ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.24(2017) No.2, pp.67-75

Element-localized Boundary Integral Formulations for Transient Nonlinear Problems

Okey Oseloka Onyejekwe *
Computational Science Program, Addis Ababa University, Arat Kilo Campus,1176, Addis Ababa Ethiopia.

Abstract: Boundary value problems are ubiquitous in engineering and mathematical physics. A good many of such problems are amenable to boundary integral formulations which from a numerical point of view have proven to be very competitive. The same can not be said when the problem is nonlinear because the appearance of volume integrals and an unavoidable encounter with the problem domain, pose a considerable numerical challenge. In this paper, we aim at providing further insight into the development of numerical background of an emerging family of computational techniques for handling nonlinear partial differential equations (PDEs). This is based essentially on converting the original nonlinear PDE into its discrete, element-localized boundary-domain analog resulting in system of nonlinear system of algebraic equations. The occurrence of domain integrals arising from nonlinearity as well as those from the approximation of the time derivative are encountered but unlike the classical approach, they are resolved within the discrete-element domain. Comparisons of numerical results with those obtained analytically or from literature confirm the utility of the proposed formulation in handling nonlinearity in an unambiguous, and yet elegant manner, without much numerical complications

Keywords: Nonlinearity, boundary value problems, boundary element method, volume integrals, element-localized, boundary-domain, domain integrals

1 Introduction

The boundary element method (BEM) started making a serious impact into the computational field from the 1960s following the work of research pioneers like Rizzo [1], Kupradze [2], Rizzo and Shippy [3], Brebbia[4], Kupradze and Gegelia [5]. Its popularity rests on one of its major advantages, namely its dimensionality reduction i.e. its ability to reduce a boundary value problem (BVP) specified in the problem domain to an integral equation on the problem boundary. This conversion leads to a simple discretization procedure with boundary elements only; and as a result, smaller system of equations as well as considerable reduction of the data requirement are needed for discretization. These attributes coupled with the ability to handle singularities and boundary conditions especially those situated at infinity have made BEM a powerful and elegant technique in computational continuum mechanics (Brebbia and Dominguez [6]).

It will therefore come as a surprise that despite these attractive features BEM has continuously lagged behind FEM as a competitive numerical tool. Some of these reasons are obvious. For example a major requirement of partial differential equation (PDE) reduction to a boundary equation is that a fundamental solution to the PDE must exist. Unfortunately this condition is only satisfied when the coefficients for the governing PDE are non-variable. It is also well known that for most practical problems involving nonlinearity, heterogeneity, body force terms, transient terms, initial conditions , fundamental solutions are not easily available in cheaply computable forms, even if they do , they come as a result of adopting them for simplified, linear, steady state problem with constant coefficients. In addition, since both the problem domain as well as the boundary need to be discretized for virtually all boundary value problems, domain integration as well as a loss of dimensional reducibility can not be completely avoided. As a result, discretization is often accompanied by a dense and fully populated coefficient matrix even for relatively simple applications. Therefore, the underlying mathematics becomes relatively complicated and BEM becomes inadequate for tackling field problems.

* *E-mail address*: okuzaks@yahoo.com

IJNS.2017.10.15/972

For well over two decades BEM research has been focused on finding 'boundary-driven' methods of overcoming or alleviating these difficulties. Though the intent of this work precludes offering a detailed literature review, a few of them is worthy of attention. The dual reciprocity method (DRM) without domain discretization was developed by Nardini and Brebbia [7]. It consists of approximating the source term with a series of basis functions and using their particular solutions. This idea was extended further by Wrobel et al. [8], Wrobel and Brebbia [9] to handle transient problems, as well as nonlinear problems. A method involving direct conversion of domain to boundary integral involving a local boundary integral formulation (LBIE) was adopted by Sladek et al [10] for solving problems of elasticity with nonhomogeneous material properties. They applied a Kelvin type fundamental solution for each small sub-domain and succeeded in substantially eliminating the dominance of the domain integral.

