{R}^+ \times \mathbb{R}$ and is such that

$$|f(t,x)| \leq t^{\mu} e^{-\sigma t} \varphi(t)|x|^m, \mu \geq 0, m > 1, \sigma > 0, \tag{1}$$

where $\varphi(t)$ is a continuous function on $\mathbb{R}^+$.

H2: $f(t,x)$ is continuous on $\mathbb{R}^+ \times \mathbb{R}$ and is such that

$$|f(t,x)| \leq t^{\mu} \varphi(t)|x|^m, \mu \geq 0, m > 1, \tag{2}$$

where $\varphi(t)$ is a continuous function on $\mathbb{R}^+$.

Lemma 1 *[1] Given $A \in M_n$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ in any prescribed order, there is a unitary matrix $U \in M_n$ such that $U^*AU = T = [t_{ij}]$ is upper triangular with diagonal entries $t_{ii} = \lambda_i, i = 1, 2 \cdots n$. That is, every square matrix A is unitarily equivalent to triangular matrix whose entries are the eigenvalues of A in a prescribed order. Further more, if $A \in M_n(R)$ and if all the eigenvalues of A are real, then U may be chosen to be real and orthogonal.*

Lemma 2 *Let $\frac{1}{2} < \alpha < 1$. If we assume that $0 < q < \frac{1}{1-\alpha}$ then the initial value problem*

$$\begin{cases} D^{\alpha}x(t) = x^q(t) + y(t), \\ t^{1-\alpha}x(t)|_{t=0} = b \end{cases} \tag{3}$$

where $y(t) \in C^0_{1-\alpha}([0,h]) \bigcap L^1(0,h)$, has at least a solution $x(t) \in C^0_{1-\alpha}([0,h])$ for $h > 0$ sufficiently small.

Proof. If $C^0_{1-\alpha}([0,h])$, and $q(\alpha-1) > -1$, then $x^q \in L^1(0,h)$. We are therefore reduced again to the nonlinear integral equation

$$x(t) = bt^{\alpha-1} + \frac{1}{\Gamma(\alpha)}(\int_0^t (t-s)^{\alpha-1}x^q(s)ds + \int_0^t (t-s)^{\alpha-1}y(s)ds) \tag{4}$$

The existence of a solution to the problem(3) can be formulated as a fixed point equation $Tx = x$, where

$$(Tx)(t) = bt^{\alpha-1} + \frac{1}{\Gamma(\alpha)}(\int_0^t (t-s)^{\alpha-1}x^q(s)ds + \int_0^t (t-s)^{\alpha-1}y(s)ds) \tag{5}$$

in the space $C^0_{1-\alpha}([0,h])$. To prove existence of the solution to the problem given by (3) we use Schauder's fixed point theorem.

Define $S = \{x \in C^0_{1-\alpha}([0,h]) : \|x - bt^{\alpha-1}\|_{1-\alpha} \leq r + \frac{1}{\alpha}h^{2\alpha-1}\|y\|_{1-\alpha}\}$. clearly, it is closed, convex and nonempty.

Step I. We shall prove that $TS \subseteq S$. we note that

$$\begin{aligned} \|Tx - bt^{\alpha-1}\|_{1-\alpha} &= \max_{t\in[0,h]} \tfrac{t^{1-\alpha}}{\Gamma(\alpha)} |\textstyle\int_0^t (t-s)^{\alpha-1}x^q(s)ds + \int_0^t (t-s)^{\alpha-1}y(s)ds| \\ &= \max_{t\in[0,h]} \tfrac{t^{1-\alpha}}{\Gamma(\alpha)} \{|\textstyle\int_0^t (t-s)^{\alpha-1}s^{q(\alpha-1)}s^{q(1-\alpha)}x^q(s)ds| \\ &+ |\textstyle\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}y(s)ds|\} \\ &\leq \tfrac{\Gamma(q(\alpha-1)+1)}{\Gamma(q(\alpha-1)+1+\alpha)\Gamma(\alpha)} h^{q(\alpha-1)+1}\|x\|^q_{1-\alpha} + \tfrac{1}{\alpha}h^{2\alpha-1}\|y\|_{1-\alpha}. \end{aligned}$$

Since $\|x\|_{1-\alpha} \leq r + |b|$, it will be sufficient to impose

$$\|x - bt^{\alpha-1}\|_{1-\alpha} \leq \text{const.} h^{q(\alpha-1)+1}(r + |b|)^q \leq r$$

In view of the assumption $q(\alpha-1)+1 > 0$, the second estimate is satisfied if say $r = |b|$ and h is chosen sufficiently small.

