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The Property of Operators in a Non-Wandering C_0-Semigroup

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Abstract: In this paper, we studied the property of operators in a non-wandering semigroup on Banach s-pace. We obtain the necessary and sufficient conditions that discrete semigroup is non-wandering. By a new method, a sufficient conditions for strongly continuous semigroup being non-wandering is given.

Keywords: non-wandering operator; semigroup; discrete semigroup; continuous semigroup; strongly continuous semigroup

1 Introduction

In the infinite-dimensional linear space, chaotic operators, supercyclic operators and Hypercyclic operators have been intensively studied recently. In 1929, Birkhoff firstly introduced the concept of hypercyclic operators [2]. From then on, mathmatics from all over the world have great interests in this field. For more details about hypercyclic operators see the survey [3,4,5].

In 1996, Lixin Tian and Diancheng Lu introduced the concept of non-wandering operators [6]. They are new linear chaotic operators and are relative to hypercyclic operators, but different from them. For more details about non-wandering operators see the survey[8,11,12].

The investigation of hypercyclic semigroups was initiated by Desh, Schappacher and Webb in [7]. In 2007, Jose A.C onjero, V.Muller, A.Peris studied the hypercyclic behavior of operators in a hypercyclic strong continuous semigroups [1].

In 2002, Xun Liu introduced non-wandering semigroups [9-10]. In his paper , he mainly introduced continuous semigroup. We can extend this defenition to discrete semigroup. It is to easy to see that if T is a non-wandering operators,then the discrete semigroup $(T^n)_{n=1}^{\infty}$ is non-wandering. And the converse situation is affirmative(if the discrete semigroup $(T^n)_{n=1}^{\infty}$ is a non-wandering operators,then T is non-wandering). In continuous semigroup ,even more strongly continuous semigroup, the relative result is not so easy. In our paper, we introduce that if $\{\mathrm{T}_t\}_{t\geq 0}$ is a strongly continuous semigroup (or C_0-semigroup) of operators in $L\left(X\right)$ and some operator T_{t_0} $(t_0 > 0)$ in the semigroup is nonwandering, then the semigroup $\{\mathrm{T}_t\}_{t\geq 0}$ itself is nonwandering.

2 Basic notations and definitions

Given an infinite-dimensional separable real or complex Banach space $(X, \|\bullet\|)$, we denote by $L\left(X\right)$ the set of all bounded linear operators in Xequipped with the operator norm $\|T\| = \sup\limits_{\|x\|\leq 1} \|Tx\|$.

We will refer to N, Z, Q, $R\left(R_{+}\right)$, C and Kas the sets of positive integers, integers, rational numbers, and the real (positive real), complex scalar fields and real number field or complex number field , respectively.

Definition 1 *Suppose $T \in L\left(X\right)$ and $x \in X$,for any neighborhood $U\left(x\right)$, if there exists $k \in N$ such that $T^k\left(U\left(x\right)\right) \cap U\left(x\right) \neq \phi$, then x is called nonwandering point of T.*

Definition 2 *Suppose $E \subset X$ is a closed linear subspace of X ,and $E_1 \subset E$, $E_2 \subset E$are also closed linear subspaces in X.For arbitrary $x \in E$,if there is a unique decomposition such that $x = x_1 + x_2$,$x_1 \in E_1$,$x_2 \in E_2$,$E_1 \cap E_2 = \{0\}$,then E is called the direct sum of E_1 and E_2,and written as $E = E_1 \oplus E_2$,where $\oplus$ represents direct sum.*

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Definition 3 *Suppose $T \in L(X)$.*

(1)Assume that there exists a closed subspace $E \subset X$,which has hyperbolic structure:$E = E_u \oplus E_s$,$TE_s = E_s$,$TE_u = E_u$,where E_u,E_s are closed subspaces. In addition,there exist constants $\tau\,(0 < \tau < 1)$ and $c > 0$,such that for any $x_u \in E_u$, $k \in N$, $\left\|T^k x_u\right\| \geq c\tau^{-k}\left\|x_u\right\|$,and for any $x_s \in E_s$,$k \in N$,$\left\|T^k x_s\right\| \leq c\tau^k\left\|x_s\right\|$;

(2)Assume also that Per(T) is dense in E.(i.e. $\overline{per\,(T)} = E$);

Then T is said to be a nonwandering operator relative to E.

From definition 1 and definition 3 , we can obtain the following proposition.

