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Existence Results for Nonhomogeneous System of Elliptic Equations with Lack of Compactness

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Abstract: We establish the existence of a nontrivial solution for inhomogeneous quasilinear elliptic systems, governed by two Pseudo-Laplacian operators

$$\begin{cases} -\Delta_p u + m(x)u|u|^{p-2} = \lambda(\alpha+1)h(x)u|u|^{\alpha-1}|v|^{\beta+1} + f & \text{in } \Omega, \\ -\Delta_q v + l(x)v|v|^{q-2} = \lambda(\beta+1)h(x)|u|^{\alpha+1}v|v|^{\beta-1} + g & \text{in } \Omega, \\ (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega). \end{cases}$$

Our result is depending on the local minimization method.

Keywords: Quasilinear Elliptic systems; Nehari manifold; Local minimization; Ekeland Variational principle
2000 Mathematics Subject Classifications: 35J60, 35B30, 35B40.

1 Introduction

In this paper we are interested in the problem

$$\begin{cases} -\Delta_p u + m(x)u|u|^{p-2} = \lambda(\alpha+1)h(x)u|u|^{\alpha-1}|v|^{\beta+1} + f & \text{in } \Omega, \\ -\Delta_q v + l(x)v|v|^{q-2} = \lambda(\beta+1)h(x)|u|^{\alpha+1}v|v|^{\beta-1} + g & \text{in } \Omega, \\ (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $R^N (N \geq 2)$, $1 < p, q < N$, $\alpha > -1$, $\beta > -1$, λ is a positive parameter, the functions $m(x), l(x)$ and $h(x) \in C(\overline{\Omega})$ are smooth functions with change sign on $\overline{\Omega}$, $(f,g) \in L^{p'}(\Omega) \times L^{q'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. We propose to show that under condition $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$, where $p^* = \frac{Np}{N-p}$, $q^* = \frac{Nq}{N-q}$ designate respectively the effective critical exponents relating to the Sobolev embedding $W_0^{1,p}(\Omega) \subset L^r(\Omega)$ and $W_0^{1,q}(\Omega) \subset L^r(\Omega)$, (1.1) admits a solution in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. Indeed, this condition represents the maximal growth that the integrability of the product term $|u|^{\alpha+1}|v|^{\beta+1}$ (which will appear in the Euler-Lagrange functional) can be guaranteed by suitable Holder estimates.

For $p \geq 1$, $\Delta_p u$ is the p-Laplacian defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ equipped with the norm $\|\nabla u\|_p$, where $\|.\|_p$ represent the norm of Lebesgue space $L^p(\Omega)$. Let $W_0^{-1,p'}(\Omega)$ be the dual space to $W_0^{1,p}(\Omega)$ and we will write $||f||_{-1,p'}$ for the norm in $W_0^{-1,p'}(\Omega)$. We denote by $\langle,\rangle_{-1,1}$ or more simply $\langle,\rangle$ the natural duality paring between $W_0^{1,p}(\Omega)$ and $W_0^{-1,p'}(\Omega)$. For all $p > 1$, $\mathrm{S_p} = \inf\{\|\nabla u\|_p^p;\ \|u\|_{p^*}^{p^*} = 1,\ u \in W_0^{1,p}(\Omega)\}$ is the best Sobolev constant of immersion $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

In nonlinear elliptic variational problems involving critical nonlinearities, one of the major difficulties is to recover the compactness of Palais-Smale sequences of the associated Euler-Lagrange functional. The concentration-compactness principle due to Lions [10] is widely used to overcome these difficulties. In [11], the author considered the system

$$\begin{cases} -\Delta_p u = u|u|^{\alpha-1}|v|^{\beta+1} + f & \text{in } \Omega, \\ -\Delta_q v = |u|^{\alpha+1}v|v|^{\beta-1} + g & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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where Ω is a regular bounded set of $\mathbf{R}^N$, $\alpha > -1$, $\beta > -1$, $(f,g) \in W_0^{-1,p'}(\Omega) \times W_0^{-1,q}(\Omega)$. He show that under condition $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$, (1.1) admits a solution.