Step II. We shall prove that the operator T is compact. To prove the compactness of

$$T : C^0_{1-\alpha}([0,h]) \to C^0_{1-\alpha}([0,h])$$

defined by (5), it will be sufficient to argue on the operator

$$T_* : C^0([0,h]) \to C^0([0,h])$$

defined in this way:

$$(T_*x)(t) = t^{1-\alpha}T(t^{1-\alpha}x(t)).$$

We have $T_*x = b + T^*x$ where the operator

$$(T^*x)(t) = \frac{t^{1-\alpha}}{\Gamma(\alpha)}(\int_0^t (t-s)^{\alpha-1}x^q(s)ds + \int_0^t (t-s)^{\alpha-1}y(s)ds)$$

turn out be compact from classical sufficient conditions, since $q(\alpha-1) > -1$,$s - 1 > -1$. ∎

Lemma 3 *[4] Suppose that $f(t,x)$ satisfies H1, $\mu-(m-1)(1-\alpha)>0$ and $\alpha>\frac{1}{2}$. If $\|\varphi\|_q<L$ for some $q>\frac{1}{\alpha}$, then the problem*

$$\begin{cases} D^\alpha x = f(x), \\ t^{1-\alpha}x(t)|_{t=0} = b \end{cases} \tag{6}$$

exists a positive constant C such that $|x(t)|\le Ct^{\alpha-1}, t>0$.

Lemma 4 *[4] Let $x\in C^{\alpha}_{1-\alpha}([0,h])$ with $\alpha>\frac{1}{2}$. Suppose further that $\mu-(m-1)(1-\alpha)>0$. Then problem (6) and its associated integral equation*

$$x(t) = bt^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds \tag{7}$$

are equivalent.

Lemma 5 *[4] Assume that $\alpha>\frac{1}{2}$, $f(t,x)$ satisfies H2, and $\|\varphi\|_q<K$ for some $q>\frac{1}{\alpha.}$ Suppose further that $\mu+\frac{1}{p}<m(1-\alpha)$, then there exists $C>0$ and $0<\delta<1-\alpha$ such that any solution of (6) exists globally and satisfies*

$$|x(t)|\le Ct^{-\delta}, t\ge a>0. \tag{8}$$

3 Main results

Theorem 6 *Let $A\in M_n$, then initial problem $(*)$ has a unique solution $\bar{x}(t)\in\mathbb{R}^n$, where $\bar{x}(t)=[\bar{x}_1(t),\bar{x}_2(t),\cdots,\bar{x}_n(t)]^T$ and $x_i(t)\in C^0_{1-\alpha}(([0,h])\bigcap L^1(0,h))$ for all $i=1,2,\cdots n$ and sufficiently small $h>0$.*

Proof. Given $A\in M_n$ with eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$, By Lemma 1, there is a unitary matrix $U\in M_n$ such that $U^*AU=T=[t_{ij}]$ is upper triangular with diagonal entries $t_{ii}=\lambda_i, i=1,2\cdots n$. Let $\bar{y}(t)=U^*\bar{x}(t)$, we have

$$D^\alpha\bar{y}(t) = U^*D^\alpha\bar{x}(t) = U^*A\bar{x}(t) = U^*AU\bar{y}(t) = T\bar{y}(t).$$

At the same time, the initial problem $(*)$ changed into

$$\begin{cases} D^\alpha\bar{y}(t) = T\bar{y}(t), \\ t^{\alpha-1}\bar{y}(t) = U^*\bar{b} \end{cases} \tag{$**$}$$

Now, let's consider the problem $(**)$.

Clearly, the problem $(**)$ is equivalent to the following n problems

$$\begin{cases} D^\alpha y_i(t) = \sum_{j=i}^n t_{ij}y_j(t), \\ t^{\alpha-1}y_i(t) = b_i \end{cases}$$

for $i,j=1,2,\cdots n$. where, b_i is the ith entries of the vector $U^*\bar{b}$.

Consider the system of equations

$$\begin{cases} D^\alpha y_n(t) = t_{nn}y_n(t), \\ t^{\alpha-1}y_n(t) = b_n. \end{cases}$$

In Lemma 2, take $q=1, y(t)=0$. Then by Lemma 2, $\exists h>0$, s.t. the above problem has at least a solution $y_n(t)\in C^0_{1-\alpha}(([0,h])\bigcap L^1(0,h)$.

Consider the system of equations

$$\begin{cases} D^\alpha y_{n-1}(t) = t_{n-1,n-1}y_{n-1}(t) + t_{n-1,n}y_n(t), \\ t^{\alpha-1}y_{n-1}(t) = b_{n-1}. \end{cases}$$

In Lemma 2, take $q=1, y(t)=t_{n-1n}y_n(t)$. Then by Lemma 2, $\exists h>0$, s.t. the above problem has at least a solution $y_{n-1}(t)\in C^0_{1-\alpha}(([0,h])\bigcap L^1(0,h)$.

