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Dynamics Analysis and Synchronization of the Nonlinear Schrödinger Equation with External Perturbation

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Abstract: In this paper, the Melnikov method is applied to the nonlinear Schrödinger equation with external perturbation. Melnikov theoretical analysis obtain a new simple chaotic system. Based on the stability theorems of Lyapunov, backstepping design is proposed for synchronization of the new chaotic system. At the end, the effectiveness of the proposed synchronization scheme is verified by numerical simulation.

Keywords: Fiber-optic signal; Nonlinear Schrödinger equation; four-order Runge-Kutte method;synchronization

1 Introduction

In the last few years, since the pioneering work of [1], synchronization in chaotic dynamical systems has received a great deal of interest among scientists from various fields [2– 9].The phenomenon of synchronization observed in computer simulations of various systems was and is widely discussed in the physical literature. The concept of synchronization goes back to the 17th century. In 1673, Huygens [10] described the synchronization of two pendulum clocks with a weak interaction. In fact, Huygens discussed the synchronization of two modal shapes of vibration. The phenomenon of synchronization of two chaotic systems is fundamental in science and has a wealth of applications in technology. In recent years, more and more applications of chaos synchronization were proposed. There are many control techniques to synchronize chaotic systems, such as linear error feedback control, adaptive control, active control.

The nonlinear Schrödinger equation

$$iu_t + u_{xx} + u\,|u|^2 = 0 \tag{1}$$

is widely used in many areas of physics, such as the evolution of nearly monochromatic, high intensity laser beam propagation and one-dimensional waves in deep wave.It also describes the evolution of the slowly varying envelope of an optical plus. The NLS equation plays an important role in understanding optic fibers which is of importance to the fiber-based telecommunications.

In this paper, a new chaotic system is discussed. It is the nonlinear Schrödinger equation with external perturbation. In some certain conditions, the process of fiber-optic signal transmission with externalSchrödinger equation with external perturbation can be depicted as the following equation

$$iu_t + u_{xx} + u\,|u|^2 = d\cos(\omega x)\exp(ict), \tag{2}$$

where d and ω are the amplitude and frequency of a certain perturbation respectively. c is the coefficient of liner term.

The organization of the paper is as follows. In Section 2, we studied the chaotic behavior of system (2) and fixed points and phase portraits for unperturbed system. In Section 3, the conditions of existence of chaos under periodic perturbation resulting from the homoclinic bifurcations are performed.For chaotic synchronization of the system, a backstepping control scheme has beenproposed based on the Lyapunov stability theory are given in Section 4. Last section is the conclusion.

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2 Analysis of perturbed system

Let $u = \phi(x)e^{ict}$ and substituting it into equation (2), we get

$$\phi'' + \phi^3 + c\phi = d\cos(\omega x) \quad . \tag{3}$$

We make the transformation $\phi \to x_1, \phi' \to x_2$ then equation (3) can be transformed into first-order non-autonomous equations

$$x_1' = -x_2, \quad x_2' = -x_1^3 + cx_1 + d\cos(\omega t). \tag{4}$$

Let $c = 1$, $x_3' = 3x_1^2x_2 - x_2$, If the sides do integral can get the following equation

$$\int x_3'\mathrm{d}t = \int (3x_1^2x_2 - x_2)\mathrm{d}t \tag{5}$$

the result of the integral equation

$$x_3 = x_1^2 - x_1 + n \tag{6}$$

where n is undetermined constants, it value can be determined by the system initial state values and 0. Through the definition of the variable, the equation (4) can be transformed, suppose the drive system is defined as follows

$$x_1' = -x_2, \quad x_2' = -x_3 + d\cos(\omega t), \quad x_3' = 3x_1^2x_2 - x_2. \tag{7}$$

We have performed computer simulations on the perturbed system(7)(see Fig1).

