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On Multi-Objective Optimization and Game Theory in Production Management

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Abstract: Service management includes two basic concepts: competition and multi-objective optimization. In this paper, a dynamic model for multi-objective optimization Cournot game where there are two objectives is introduced. The two objectives to be simultaneously optimized in this model are profit and risk. Here, the profit is to be maximized and, at the same time, the risk is to be minimized.

Keywords: Multi-objective optimization; Keeny-Raiffa; compromise methods; Oligopoly

1 Introduction

Service management includes two basic concepts: competition and multi-objective optimization [1, 2]. Competition is typically modeled by game theory [3]. Therefore, studying multi-objective optimization games is an important topic. Here, a dynamic model for multi-objective optimization Cournot game, where there are two objectives, is proposed and discussed. The first to maximize profit and the second is to minimize risk.

In the reminder of the paper, some mathematical preliminaries required throughout the paper are presented in Section 2. In Section 3, a brief discussion on some multi-objective optimization analytical approaches is given. Thereafter, a multi-objective oligopoly model is introduced and studied in Section 4. Finally, conclusion of the study is made.

2 Mathematical Preliminaries

This section introduces some definitions and notations that are fundamental when tacking multi-objective optimization problems (MOOP) [1] and [2] .The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. For any two vectors $x, y \in \mathbb{R}^n$ the following notations are defined:

$$\begin{aligned} x &= y \Leftrightarrow x_i = y_i, x = (x_1, ..., x_i, ..., x_n)^T \\ x &\leq y \Leftrightarrow x_i \leq y_i, i = 1, 2, ..., n \\ x &< y \Leftrightarrow x_i \leq y_i, i = 1, 2, ..., n; x \neq y \end{aligned} \tag{1}$$

Now consider the following constrained multi-objective optimization problem:

$$\begin{aligned} &\textit{minimize} \quad F(x) \\ &\textit{subject to} \quad g_j(x) \leq 0, j = 1, 2, ..., m \end{aligned} \tag{2}$$

where, $F(x) = [f_1(x), f_2(x), ..., f_r(x)]^T$ is a $r - dimensional$ vector of objective functions to be minimized $g_j(x)$ designates the j^{th} constraint function. The main goal here is to simultaneously minimize all given objective functions

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according to to the constraints imposed. Because of the contradiction and possible incommensurability of the objective functions, it is not possible to find a single solution that would be optimal for all the objectives simultaneously. But instead there may be set of many alternative solutions. These solutions have special features that are given below [1, 2].

Definition 1 *A solution x^* is called Pareto-optimal solution to problem (2) if there does not exist x such that $f_q(x) \leq f_k(x^*), k = 1, 2, ..., r$ and $f_q(x) < f_k(x^*)$ for at least one index q.*

Definition 2 *An objective vector $F(x)$ is pareto-optimal front at x if x is Pareto-optimal solution.*

Definition 3 *A solution x is locally Pareto-optimal solution to problem (2) if there does exist $\delta > 0$ such that x is Pareto-optimal in $\mathbb{X} \cap \mathcal{B}(x, \delta)$. where, $\mathcal{B}(x, \delta)$ is the neighborhood of x.*

Definition 4 *An objective vector $F(x)$ is locally Pareto-optimal front at x if x is locally Pareto-optimal solution.*

For the Pareto-optimal solutions of problem (2), the Kuhn-Tucker optimality conditions are stated as follows [1]: If a feasible solution x^* is the Pareto-optimal solution of problem (2), then there exist multipliers $\lambda_j \geq 0, j = 1, 2, .., m$ and $\omega_k \geq 0, k = 1, 2, ..., r$ satisfying the following conditions

$$\sum_{k=1}^{r} \omega_k \nabla f_k(x^*) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x^*) = 0$$