Other methods, based on the convergence almost everywhere of the gradients of Palais-Smale sequences, can be also used to recover the compactness.

J. Chabrowski [7] studied the following system

$$\begin{cases} -\Delta_p u = \lambda u|u|^{\alpha-1}|v|^{\beta+1} + f & \text{in } \Omega, \\ -\Delta_q v = \lambda |u|^{\alpha+1} v|v|^{\beta-1} + g & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain, $\lambda \in \mathbf{R}$ with $\lambda \neq 0$. In the case $p = q$, he developed a method that can be used to find norm-estimates of f and g guaranteeing the solvability of system.

K. Adriouch et al. [1] considered the system

$$\begin{cases} -\Delta_p u = \lambda f(x,u) + u|u|^{\alpha-1}|v|^{\beta+1} & \text{in } \Omega, \\ -\Delta_q v = \lambda g(x,u) + |u|^{\alpha+1} v|v|^{\beta-1} & \text{in } \Omega, \end{cases}$$

in bounded domain with Dirichlet or mixed boundary conditions The functions f and g are two Caratheodory functions with subcritical conditions on the level corresponding to the energy of Palais-Smale sequences which guarantees their relative compactness.

In [5] S. Benmouloud et al. studied system (1.2) in open subset of $\mathbf{R}^N$ with lack of compactness. They used the method based on preliminary results on the convergence almost everywhere of the gradients of Palais-Smale sequences.

Motivated by paper [5], the object of this article is to study the existence of weak solution of system (1.1). Here, we borrow some ideas from that work.

Let us define $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ equipped with the norm $\|(u,v)\|_X = \max(\|\nabla u\|_p, \|\nabla v\|_q)$ which gives to X Banach space properties, reflexivity and separability ([11]).

Definition 1 *We say that $(u,v) \in X$ is a weak solution of system (1.1) if and only if*

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla w_1 dx + \int_\Omega m(x)u|u|^{p-2}w_1 dx = \lambda(\alpha+1)\int_\Omega h(x)u|u|^{\alpha-1}|v|^{\beta+1}w_1 dx + \int_\Omega f w_1 dx,$$
$$\int_\Omega |\nabla v|^{q-2}\nabla v.\nabla w_2 dx + \int_\Omega l(x)v|v|^{q-2}w_2 dx = \lambda(\beta+1)\int_\Omega h(x)|u|^{\alpha+1}v|v|^{\beta-1}w_2 dx + \int_\Omega g w_2 dx,$$

for all $(w_1, w_2) \in X$.

The associated Euler-Lagrange functional to system (1.1) $J : X \to \mathbf{R}$ is defined by

$$J(u,v) = \frac{1}{p}P(u) + \frac{1}{q}Q(v) - \lambda R(u,v) - \langle f,u\rangle - \langle g,v\rangle,$$

where

$$P(u) = \|\nabla u\|_p^p + \int_\Omega m(x)|u|^p dx, \qquad Q(v) = \|\nabla v\|_q^q + \int_\Omega l(x)|v|^q dx,$$

and

$$R(u,v) = \int_\Omega h(x)|u|^{\alpha+1}|v|^{\beta+1}dx.$$

It is well known if J is bounded below and J has a minimizer on X, then this minimizer is a critical point of J. However, the Euler function $J(u,v)$, associated with the problem (1.1), is not bounded below on the whole space X, but is bounded on an appropriate subset, and has a minimizer on this set (if it exists) gives rise to solution to (1.1). Clearly, the critical points of J are the weak solutions of problem (1.1).