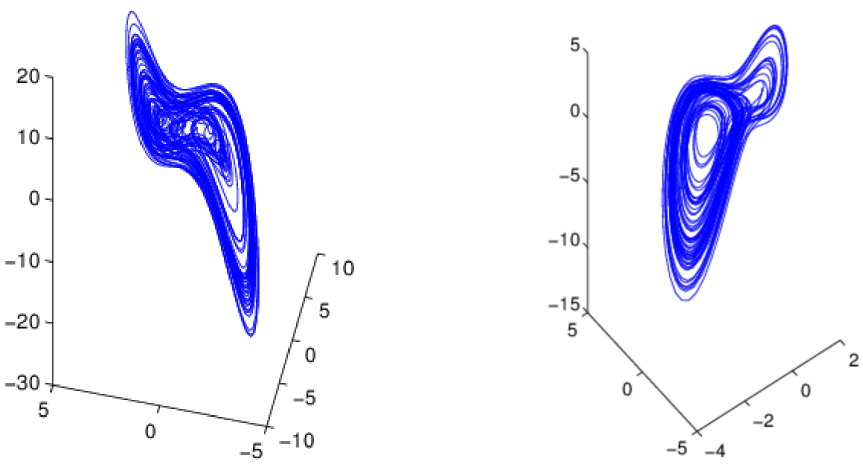

Figure 1: Phase trajectories;(a)chaotic regime $d = 1, \omega = 0.5$;(b)chaotic regime $d = 3, \omega = 1$

If $d = 0$, Eq. (4) is considered as an unperturbed system and can be written as

$$x_1' = -x_2, \quad x_2' = -x_1^3 + cx_1 \tag{8}$$

The system (8) is a Hamiltonian system with Hamiltonian function

$$H(x_1, x_2) = \frac{1}{2}x_2^2 - \frac{1}{2}cx_1^2 + \frac{1}{4}x_1^4 \tag{9}$$

And the function

$$V(x_1) = -\frac{1}{2}cx_1^2 + \frac{1}{4}x_1^4 \tag{10}$$

is called the potential function. By the analysis of the fixed points (x_1, x_2) and their stabilities for system (8), we can easily obtain the following results.

Lemma 1 *When $c > 0$, the unperturbed system has three equilibrium points: two centers $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$ and one saddle (0, 0). The saddle is connected to itself by two symmetric homoclinic orbits as shown in Fig.2(b).*

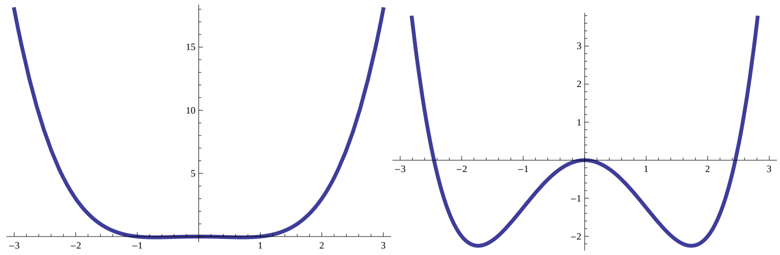

Figure 2: (a) and (b) The two-well potential function of the system

3 Melnikov theoretic analysis

In this section, we discuss the chaotic behaviors of Eq. (4). Melnikov theory has proved to be a simple, elegant, and successful alternative to characterizing the complex dynamics of multi-stable oscillators.

We discuss the chaotic behaviors of Eq. (4) in which d is assumed to be small parameters. A transformation of $d \to \epsilon d$ is done in order to apply the ϵ first-order perturbation scheme of the Melnikov theory. Hence, the system of Eq. (4) may be written as

$$x_1' = -x_2, \quad x_2' = -x_1^3 + cx_1 + d\cos(\psi), \quad \psi' = \omega. \tag{11}$$

The Melnikov method derives a function to describe the first order distance between perturbed stable and unperturbed manifolds. Satisfying the conditions for a double-well potential gives rise to a homoclinic orbit in the phase space for $\epsilon = 0$. Solving for the resulting displacement and differentiating to determine velocity, the homoclinic trajectory is given as follows:

$$(x_1, x_2) = (\sqrt{2c}sech(\sqrt{c}t), -c\sqrt{2}sech(\sqrt{c}t)tanh(\sqrt{c}t)) \tag{12}$$

then the Melnikov function for system (7) can be given by:

$$M(t_0) = \int_{-\infty}^{+\infty} x_2(t) d\cos(\omega(t+t_0))\mathrm{d}t, \tag{13}$$

where t_0 is the cross section time of the Poincaré map and t_0 can be interpreted as the initial time of the forcing term.Because it is difficult to give analytical expression of $x_2(t)$, we will compute $x_2(t)$ numerically in section 5.For the homoclinic orbits $\Gamma_{ho}^{\pm}$, the Melnikov function can be simplified as

$$M(t_0) = \int_{-\infty}^{+\infty} x_2(t) d\cos(\omega(t+t_0))\mathrm{d}t = -2d\sin(\omega t_0)\int_{0}^{+\infty} x_2(t) d\sin(\omega t)\mathrm{d}t = -2d\sin(\omega t_0)I_{hom}(\omega), \tag{14}$$

where $I_{hom}(\omega) = \int_0^{+\infty} x_2(t) d\sin(\omega t)\mathrm{d}t$ is functions of the frequency ω.