In line with this area of BEM research, another 'domain-to-boundary' integration technique known as the radial integration method (RIM) was developed by Gao [11,12]. RIM has been shown to present some advantages over DRM in terms of accuracy in the solution of static and dynamic problems. Its major advantage over other methods lies in the claim that it can handle all types of domain integration in a unified way unlike the DRM which relies on particular solutions. It has been applied in such diverse areas as elastoplacity and thermoelastic problems coupled with nonlinearly [13,14]. The work of AL-Jawary and Wrobel [15,16] falls into this category. They constructed localized paramerices to reduce a BVP to a localized boundary-domain integro-differential equation (LBDIE or LBDIDE). The use of these parametrices led to sparsely populated systems of algebraic equations. Mikhailov [17] went a step further to convert the domain integrals appearing in RIM to equivalent boundary integrals.

The general trend for all these rescue techniques is that they are all boundary-driven and close observation reveals a persistent inherent numerical difficulty in extending and applying these methods to non-homogeneous, transient, nonlinear problems that abound in real world engineering practice. What has become clear in this field of BEM research over the years is the employment of different types of techniques ranging from heuristic to rigorous mathematical artefacts to arrive at a boundary-only formulation for problems whose physics justifies an intrinsic and dynamic link between the boundary and the problem domain. One would wonder when the adoption a boundary-only concept for all problems remains an advantage instead of a disadvantage. With this in mind, there needs to be a rethink in the primary mathematical concept that lies in the heart of BEM formulation. Can we only arrive at acceptable BEM numerical solutions of PDEs by first searching for the solution on the boundary and later utilize this information to determine the solutions in the problem domain? Must all solutions follow this path even if the physics dictates otherwise? Again when should domain discretization become an advantage instead of a disadvantage? Experience has shown that except for the Laplaces equation which exhibits a strong link between the problem domain and the boundary, determination of the scalar field over the boundary does not necessarily lead to an accurate expression of the scalar profile inside the problem domain. Mathematical manipulation alone without sufficiently addressing the issues related to the dynamic link between the two entities has often led to unnecessary complications of an otherwise a useful numerical tool. In the light of all these challenges , an efficient coupling algorithm that will allow for the optimal exploitation of two cooperating numerical technique is needed.

This idea is not new, it goes way back to Zienkiewicz et al.[18] who identified the complementary characters of numerical techniques in dealing with these types of problems and the benefits of their combined application in the development of hybrid numerical algorithms (Ruberg [19], Hsiao [20], Steinbach [21]). A comprehensive list of such methods can be found in Estorff and Hagen [22]. The drawback of such existing methods comes from the complex substructuring of the problem domain into different areas that are handled independently by BEM and FEM. Some of the old problems are solved but more often than not, new ones arise; especially those arising from nonconformity of interface discretizations, non-sparsity of the system matrix, and further complication of the numerical technique.

2 Theoretical Background

Our governing differential equation is a nonlinear conduction equation represented as:

$$\frac{\partial}{\partial x}\left(\phi(\theta)\frac{\partial\theta}{\partial t}\right) = \eta\frac{\partial\theta}{\partial t} + F(x,t,\theta) \tag{1}$$

where θ, η, F are scalar dependent. Equation (1) has continued to receive great attention in diverse areas of engineering and life science. Its solution requires information at the boundaries of the problem domainx_0 and x_L as well as the data specification $\theta(x,t)$ at initial time t_0. Integral representation of equation (1) starts with recasting it into a proper Poisson form i. e.

$$\frac{\partial^2\theta}{\partial^2 x} = \frac{1}{\phi(\theta)}\left[-\frac{\partial\ln\theta}{\partial x}\frac{\partial\theta}{\partial x} + \frac{1}{\phi(\theta)}\left\{\frac{\partial\theta}{\partial t} + F(x,t,\theta)\right\}\right] \tag{2}$$

which can be now put appropriately via the Greens second identity as:

$$\begin{aligned}&\int_{x_0}^{x_l}\left\{\theta\frac{d^2G}{dx^2}-G\frac{d^2\theta}{dx^2}\right\}dx=\theta\frac{dG}{dx}|_{x_0}^{x_L}=\lambda\theta_i+\theta\frac{dG}{dx}|_{x_0}^{x_L}-G\frac{d\theta}{dx}|_{x_0}^{x_L}+\int_{x_0}^{x_L}G(\xi,\xi_i)\frac{d^2\theta}{dx^2}dx=-2\lambda\theta_i+[H(x_2-x_i)\\&-H(x_i-x_2)]\theta_2-[H(x_1-x_i)-H(x_i-x_1]\theta_1-(|x_2-x_i|+l_m)\varphi_2+(|x_1-x_i|+l_m)\varphi_1+\\&\int_{x_0}^{x_L}\left[-\frac{\partial LnD}{\partial\xi}\varphi+\frac{1}{\phi(\theta}\left\{\frac{\partial\theta}{\partial t}+F(\theta,t,x)\right\}\right]dx\quad i=1,2\end{aligned}\tag{3}$$

where φ is the spatial derivative of the dependent variable, l_m is the maximum length of a generic element in the problem domain. The complementary differential equation is given by $d^2G/dx^2=\delta(x-x_i)$ with a fundamental solution and its derivative given by $G(x,x_i=|x-x_i|+l_m/2$, $dG(x.x_i)/dx=0.5\left[H(x-x_i)-H(x_i-x)\right]$. H is the Heaviside function which can be represented as

$$H(x-x_i)=\begin{cases}1,x>x_i\\0,x<x_i\end{cases}$$

We note in passing that the above formulation admits a relatively simple complimentary differential equation as well as a fundamental solution that guarantees that domain integration is implemented straightforwardly. Several authors have relied on more complicated techniques for example the use of parametrix to deal with domain integration (Estorff and Hagen [22]) . To prevail over these problems, there should be a radical departure from the classical boundary-only approach to a method that accommodates and deals with the problem domain. An elementization of the problem domain allows a boundary-domain formulation of equation (3) and leads to a FEM-like sparse system of equations but not the dimensional diminution of BEM. In order to facilitate this procedure, both the dependent variable and its functions in equation (3) are interpolated within an element to yield:

$$\begin{aligned}&\frac{\partial\theta_j}{\partial x}\approx\{\Omega_j(\xi)\varphi_j(t)\}=\{\Omega_1(\xi)\varphi_1(t)+\Omega_2(\xi)\varphi_2(t)\}\\&\frac{\partial\theta_j}{\partial t}\approx\{\Omega_j(\xi)\theta_j(t)\}=\frac{\partial}{\partial\xi}\{\Omega_1(\xi)\theta_1(t)+\Omega_2(\xi)\theta_2(t)\}\\&LnD=\Theta(x,t)\approx\{\Omega_j(\xi)\Theta_j(t)\}=\{\Omega_1(\xi)\Theta_1(t)+\Omega_2(\xi)\Theta_2(t)\}\\&\frac{\partial\Theta_j}{\partial x}\approx\frac{\partial}{\partial xi}\{\Omega_j(\xi)\Theta_j(t)\}=\frac{\partial}{\partial\xi}\{\Omega_1(\xi)\theta_1(t)+\Omega_2(\xi)\theta_2(t)\}\\&\frac{1}{\eta(\theta)}=\chi(\xi,t)\approx\{\Omega_1(\zeta)\chi_1(t)+\Omega_2(\zeta)\chi_2(t)\}\end{aligned}\tag{4}$$

where $\Omega_1=1-\zeta$, $\Omega_2=\zeta$, $\zeta=(x-x_1)/l$. The same procedure will apply to all other items inside the integral sign. Following this interpolation, the discrete element equation becomes:

$$\begin{aligned}&-2\lambda\theta_i+[H(\xi_2-\xi_2)-H(x_i-x_2)]\theta_2-[H(x_1-x_i)-H(x_i-x_1)]\theta_1-(|x_2-x_1|+l_m)\varphi_2+\\&(|x_1-x_i|+l_m)\varphi_1+G(x,x_1)\int_{x_1}^{x_2}\left[\left(-\frac{1}{l}\frac{\partial\Omega_n}{\partial\zeta}\Theta_n\right)(\Omega_j\varphi_j+\left\{\chi_n\Omega_j\frac{d\theta_j}{dt}+\chi_n\Omega_jF_j\right\}\right]dx\end{aligned}\tag{5}$$