Consider the Nehari manifold associated to problem (1.1) given by

$$\Lambda = \{(u,v) \in X\backslash\{(0,0)\}; \langle J'(u,v),(u,v)\rangle = 0\}$$

We set

$$m_1 = \inf_{(u,v)\in\Lambda} J(u,v),$$

and for all $r > 0$ and $t > 0$

$$a(t) = \frac{1}{t} - \frac{1}{\alpha+\beta+2}, \qquad b(t) = \frac{t-1}{(\alpha+\beta+2)(\alpha+\beta+1)},$$
$$c(t) = \frac{\alpha+\beta+2-t}{\alpha+\beta+1}, \qquad d(r,t) = \frac{1}{\frac{1}{pr^p} + \frac{1}{q't^{q'}}},$$

and

$$\varepsilon_1 = d(\theta,\gamma)[c(p) - \frac{\theta^p}{p}][\frac{b(p)\min(S_p^{\frac{p^*}{p}}, S_q^{\frac{q^*}{q}})}{c_0\lambda}]^{\frac{p}{p^*-p}}, \qquad \varepsilon_2 = d(\theta,\gamma)[c(q) - \frac{\gamma^q}{q}][\frac{b(q)\min(S_p^{\frac{p^*}{p}}, S_q^{\frac{q^*}{q}})}{c_0\lambda}]^{\frac{q}{q^*-q}},$$

where $c_0 = \max_{x\in\overline{\Omega}} h(x)$ and θ, γ are fixed numbers such that

$$0 < \theta < [pc(p)]^{\frac{1}{p}}, \qquad \text{and} \qquad 0 < \gamma < [qc(q)]^{\frac{1}{q}}.$$

2 Main result

Our main result is the following:

Theorem 1 *Suppose that $(f,g) \in W_0^{-1,p'}(\Omega) \times W_0^{-1,q'}(\Omega)$, non of the functions f and g is identically to zero on Ω and*

$$(\mathbf{a})\frac{\alpha+\mathbf{1}}{\mathbf{p}^*} + \frac{\beta+\mathbf{1}}{\mathbf{q}^*} = \mathbf{1}, \qquad (\mathbf{b})\max(\mathbf{p},\mathbf{q}) < \alpha+\beta+\mathbf{2}, \qquad (\mathbf{c})\mathbf{0} < \|\mathbf{f}\|_{-\mathbf{1},\mathbf{p}'} + \|\mathbf{g}\|_{-\mathbf{1},\mathbf{q}'} < \min(\varepsilon_\mathbf{1}, \varepsilon_\mathbf{2}, \mathbf{1}).$$

Then for any $\lambda > 0$ there exists a pair $(u^*, v^*) \in \Lambda$ such that (u^*, v^*) is a solution of system (1.1) satisfies the property $J(u^*, v^*) < 0$.

Definition 2 *We say that the functional J satisfies the Palais-Smale condition at level $c \in \boldsymbol{R}$ (in short form $(PS)_c$) if every sequence $\{(u_m, v_m)\} \subset X$ such that $J(u_m, v_m) \to c$ and $J'(u_m, v_m) \to 0$ in X^* as $m \to \infty$ is relatively compact in X.*

Lemma 2 *Suppose $\alpha+\beta+2 > \max(p,q)$. Then, there exists a sequence $(u_m, v_m) \in \Lambda$ such that $\lim_{m\to\infty} J(u_m, v_m) = \inf_{(u,v)\in\Lambda} J(u,v)$ and*

$$\|J'_{|\Lambda}(u_m, v_m)\|_{X^*} \le \frac{1}{m}.$$

Proof. We show that J is bounded below on Λ. Let (u,v) be an arbitrary element in Λ. We have

$$J_{|\Lambda}(u,v) = a(p)P(u) + a(q)Q(v) - a(1)\langle f,u\rangle - a(1)\langle g,v\rangle.$$

Using successively the Holder's inequality and the Young's inequality on the terms $\langle f,u\rangle$ and $\langle g,v\rangle$, we can write

$$J_{|\Lambda} \ge [a(p)\|\nabla u\|_p^p - \theta^p\|\nabla u\|_p^p] + [a(q)\|\nabla u\|_q^q - \gamma^q\|\nabla u\|_q^q] - \theta^{-p'}[a(1)\|f\|_{-1,p'}]^{p'}$$
$$- \gamma^{-q'}[a(1)\|g\|_{-1,q'}]^{q'}.$$