Using the previous results and Melnikov's theorem [10,11] the following is stated. If $M(t_0) = 0$ and $M(t_0)' \neq 0$ for some t_0 and some set of parameters, then horseshoes exist, and chaos occurs [12, 13]. If $M(t_0)$ has a simple zero and the corresponding critical parameter value is

$$R = \frac{|dI_{hom(\omega)}|}{A}, \tag{15}$$

$A = \int_0^{+\infty} x_2^2(t)\mathrm{d}t$ is a constant once $x_2(t)$ is given, then in the system with fractional order displacement Eq.(12) the deterministic chaos may appear for certain parameter values which satisfy the relation.

Lemma 2 *Using the Melnikov criterion for the appearance of the intersection between the perturbed and unperturbed separatrixes. Therefore, for fixed frequency ω, the system always produces Smale commutation of chaos.*

We will discuss the behaviors of the fiber-optic signal transmission under perturbation by the four-order Runge-Kutte method. The study is carried out by taking $c = 1, \omega = 0.5$ and setting d as the variable. The results are given by maximum Lyapunov exponents and Phase plane diagram.

According to (see Fig. 3) we can observe that the value of Lyapunov exponents is positive (a positive top Lyapunov exponent for a bounded attractor is usually a sign of chaos), so the system easily converts to chaos even if there is small perturbation. It indicates that the system as the optical fiber transmission signal very vulnerable to is outside perturbed, the phenomenon of chaos.

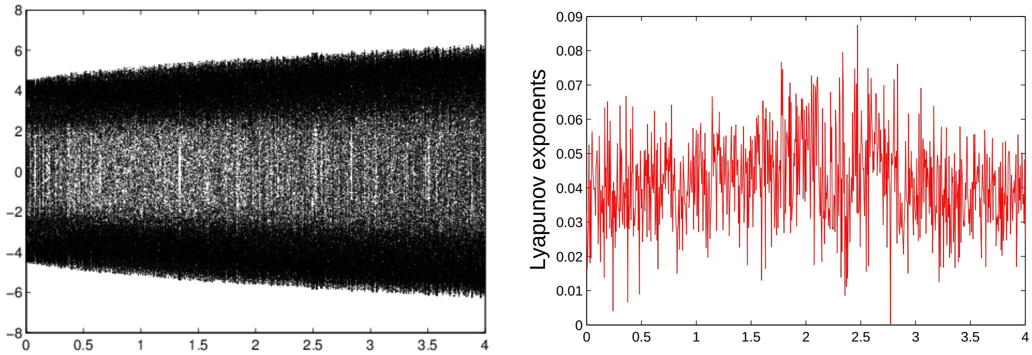

Figure 3: (a) Bifurcation diagram of system (7) in (d, x_2), plane $(0 \leq d \leq 4)$. (b) Maximum Lyapunov exponents corresponding to (a)

4 Synchronization of identical new chaotic systems

In this section, the problem of synchronization between two identical new chaotic systems is formulated and main results are presented.

Suppose the response system takes as follows:

$$y_1' = -y_2, \quad y_2' = -y_3 + d\cos(\omega t), \quad y_3' = 3y_1^2 y_2 - y_2 + u, \tag{16}$$

where y_1, y_2, y_3 are the states of the response system, u is the controller to be designed.

The errors between the drive system (7) and the controlled response system (16) is defined as

$$e_1' = y_1 - x_1, \quad e_2' = y_2 - x_2, \quad e_3' = y_3 - x_3, \tag{17}$$

Then the error dynamics is easily obtained as

$$e_1' = e_2, \quad e_2' = -e_3, \quad e_3' = 3y_1^2 y_2 - 3x_1^2 x_2 - e_2 + u. \tag{18}$$

In this paper, we introduce the backstepping procedure to design the controller $u(t)$.

