The Property of Operators in a Non-Wandering $C_0$-Semigroup

Jianmei Zhang *,
Department of Mathematics, Jiangsu University Nonlinear Scientific Research Center, Jiangsu University Zhenjiang, Jiangsu, 212013, P.R.China
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Abstract: In this paper, we studied the property of operators in a non-wandering semigroup on Banach space. We obtain the necessary and sufficient conditions that discrete semigroup is non-wandering. By a new method, a sufficient conditions for strongly continuous semigroup being non-wandering is given.

Keywords: non-wandering operator; semigroup; discrete semigroup; continuous semigroup; strongly continuous semigroup

1 Introduction

In the infinite-dimensional linear space, chaotic operators, supercyclic operators and Hypercyclic operators have been intensively studied recently. In 1929, Birkhoff firstly introduced the concept of hypercyclic operators [2]. From then on, mathematicians from all over the world have great interests in this field. For more details about hypercyclic operators see the survey [3,4,5].

In 1996, Lixin Tian and Diancheng Lu introduced the concept of non-wandering operators [6]. They are new linear chaotic operators and are related to hypercyclic operators, but different from them. For more details about non-wandering operators see the survey [8,11,12].

The investigation of hypercyclic semigroups was initiated by Desh, Schappacher and Webb in [7]. In 2007, Jose A.C. Conero, V. Muller, A. Peris studied the hypercyclic behavior of operators in a hypercyclic strong continuous semigroups [1].

In 2002, Xun Liu introduced non-wandering semigroups [9-10]. In his paper, he mainly introduced continuous semigroup. We can extend this definition to discrete semigroup. It is easy to see that if $T$ is a non-wandering operators, then the discrete semigroup $(T^n)_{n=1}^\infty$ is non-wandering. And the inverse situation is affirmative (if the discrete semigroup $(T^n)_{n=1}^\infty$ is a non-wandering operators, then $T$ is non-wandering). In continuous semigroup, even more strongly continuous semigroup, the relative result is not so easy. In our paper, we introduce that if $\{T_t\}_{t \geq 0}$ is a strongly continuous semigroup (or $C_0$-semigroup) of operators in $L^1(X)$ and some operator $T_{t_0}$ ($t_0 > 0$) in the semigroup is nonwandering, then the semigroup $\{T_t\}_{t \geq 0}$ itself is nonwandering.

2 Basic notations and definitions

Given an infinite-dimensional separable real or complex Banach space $(X, \| \cdot \|)$, we denote by $L^1(X)$ the set of all bounded linear operators in $X$ equipped with the operator norm $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$.

We will refer to $N$, $Z$, $Q$, $R$ ($R_+$), $C$ and $K$ as the sets of positive integers, integers, rational numbers, and the real (positive real), complex scalar fields and real number field or complex number field, respectively.

Definition 1 Suppose $T \in L^1(X)$ and $x \in X$, for any neighborhood $U(x)$, if there exists $k \in N$ such that $T^k(U(x)) \cap U(x) \neq \emptyset$, then $x$ is called nonwandering point of $T$.

Definition 2 Suppose $E \subset X$ is a closed linear subspace of $X$ and $E_1 \subset E$, $E_2 \subset E$ are also closed linear subspaces in $X$. For arbitrary $x \in E$, if there is a unique decomposition such that $x = x_1 + x_2$, $x_1 \in E_1$, $x_2 \in E_2$, $E_1 \cap E_2 = \{0\}$, then $E$ is called the direct sum of $E_1$ and $E_2$, and written as $E = E_1 \oplus E_2$, where $\oplus$ represents direct sum.

* Corresponding author. E-mail address: leo_zsc@ujs.edu.cn
Definition 3 Suppose \( T \in L(X) \).

(1) Assume that there exists a closed subspace \( E \subset X \), which has hyperbolic structure: \( E = E_u \oplus E_s, T E_s = E_s, T E_u = E_u \), where \( E_u, E_s \) are closed subspaces. In addition, there exist constants \( \tau (0 < \tau < 1) \) and \( c > 0 \), such that for any \( x_u \in E_u, k \in \mathbb{N}, \| T^k x_u \| \geq c \tau^{-k} \| x_u \| \), and for any \( x_s \in E_s, k \in \mathbb{N}, \| T^k x_s \| \leq c \tau^k \| x_s \| \);

(2) Assume also that \( \text{Per}(T) \) is dense in \( E \), i.e. \( \text{per}(T) = E \).

Then \( T \) is said to be a nonwandering operator relative to \( E \).

From definition 1 and definition 3, we can obtain the following proposition.

Proposition 1 If \( T \) is a nonwandering operator relative to \( E \), then for any \( x \in E \) is a nonwandering point of \( T \).