Since the real numbers θ and γ being arbitrary, a suitable choose of θ and γ assure that J is bounded below on Λ. The Ekeland Variational principle ensures the existence of such sequence. ∎

We shall show that each minimizing sequence contains a Palais-Smale sequence when f, g satisfied in condition $(\mathbf{c})$.
For all $(u,v) \in X$ we consider

$$I(u,v) = \langle J'(u,v), (u,v)\rangle$$
$$= P(u) + Q(v) - \lambda(\alpha+\beta+2)R(u,v) - \langle f,u\rangle - \langle g,v\rangle.$$

We want to establish that $J'(u_m, v_m) \to 0$ in X^* as $m \to \infty$. It suffice to show that

Lemma 3 (see [11], Lemma 4.6, Proposition 5.1) *Under condition* $(\mathbf{c})$, *we have*
$(\mathbf{i})$ $\langle I'(u,v),(u,v)\rangle \neq 0$ *for all* $(u,v)\in\Lambda$.
$(\mathbf{ii})$ *There exists* δ *such that* $|\langle I'(u_m,v_m),(u_m,v_m)\rangle| > \delta > 0,\ \forall n \geq n_0$ *for some* $n_0 \in \mathbb{N}$.

Lemma 4 (see [5]) *The critical value of* J *on* Λ, $m_1 = \inf_{(u,v)\in\Lambda} J(u,v)$, *has the following property*

$$m_1 < min[-\frac{\alpha+1}{p'}\|f\|^{p'}_{-1,p'}, -\frac{\beta+1}{q'}\|g\|^{q'}_{-1,q'}].$$

Lemma 5 *Let* $c \in \boldsymbol{R}$. *Then each* $(PS)_c-$*sequence for* J *is bounded.*

Proof. Let $\{(u_m,v_m)\}$ be such sequence, that is

$$J(u_m,v_m) = c + o_m(1), \qquad \text{and} \qquad J'(u_m,v_m) = o_m\Big(\|(u_m,v_m)\|_X\Big).$$

We can write

$$\begin{aligned} &J(u_m,v_m) - \frac{1}{\alpha+\beta+2}\langle J'(u_m,v_m),(u_m,v_m)\rangle \\ &\quad = a(p)\|\nabla u_m\|_p^p + a(q)\|\nabla v_m\|_q^q - a(1)\langle f,u_m\rangle - a(1)\langle g,v_m\rangle. \end{aligned}$$

Using successively the Holder's inequality and the Young's inequality on the terms $\langle f,u_m\rangle$ and $\langle g,v_m\rangle$, we have

$$\begin{aligned} &[a(p)\|\nabla u_m\|_p^p - \theta^p\|\nabla u_m\|_p^p] + [a(q)\|\nabla v_m\|_q^q - \gamma^q\|\nabla v_m\|_q^q] \\ &\quad \leq \theta^{-p'}[a(1)\|f\|_{-1,p'}]^{p'} + \gamma^{-q'}[a(1)\|g\|_{-1,q'}]^{q'} + c + o_m\|(u_m,v_m)\| + o_m(1) \end{aligned}$$

This inequality follows from $m(x)$ and $l(x)$ are sign chaining functions and we can choose $(u,v)\in X$ with these properties that $supp\ u \subset \Omega_1 = \{x\in\Omega;\ m(x)>0\}$ and $supp\ v \subset \Omega_2 = \{x\in\Omega;\ l(x)>0\}$. By choosing $\theta = \{a(p)\}^{\frac{1}{p}}$ and $\gamma = \{a(q)\}^{\frac{1}{q}}$ we get the boundedness of the sequence $\{(u_m,v_m)\}$. ■
At this stage we can assume, up to a subsequence, that

$$\begin{aligned} &u_m \rightharpoonup u \quad \text{in} \quad W_0^{1,p}(\Omega), \\ &v_m \rightharpoonup v \quad \text{in} \quad W_0^{1,q}(\Omega), \\ &u_m \to u \quad \text{a.e. in} \quad \Omega, \qquad (2.1) \\ &v_m \to v \quad \text{a.e. in} \quad \Omega. \qquad (2.2) \end{aligned}$$