Proposition 1 *If T is a nonwandering operator relative to E, then for any $x \in E$ is a nonwandering point of T.*

Proof. Since T is a nonwandering , for any $x \in E = \overline{per\,(T)}$, there exist $\{x_n\} \subset per\,(T)$ such that $\lim\limits_{n\to\infty} x_n = x$,and for all n, there exist $k_n \in N$ such that $T^{k_n}x_n = x_n$.

For any $\varepsilon > 0$,there exist $n_0 \in N$ such that $x_{n_0} \in U(x,\varepsilon)$.

We have

$$x_{n_0} \in T^{k_{n_0}}\left(U(x,\varepsilon)\right) \cap U(x,\varepsilon).$$

i.e.

$$T^{k_{n_0}}\left(U(x,\varepsilon)\right) \cap U(x,\varepsilon) \neq \phi$$

So x is a non-wandering point of T. ■

Definition 4 *A non-empty subset $A \subseteq L(X)$ is called a semigroup if*

$$T, S \in A \Rightarrow T \circ S \in A \quad (\forall T, S \in A).$$

A semigroup A is called abelian if

$$T \circ S = S \circ T \quad (\forall T, S \in A).$$

Remark 2 *we shall study mainly operator semigroups indexed by non-negative integers or non-negative reals (we call them one-parameter semigroups).Obviously, any one-parameter semigroup is abelian.*

Definition 5 *If a semigroup is indexed by R_+, we call it is continuous. Let us use the notation $\{T_t\}_{t\geq 0}$; If a semigroup is indexed by N, we call it is discrete . Let us use the notation $\{T^n\}_{n=1}^{\infty}$, generated by a single operator T.*

In the following, we introduced detailly defenition of C_0-semigroup:

Definition 6 *In the continuous case ,a one-parameter family $\{T_t\}_{t\geq 0}$ of continuous linear operators in $L(X)$ is a strongly continuous semigroup (or C_0-semigroup) of operators in $L(X)$ if $T_0 = I$,$T_tT_s = T_{t+s}$ for all $t, s \geq 0$,and $\lim\limits_{t\to s} T_t x = T_s x$ for all $s \geq 0$,$x \in X$.*

Definition 7 *A one-parameter semigroup $\{T^n\}_{n=1}^{\infty}$ in $L(X)$ is nonwandering relative to E if E has hyperbolic structure:$E = E_u \oplus E_s$,$T^nE_s = E_s$,$T^nE_u = E_u$, where E_u,E_s are closed subspaces. In addition,there exist constants $\tau\,(0 < \tau < 1)$ and $c > 0$,such that for any $x_u \in E_u$,,$\|T^n x_u\| \geq c\tau^{-n}\|x_u\|$,and for any $x_s \in E_s$, $\|T^n x_s\| \leq c\tau^n\|x_s\|$for all $n \in N$;and if $\overline{per\,(T^n)} = E$. where $per\,(T^n)$ be the set of all period points,i.e.,if $x \in per\,(T^n)$, there exists $n_0 \in N$ such that $T^{n_0}x = x$.*

Definition 8 *A one-parameter semigroup $\{T_t\}_{t\geq 0}$ in $L(X)$ is nonwandering relative to E if E has hyperbolic structure:$E = E_u \oplus E_s$,$T_tE_s = E_s$,$T_tE_u = E_u$, where E_u,E_s are closed subspaces. In addition,there exist constants $\tau\,(0 < \tau < 1)$ and $c > 0$,such that for any $x_u \in E_u$,,$\|T_t x_u\| \geq c\tau^{-t}\|x_u\|$,and for any $x_s \in E_s$, $\|T_t x_s\| \leq c\tau^t\|x_s\|$for all $t \geq 0$;and if $\overline{per\,(T_t)} = E$, where $per\,(T_t)$ be the set of all period points,i.e.,if $x \in per\,(T_t)$, there exists $t > 0$ such that $T_t x = x$.*

3 Main results

From the defenition of nonwandering discrete semigroup and nonwandering operator,we have the following theorem.

Theorem 3 *The discrete semigroup $\{T^n\}_{n=1}^{\infty}$ is nonwandering relative to E if and only if operator T is nonwandering relative to E.*

Theorem 4 *let $\{T_t\}_{t\geq 0}$ is a strongly continuous semigroup (or C_0-semigroup) of operators in $L(X)$,and $E = E_u \oplus E_s$,$T_tE_s = E_s$,$T_tE_u = E_u$ for all $t \geq 0$,where E_u,E_s are closed subspaces, and some operator T_{t_0} $(t_0 > 0)$ in the semigroup is nonwandering,then the semigroup $\{T_t\}_{t\geq 0}$ itself is nonwandering.*

Proof. Without loss of generality, we may assume that $t_0 = 1$. Indeed, we can consider the semigroup $\{\mathrm{T}_t\}_{t\geq 0}$ in $L(X)$, with $\widetilde{T}_t := T_{tt_0}$ for every $t \geq 0$. Clearly, $T_1 := T_{t_0}$.