Proof. Since \( T \) is a nonwandering, for any \( x \in E = \text{per}(T) \), there exist \( \{ x_n \} \subset \text{per}(T) \) such that \( \lim_{n \to \infty} x_n = x \), and for all \( n \), there exist \( k_n \in \mathbb{N} \) such that \( T^{k_n} x_n = x_n \).

For any \( \varepsilon > 0 \), there exist \( n_0 \in \mathbb{N} \) such that \( x_{n_0} \in U(x, \varepsilon) \).

We have

\[ x_{n_0} \in T^{k_n} (U(x, \varepsilon)) \cap U(x, \varepsilon). \]

i.e.

\[ T^{k_n} (U(x, \varepsilon)) \cap U(x, \varepsilon) \neq \emptyset. \]

So \( x \) is a non-wandering point of \( T \).

Definition 4 A non-empty subset \( A \subset L(X) \) is called a semigroup if

\[ T, S \in A \Rightarrow T \circ S \in A \quad (\forall T, S \in A). \]

A semigroup \( A \) is called abelian if

\[ T \circ S = S \circ T \quad (\forall T, S \in A). \]

Remark 2 We shall study mainly operator semigroups indexed by non-negative integers or non-negative reals (we call them one-parameter semigroups). Obviously, any one-parameter semigroup is abelian.

Definition 5 If a semigroup is indexed by \( R^+ \), we call it continuous. Let us use the notation \( \{ T_t \}_{t \geq 0} \). If a semigroup is indexed by \( \mathbb{N} \), we call it discrete. Let us use the notation \( \{ T_n \}_{n=1}^{\infty} \), generated by a single operator \( T \).

In the following, we introduced detailly definition of \( C_0 \)-semigroup:

Definition 6 In the continuous case, a one-parameter family \( \{ T_t \}_{t \geq 0} \) of continuous linear operators in \( L(X) \) is a strongly continuous semigroup (or \( C_0 \)-semigroup) of operators in \( L(X) \) if \( T_0 = I, T_t T_s = T_{t+s} \) for all \( t, s \geq 0 \), and \( \lim_{t \to s} T_t x = T_s x \) for all \( s \geq 0, x \in X \).

Definition 7 A one-parameter semigroup \( \{ T_n \}_{n=1}^{\infty} \) in \( L(X) \) is nonwandering relative to \( E \) if \( E \) has hyperbolic structure: \( E = E_u \oplus E_s, T T^n E_s = E_s, T^n T E_u = E_u \), where \( E_u, E_s \) are closed subspaces. In addition, there exist constants \( \tau (0 < \tau < 1) \) and \( c > 0 \), such that for any \( x_u \in E_u, \| T^n x_u \| \geq c \tau^{-n} \| x_u \| \), and for any \( x_s \in E_s, \| T^n x_s \| \leq c \tau^n \| x_s \| \) for all \( n \in \mathbb{N} \); and if \( \text{per}(T^n) = E \), where \( \text{per}(T^n) \) be the set of all period points, i.e., if \( x \in \text{per}(T^n) \), there exists \( n_0 \in \mathbb{N} \) such that \( T^{n_0} x = x \).

Definition 8 A one-parameter semigroup \( \{ T_t \}_{t \geq 0} \) in \( L(X) \) is nonwandering relative to \( E \) if \( E \) has hyperbolic structure: \( E = E_u \oplus E_s, T T_e = E_s, T E_u = E_u \), where \( E_u, E_s \) are closed subspaces. In addition, there exist constants \( \tau (0 < \tau < 1) \) and \( c > 0 \), such that for any \( x_u \in E_u, \| T_t x_u \| \geq c \tau^{-t} \| x_u \| \), and for any \( x_s \in E_s, \| T_t x_s \| \leq c \tau^t \| x_s \| \) for all \( t \geq 0 \); and if \( \text{per}(T_t) = E \), where \( \text{per}(T_t) \) be the set of all period points, i.e., if \( x \in \text{per}(T_t) \), there exists \( t > 0 \) such that \( T_t x = x \).

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3 Main results

From the definition of nonwandering discrete semigroup and nonwandering operator, we have the following theorem.

**Theorem 3** The discrete semigroup \( \{T^n\}_{n=1}^{\infty} \) is nonwandering relative to \( E \) if and only if operator \( T \) is nonwandering relative to \( E \).

**Theorem 4** Let \( \{T_t\}_{t \geq 0} \) be a strongly continuous semigroup (or \( C_0 \)-semigroup) of operators in \( L(X)_1 \) and \( E = E_u \oplus E_s, T_sE_s \oplus E_sT_sE_s = E_s, T_sE_u = E_u \) for all \( t \geq 0 \), where \( E_u, E_s \) are closed subspaces. Let operator \( T_t \) for any \( t > 0 \) in the semigroup is nonwandering, then the semigroup \( \{T_t\}_{t \geq 0} \) itself is nonwandering.