$$\lambda_j g_j(x^*) = 0, j = 1, 2, ..., m.$$

3 Multi-Objective Classical Approaches

Almost every real life problem is MOOP [1, 2]. Methods for MOOP are mostly intuitive. Many attempts have been found to solve MOOP. Apart from using set theory approaches, there exists a variety of secularization approaches such as the weighting method, lexicographic method, the compromise method and Keeny-Raiffa. Although the weighting method has some intrinsic inconveniences, i.e. the incapacity to capture Pareto-optimal solutions of the non-convex attainable regions; the uniform discrete weighting Pareto-optimal solutions of the non-convex attainable regions; the uniform discrete weighting often leading to very uneven distributions of Pareto-optimal solutions [4], this method is still considered as a usual method. The reason perhaps lies in that the formulation is straightforward and the solution of the weighting problem is mathematically ensured to be a Pareto-optimal solution. Mathematically, this method proceeds by assuming that it is required to minimize the objectives $f_k(x),\ k = 1, 2, ..., r$ (The problem of maximization is obtained via replacing $f_k(x)$ by $-f_k(x)$). Therefore, a scalar and linear weighting summation is defined as follows:

$$Z = \sum_{k=1}^{r} \omega_k f_k(x), \quad 0 \leq \omega_k \leq 1, \quad \sum_{k=1}^{r} \omega_k = 1$$

The second method is the lexicographic method. In this method, objectives are ordered according to their importance. Then, a single objective problem is solved while completing the problem gradually with constraints. Mathematically, this method proceeds by beginning with the first objective function as follows:

$$\begin{array}{ll} \textit{minimize} & f_1(x), \\ \textit{subject to} & g_j(x) \leq 0, \ \ j = 1, 2, ..., m \end{array} \tag{3}$$

Let $f_1(x^*)$ denotes the solution of problem (3). Then, the first objective function is transformed into an equality constraint and the second objective function is taken as follows:

$$\begin{array}{ll} \textit{minimize} & f_2(x), \\ \textit{subject to} & f_1(x) = f_1(x^*), \\ & g_j(x) \leq 0, \ \ j = 1, 2, ..., m \end{array} \tag{4}$$

This approach is repeated until the last objective function and therefore, the last value of x is the one that minimizes all the objective functions. A famous application of this approach is in university admittance where students with highest grades are allowed in any college they choose. The second best group is allowed only the remaining places and so on. This method is useful but in some cases it is not applicable.

The third method is the compromise method. In this case one minimizes only one objective while setting the other objectives as constraints. Mathematically, this method proceeds as follows:

$$\begin{array}{rl} \textit{minimize} & f_1(x), \\ \textit{subject to} & f_2(x) \leq \epsilon_2, \\ & \vdots \\ & f_r(x) \leq \epsilon_r, \\ & g_j(x) \leq 0, \quad j = 1, 2, ..., m \end{array} \tag{5}$$

where $\epsilon_k, \; k = 1, 2, ..., r$ are parameters to be gradually decreased till no solution is found. The problem with this method is the choice of the thresholds ϵ_k. In the case of equality constraints, this method is guaranteed to give a Pareto-optimal solution.

A fourth method is Keeny-Raiffa method which uses the product of objective functions to build an equivalent single objective one. An elaborate description about this approach is in [2].

In the next section, both Keeny-Raiffa and compromise methods are modified and applied to portfolio management problem.

4 Proposed Multi-Objective Oligopoly Model

Oligopoly [5] is the case where a market is controlled by a few number of firms producing similar products. In oligopoly models, one typically maximizes the profits. This is not realistic since in many cases one has more than one objective. Here, a multi-objective oligopoly is modeled, where both profit maximization and risk minimization are included. A modification of Keeny-Raiffa method is presented.

Consider $n-$firms each producing $q_i^t = 1, 2, ..., n$ of a certain product at time t with profit function:

$$\Pi_i^t = q_i^t(a - c_i - bQ^t), \quad Q^t = \sum_{j=1}^{n} q_j^t$$

The multi-objective goals for each firm are to maximize the profit and to minimize the risk.