Let

$$J_0(u,v) = \frac{1}{p}P(u) + \frac{1}{q}Q(v) - \lambda R(u,v),$$

and

$$\Lambda_0 = \{(u,v)\in X\backslash\{(0,0)\};\ D_1J_0(u,v) = D_2J_0(u,v) = 0\},$$

where D_1J_0 (resp. D_2J_0) denotes the Gateaux derivative of J_0 with respect to its first (resp. second) variable. Set

$$m_0 = \inf_{(u,v)\in\Lambda_0} J_0(u,v).$$

Theorem 6 *The functional* J *satisfies* $(PS)_c$ *with*

$$c \in (-\infty, m_0 + m_1). \qquad (2.3)$$

Proof. By standard argument, we can show that the pair (u,v) in (2.1) and (2.2) is a critical point of J. Now, we set
$X_m = u_m - u$
and

$Y_m = v_m - v$.
From Brezis-Lieb's lemma [6], we have

$$P(X_m) = P(u_m) - P(u) + o_m(1),$$
$$Q(Y_m) = Q(v_m) - Q(v) + o_m(1),$$
$$R(X_m, Y_m) = R(u_m, v_m) - R(u, v) + o_m(1).$$

It follows that

$$P(X_m) - R(X_m, Y_m) = o_m(1),$$
$$Q(Y_m) - R(X_m, Y_m) = o_m(1),$$
$$J_0(X_m, Y_m) = c - J(u, v) + o_m(1). \tag{2.4}$$

Let $P(X_m),\ Q(Y_m)$ and $R(X_m, Y_m)$ have the same limit l. We will show that $l = 0$. Assume for the sake of contradiction, $l \neq 0$. Let $(s_0(u_m, v_m), t_0(u_m, v_m)) \in \mathbf{R}^2$ satisfy the following system

$$\begin{cases} \frac{\partial}{\partial s_0} J_0(s_0 X_m, t_0 Y_m) = 0 \\ \frac{\partial}{\partial t_0} J_0(s_0 X_m, t_0 Y_m) = 0. \end{cases}$$

Let $r = \frac{q(\alpha+1)}{q-(\beta+1)}$, we get $p < r$. An easy computation shows that
$s_0(u_m, v_m) = \left[\frac{P(X_m) Q(Y_m)^{\frac{r(\beta+1)}{q(\alpha+1)}}}{(\alpha+1)(\lambda R(X_m, Y_m))^{\frac{r}{\alpha+1}}}\right]^{\frac{1}{r-p}}$
and
$t_0(u_m, v_m) = s_0^{\frac{r}{q}}(u_m, v_m)\left[\frac{\lambda(\beta+1) R(X_m, Y_m)}{Q(Y_m)}\right]^{\frac{r}{q(\alpha+1)}}$.
It is clear that for suitable choose of λ when $\alpha,\ \beta$ are sufficiently small, we have
$\lim_{m\to\infty} s_0(u_m, v_m) = 1 = \lim_{m\to\infty} t_0(u_m, v_m)$
and the pair $(s_0 X_m, t_0 Y_m) \in \Lambda_0$ which together with (2.4), implies that

$$c - J(u, v) = \lim_{m\to\infty} J_0(X_m, Y_m) = \lim_{m\to\infty} J_0(s_0 X_m, t_0 Y_m) \geq m_0,$$

and consequently

$$c \geq m_0 + m_1.$$

This leads to contradiction with (2.3) . ■

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