**Proof.** Without loss of generality, we may assume that \( t_0 = 1 \). Indeed, we can consider the semigroup \( \{T_t\}_{t \geq 0} \) in \( L(X)_1 \) with \( \bar{T}_t := T_{t_0} \) for every \( t \geq 0 \). Clearly, \( \bar{T}_1 := T_0 \).

Let \( \bar{T}_t \) be a non-wandering operator relative to \( E \); we have \( E \) has hyperbolic structure: \( E = E_u \oplus E_s, T_1 E_s = E_s, T_1 E_u = E_u \), where \( E_u, E_s \) are closed subspaces.

In addition, there exist constants \( \tau (0 < \tau < 1) \) and \( c > 0 \), such that for any \( x_u \in E_u, k \in N, \|T_t^k x_u\| \geq c \tau^{-k} \|x_u\| \), and for any \( x_s \in E_s, k \in N, \|T_t^k x_s\| \leq c \tau^k \|x_s\| \).

And \( \{T_t\} \) is dense in \( E \).

For any \( t \geq 0 \), we have \( t = t' + k \), where \( t' \in [0, 1], k \in N \).

So we firstly consider \( t \in [0, 1] \).

Since \( \lim_{t \to s} T_t x = T_s x \) for all \( s \geq 0, x \in X \), we have \( \lim_{t \to s} \|T_t\| = \|T_s\| \).

\( f(t) = \|T_t\| \) is continuous in \([0, +\infty)\). There exists constant \( M > 0 \) which satisfies \( \|T_t\| \leq M \), for any \( t \in [0, 1] \).

\[ \|T_t x_u\| \leq \|T_t\| \|x_u\| \leq M \|x_u\| = M \frac{\tau}{\tau} \|x_u\| \leq c_1 \tau^t \|x_u\| , \]

where \( c_1 = \frac{M}{\tau} \).

\[ \|T_{t+1}^t \| T_t x_u \| \geq \|T_{t+1}^t T_t x_u\| = \|T_t x_u\| \geq c \tau^{-1} \|x_u\| , \]

\[ \|T_t x_u\| \geq c \tau^{-1} \|x_u\| \leq \|T_{t+1}^t\| \|x_u\| \geq c_2 \tau^{-t} \|x_u\| , \]

where \( c_2 = \frac{c}{\tau^t} \).

Next we consider \( t > 1 \), then \( t = t' + k \), where \( t' \in [0, 1], k \in N \), for any \( x_s \in E_s \)

\[ \|T_t x_s\| = \|T_{t+1}^{t'} x_s\| \leq c \tau^{-k} \|T_{t'} x_s\| \leq c \tau^{-k} c_1 \tau^t \|x_s\| = c_3 \tau^t \|x_s\| \]

\[ \|T_t x_u\| = \|T_{t+1}^{t'} x_u\| \geq c \tau^{-k} \|T_{t'} x_u\| \geq c \tau^{-k} c_2 \tau^{-t'} \|x_u\| = c_4 \tau^{-t} \|x_u\| \]

for any \( x_u \in E_u \), where \( c_3 = c c_1, c_4 = c c_2 \).

Let \( \epsilon' = \max \left\{ c_1, \frac{1}{c_2}, c_3, \frac{1}{c_4} \right\} \),

we have \( \|T_t x_u\| \geq \epsilon' \tau^{-t} \|x_u\| , \)

for any \( x_u \in E_u \) and \( \|T_t x_s\| \leq \epsilon' \tau^t \|x_s\| , \)

for any \( x_s \in E_s \), for all \( t \geq 0 \).

For all \( x \in per(T_1) = E \), there exist \( \{x_n\} \subset per(T_1) \) such that \( \lim_{n \to \infty} x_n = x \).
For all \( n \), there exist \( k_n \in \mathbb{N} \) such that
\[
T_{k_n}^{k_n} x_n = T_{k_n} x_n = x_n.
\]
Then for all \( n, x_n \in \text{per} \{ T_t \}_{t \geq 0} \). So \( x \in \overline{\text{per} \{ T_t \}_{t \geq 0}} \). By \( x \) is arbitrary, we have
\[
\overline{\text{per} \{ T_t \}_{t \geq 0}} = E.
\]
So \( \{ T_t \}_{t \geq 0} \) is nonwandering. ■

**Remark 5** The converse situation is not affirmative. In \( C_0 \)-semigroup \( \{ T_t \}_{t \geq 0} \), we can not obtain \( \overline{\text{per} \{ T_{t_0} \}} = E \) (for any \( t_0 > 0 \)) from \( \overline{\text{per} \{ T_t \}_{t \geq 0}} = E \).

**Example 1** : Let \( T_t x = xe^{it} \), \( \{ T_t \}_{t \geq 0} \) is a \( C_0 \)-semigroup. \( \overline{\text{per} \{ T_t \}_{t \geq 0}} = E \), but \( \text{per} \{ T_t \} = \phi \).

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**References**


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