Here, the function of risk measurement is taken as follows:

$$(\sigma_t)^2 = (q_i^t - av_i^t)^2, \quad where \quad av_i^t = \frac{\sum_{s=1}^{t-1} q_j^s}{t-1}, \;\; t = 1, 2, ...$$

So, the oligopoly model can be rewritten as a multi-objective optimization problem in the following form:

$$minimize \begin{cases} \Pi_i^t = q_i^t(a - c_i - bQ^t) \\ -(\sigma_t)^2 = -(q_i^t - av_i^t)^2 \end{cases}$$

Applying Keeny-Raiffa method to the above problem, one gets a unified objective function

$$Z_i = (\gamma - q_i^t(a - c_i - bQ^t))(\alpha + q_i^t - av_i^t)^2, \;\; i = 1, 2, ..., n$$

The parameters α, γ have been introduced to prevent the objectives from becoming zero. Using the standard approach to Cournot economic dynamical systems with bounded rationality
$q_i^{t+1} = q_i^t + \beta \frac{\partial z_i}{\partial q_i^t}$ (β is a measure of the bounded rationality) one finally gets the coupled dynamical system:

$$\begin{array}{rcl} q_i^{t+1} & = & q_i^t + \beta(-a + c_i + bQ^t + bq_i^t)(\alpha + q_i^t - av_i^t)^2 + 2\beta(\gamma - q_i^t(a - c_i - bQ^t)) \\ av_i^{t+1} & = & \dfrac{(t-1)av_i^t + q_i^t}{t}, \;\; i = 1, 2, ..., n \end{array} \tag{6}$$

The second part of (6) is known as Smale's game. In the standard formulation of games, the players choose their strategies without regards to previous experiences [6]. This is not realistic since including previous experiences is important. An interesting formulation of games including memory effects has been given by Smale [7] and extended by Ahmed and Hegazi [8]. The state space is taken to be the convex closure in 2-dimensional space of possible payoffs. The repeated play yields a sequence of outcomes $(x_1, ..., x_n)$. It is assumed that at time $t+1$ the first player makes his/her decision based on his/her up to date average payoff i.e.

$$x_{t+1} = \pi(s(av_t)), \quad av_t = \frac{\sum_{i=1}^{t} x_i}{t}$$

where π is the payoff and s is the chosen strategy. The averaging (memory) process defines the dynamical system

$$av_{t+1} = f(av_t) = \frac{t(av_t) + \pi(s(av_t))}{t+1} \tag{7}$$

Definition 5 *The solution of the dynamical system (7) associated to the symmetric game is a pair (s, av^*) such that av^* is the fixed point of (7).*

The proof of the following results is in Smale [7].

Proposition 1
1. A point av^* is fixed point of the system (7) if and only if it satisfies $\pi(s(av_t)) = av^*$.
2. If the state space of the game is bounded then the system (7) has a solution for any strategy s.

Definition 6 *A solution (s^*, av^*) is globally stable if for any x_0 in the state space, the following sequence $x_{t+1} = f_t(x_t), \quad t = 0, 1, ...$ converges to x^*.*

The solution will be called locally stable if there is a neighborhood $\mathcal{B}(av^*)$ of av^* where the above is true for any $x \in \mathcal{B}(av^*)$.

Thus, the second part of (6) stabilizes the system and an equilibrium point is expected.

Numerical simulations of the model that support this conclusion are given below.

For the symmetric case $c_i = c \quad \forall i = 1, 2, ..., n$, the equilibrium solution takes the following form:

$$q = \frac{-x \pm \sqrt{x^2 - 8bny}}{4bn}, \quad x = -2(a-c) + ab(n+1), \quad y = -\alpha(a-c) + 2\gamma$$

A necessary and sufficient condition for its existence is $x \leq 0$.

5 Conclusions

To conclude, service management requires both cometition and multi-objective optimization to be combined. Here, a multi-objective oligopoly model has been proposed and discussed. It has been shown that the model is more stable and realistic than the single objective one.

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