Similarly, there has at least a solution in $C^0_{1-\alpha}(([0,h])\bigcap L^1(0,h)$ for the rest n-2 initial problem in $(**)$, denote by $y_{n-2}(t), y_{n-3}(t),\cdots,y_1(t)$ respectively. And therefore, there has at least solution $\bar{y}(t)=[y_1(t),y_2(t),\cdots,y_n)]^T$ of the problem $(**)$. Let $\bar{x}(t)=U\bar{y}(t)$. It is required for us. ■

Since the problem $(**)$ is equivalent to the following n problems

$$\begin{cases} D^\alpha y_i(t) = \sum_{j=i}^n t_{ij}y_j(t), \\ t^{\alpha-1}y_i(t) = b_i \end{cases} \tag{9}$$

for $i,j=1,2,\cdots n$. where, b_i is the ith entries of the vector $U^*\bar{b}$, we shall discuss these equations.

Theorem 7 *Assume that the righthand of these equations give by* (9) *satisfied H1,* $\mu-(m-1)(1-\alpha)>0, \alpha>\frac{1}{2}$*, and* $\|\varphi\|_q<L$ *for some* $q>\frac{1}{\alpha}$*, if the solution of the problems* $(**)$ *denoted by* $\bar{x}(t)=[x_1(t),x_2(t),\cdots,x_n(t)]^T$*, then there exists some constant* $C>0$ *such that* $|x_i(t)|\le\|U\|_\infty Ct^{\alpha-1}, t>0,$ *for all* $i=1,2,\cdots,n$*.*

Proof. Similar to the proof of the theorem 6, Consider the system of equations

$$\begin{cases} D^\alpha y_n(t)=t_{nn}y_n(t), \\ t^{\alpha-1}y_n(t)=b_n. \end{cases}$$

Then by Lemma 3,there exists some constant C_n such that $|y_n(t)|\le C_n t^{\alpha-1}, t>0.$

Consider the following system of equations

$$\begin{cases} D^\alpha y_{n-1}(t)=t_{n-1n-1}y_{n-1}(t)+t_{n-1n}y_n(t), \\ t^{\alpha-1}y_{n-1}(t)=b_{n-1}. \end{cases}$$

Then by Lemma 3, there exists some constant C_{n-1} such that $|y_{n-1}(t)|\le C_{n-1}t^{\alpha-1}, t>0.$

Similarly, there exists some constant $C_{n-3}, C_{n-4},\cdots C_1$ such that $|y_i(t)|\le C_i t^{\alpha-1}, t>0$, for all $i=n-3,n-4,\cdots,1.$

Let $\bar{x}(t)=U\bar{y}(t), C=\max_{1\le i\le n}\{C_i\}$ then $|x_i(t)|\le\|U\|_\infty Ct^{\alpha-1}, t>0$, for all $i=1,2,\cdots,n.$ ■

Theorem 8 *Assume that* $\alpha>\frac{1}{2}$*, the righthand of these equations given by* (9) *satisfied H2, and* $\|\varphi\|_q<K$ *for some* $q>\frac{1}{\alpha}$*. Suppose further that* $\mu+\frac{1}{p}<m(1-\alpha)$*. If denote solution of the problems* $(**)$ $\bar{x}(t)$ *by* $\bar{x}(t)=[x_1(t),x_2(t),\cdots,x_n(t)]^T$*, then there exists some constant* $C>0$ *and* $0<\delta<1-\alpha$*, such that* $\bar{x}(t)$ *exists globally and satisfies*

$$|x_i(t)|\le\|U\|_\infty Ct^{-\delta}, t\ge a>0$$

for all $i=1,2,\cdots,n.$

Using Lemma 1,4 and 5, the proof is similar to Theorem 7.Therefore, it is omitted.

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References

[1] ROGER A. HORN and CHARLES R. JOHNSON.Matrix analysis. *Cambridge University press*, London,1985.

[2] Varsha Daftardar-Gejji and A. Babakhani. Analysis of a system of fractional differential equations. *Journal of Mathematical Analysis and Applications*, 293(2004):511-512.

[3] Domenico Delbosco and Lurgi Rodino. Existence and uniqueness for a nonlinear fractional differential equation. *Journal of Mathematical Analysis and Applications*, 204(2)(1996):609-625.

[4] Khaled M. and Nasser-eddine Tatar.Power-type estimates for a nonlear fractional differential equation. *Journal of Nonlinear Analysis*, 62(2005):1025-1036.

[5] I. Podlubny. Fractional Differential Equations. *AcademicPress* ,New York,1999.