Theorem 3 *If the controller u is designed as*

$$u = 3e_1 + 6e_2 - e_3 - 3y_1^2 y_2 + 3x_1^2 x_2. \tag{19}$$

Then, the controlled response system (16) is globally synchronous with drive chaotic system (7).

Proof. *First, the positive Lyapunov function is constructed in the following form*

$$V_1 = \frac{1}{2} z_1^2, \tag{20}$$

where $z_1 = e_1$.

The time derivative of V_1 along trajectories of error model (18) given that

$$V_1' = z_1 z_1' = e_1 e_2 = -z_1^2 + z_2(e_1 + e_2). \tag{21}$$

Secondly, we choose the second Lyapunov candidate as

$$V_2 = V_1 + \frac{1}{2} z_2^2 = \frac{1}{2}(z_1 + z_2), \tag{22}$$

where $z_2 = e_1 + e_2$.

Differentiating V_2 with respect to time along the trajectory of synchronous error system (18) yields

$$V_2' = -z_1^2 - z_2^2 + z_2(2e_1 + 2e_2 - e_3). \tag{23}$$

Finally, the Lyapunov function is constructed in the following form

$$V = V_2 + \frac{1}{2}z_3^2 = \frac{1}{2}(z_1 + z_2 + z_3), \tag{24}$$

where $z_3 = 2e_1 + 2e_2 - e_3$ *the time derivative of* V *along the solutions of system (7) and (16) is*

$$V' = -z_1^2 - z_2^2 - z_3^2 + z_3(3e_1 + 6e_2 - e_3 - 3y_1^2 y_2 + 3x_1^2 x_2 - u). \tag{25}$$

Therefore, according to the control input (21), we have

$$V' = -z_1^2 - z_2^2 - z_3^2 = -2V < 0. \tag{26}$$

From which we can get $V \leq V(0)e^{-2t}$*. this implies that* $\lim_{t\to\infty} z_(i) = 0 \ (i = 1, 2, 3)$*, i.e.* $\lim_{t\to\infty} e_(i) = 0 \ (i = 1, 2, 3)$*. Hence, the response system (16) synchronize the drive system (7) by the controller (21). This completes the proof.* ■

We have performed computer simulations,to verify and demonstrate effectiveness and correctness of the above theoretical results, the numerical simulations will be displayed for synchronizing systems (7) and (16). In the numerical simulations, the fourth-order Runge-Kutta method is used to solve the system with time step size 0.001. Synchronization errors of systems (7) and (16) via nonlinear control law (19) is illustrated in Fig. 4.

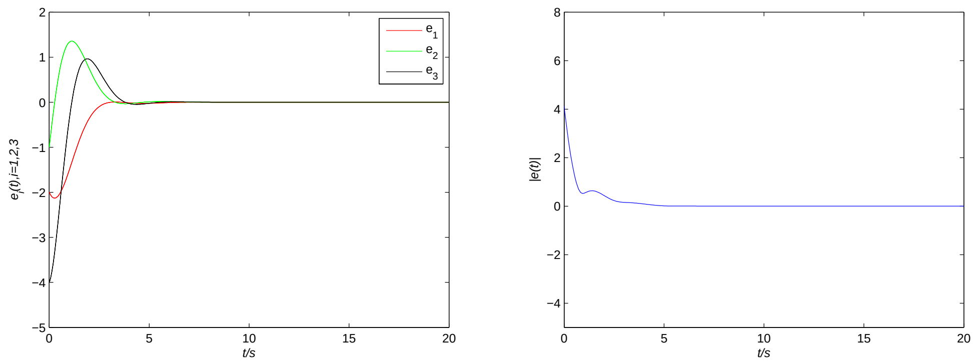

Figure 4: Synchronous errors using control law (19) with time

5 Conclusions

Recently, there has been much more interest in completely integrable the nonlinear Schrödinger equation with external perturbation. Basic dynamical properties of the new attractor are demonstrated in terms of phase portraits, Lyapunov exponents, bifurcation diagram. Based on the stability theorems of Lyapunov, backstepping design is proposed for synchronization of the new chaotic system. The numerical simulations, including bifurcation diagram of fixed points, chaos threshold diagram of system in three-dimensional space, Maximum Lyapunov exponent, phase portraits, are also plotted to illustrate theoretical analysis, and to expose the complex dynamical behaviors. Numerical results show the effectiveness of the theoretical analysis.

Acknowledgments

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