Let T_1 be a non-wandering operator relative to E,we have E has hyperbolic structure :

$$E = E_u \oplus E_s, T_1E_s = E_s, T_1E_u = E_u,$$

where E_u,E_s are closed subspaces.

In addition,there exist constants τ $(0 < \tau < 1)$ and $c > 0$,such that for any $x_u \in E_u$,$k \in N$,$\left\|T_1^k x_u\right\| \geq c\tau^{-k}\left\|x_u\right\|$,and for any

$$x_s \in E_s, k \in N, \left\|T_1^k x_s\right\| \leq c\tau^k \left\|x_s\right\|;$$

And per (T_1) is dense in E.

For any $t \geq 0$,we have $t = t' + k$,where $t' \in [0,1]$,$k \in N$.

So we firstly consider $t \in [0,1]$

Since $\lim\limits_{t\to s} T_tx = T_sx$ for all $s \geq 0$,$x \in X$, so we have $\lim\limits_{t\to s}\|T_t\| = \|T_s\|$.

$f(t) = \|T_t\|$is continuous in$[0,+\infty)$. There exists constant $M > 0$ which satisfies $\|T_t\| \leq M$,for any $t \in [0,1]$.

$$\|T_tx_s\| \leq \|T_t\|\|x_s\| \leq M\|x_s\| = \frac{M}{\tau}\tau\|x_s\| \leq c_1\tau^t\|x_s\|,$$

where $c_1 = \frac{M}{\tau}$.

$$\|T_{1-t}\|\|T_tx_u\| \geq \|T_{1-t}T_tx_u\| = \|T_1x_u\| \geq c\tau^{-1}\|x_u\|,$$

$$\|T_tx_u\| \geq \frac{c\tau^{-1}\|x_u\|}{\|T_{1-t}\|} \geq \frac{c}{M}\tau^{-1}\|x_u\| \geq c_2\tau^{-t}\|x_u\|,$$

where $c_2 = \frac{c}{M}$.

Next we consider $t > 1$,then $t = t' + k$,where $t' \in [0,1]$,$k \in N$, for any $x_s \in E_s$

$$\|T_tx_s\| = \|T_{t'+k}x_s\| \leq c\tau^k\|T_{t'}x_s\| \leq c\tau^k c_1\tau^{t'}\|x_s\| = c_3\tau^t\|x_{s\|}$$

$$\|T_tx_u\| = \|T_{t'+k}x_u\| \geq c\tau^{-k}\|T'_tx_u\| \geq c\tau^{-k}c_2\tau^{-t'}\|x_s\| = c_4\tau^{-t}\|x_u\|$$

for any $x_u \in E_u$, where $c_3 = cc_1$,$c_4 = cc_2$.

Let

$$c' = \max\left\{c_1, \frac{1}{c_2}, c_3, \frac{1}{c_4}\right\},$$

we have

$$\|T_tx_u\| \geq c'\tau^{-t}\|x_u\|,$$

for any $x_u \in E_u$ and

$$\|T_tx_s\| \leq c'\tau^t\|x_s\|,$$

for any $x_s \in E_s$, for all $t \geq 0$.

For all $x \in \overline{\text{per}\,(T_1)} = E$, there exist $\{x_n\} \subset per\,(T_1)$ such that $\lim\limits_{n\to\infty} x_n = x$.

For all n, there exist $k_n \in N$ such that

$$T_1^{k_n} x_n = T_{k_n} x_n = x_n.$$

Then for all n ,$x_n \in per\,\{T_t\}_{t\geq 0}$.So $x \in \overline{\text{per}\,\{T_t\}_{t\geq 0}}$. By x is arbitrary,
We have

$$\overline{\text{per}\,\{T_t\}_{t\geq 0}} = E.$$

So $\{\mathrm{T}_t\}_{t\geq 0}$ is nonwandering. ■

Remark 5 *The converse situation is not affirmative. In C_0-semigroup $\{T_t\}_{t\geq 0}$, we can not obtain $\overline{per\,\{T_{t_0}\}} = E$(for any $t_0 > 0$) from $\overline{per\,\{T_t\}_{t\geq 0}} = E$.*

Example 1 *: Let $T_t x = xe^{it}$, $\{T_t\}_{t\geq 0}$ is a C_0-semigroup. $\overline{per\,\{T_t\}_{t\geq 0}} = E$,but $per\,(T_1) = \phi$.*

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