Equation (5) is solved in each element of the problem domain and are later assembled to yield a sparse coefficient matrix. For a linear element the solutions at nodes 1 and 2 are:

At node 1:

$$\begin{aligned}&x_i=x_1\\&-\theta_1+\theta_2+l_m\varphi_1-(l_m+l)\varphi_2+\\&-\theta_1+\theta_2+l_m\varphi_1-(l_m+l)\varphi_2+\\&l\int_0^1(|\xi-\xi_1|+_m)\Omega_j\varphi_j\left\{-\frac{1}{l}\frac{d\Omega_n}{d\zeta}\Theta_n\right\}d\xi+l\int_0^1(|\xi-\xi_1|+l_m)\Omega_n\chi_n\Omega_j\left\{\frac{d\theta_j}{dt}+F_j\right\}d\xi=0\end{aligned}\tag{6}$$

Similarly at node 2:

$$x_i = x_2$$

$$-\theta_1 + \theta_2 + l_m\varphi_1 - (l_m)\varphi_2 + \tag{7}$$

$$l\int_0^1 (|\xi-\xi_1| +_m)\Omega_j\varphi_j\left\{-\frac{1}{l}\frac{d\Omega_n}{d\zeta}\Theta_n\right\}d\xi + l\int_0^1(|\xi-\xi_1|+l_m)\Omega_n\chi_n\Omega_j\left\{\frac{d\theta_j}{dt}+F_j\right\}d\xi = 0$$

Equations (6) and (7) are expressed in a matrix form to read:

$$R_{ij}\theta_j + \{L_{ij} - \Phi_{inj}\Theta_n\}\varphi_j + P_{inj}\chi_n\left[\frac{d\theta_j}{dt} + F_f = 0\right] \tag{8}$$

Discretization of the temporal term with forward differences yields

$$\frac{d\theta_j}{dt}|_{t=t_1+\alpha\Delta t} = \frac{\theta_j^{m+1}-\theta_j^m}{\Delta t} \qquad 0 \le \alpha \le 1 \tag{9}$$

For example $\alpha = 1$,$\alpha = 0$, $\alpha = 0.5$, $\alpha = 0.65$ are implicit, explicit, Crank-Nicolson, and Galerkin respectively. All the terms in equation (8) should be evaluated at a particular time level. To accomplish this, equation (8) adopts a weighted average formulation as shown below:

$$\begin{aligned}
&\left[\alpha R_{ij} + \frac{P_{ijn}[\alpha\chi_n^{m+1}+\omega\chi_n^m]}{\Delta t} + P_{ijn}[\alpha\chi_n^{m+1} + \omega\chi_n^{(m)}]\alpha\psi^2\right]\theta_j^{(m+1)} + \\
&\left[\alpha\{L_{ij} - \Phi_{inj}\Theta_j^{(m+1)}\}\right]\varphi_j^{(m+1)} + \\
&\left[\omega R_{ij} - \frac{P_{ijn}[\alpha\chi_n^{(m+1)}+\omega\chi_n^{(m)}]0}{\Delta t} + P_{ijn}[\alpha\chi_n^{(m+1)} + \omega\chi_n^{(m)}]\omega\psi^2\right]\theta_j^{(m)} + \\
&\left[\omega\{L_{ij} - \Phi_{inj}\Theta_n^{(m)}\}\right]\varphi_j^m - \\
&\left[P_{inj}\{\alpha\chi_n^{(m+1)} + \omega\chi_n^{(m)}\}\{(\alpha(F)^{(m+1)} + \omega(F)^{(m)}\}\right] \equiv s_i = 0 \quad i,j,n = 1,2 \quad 0 \le \alpha \le 1
\end{aligned} \tag{10}$$

where $\omega = 1 - \alpha$. Equation (10) is a system of nonlinear discrete equations whose Jacobian matrix is given by :

$$\begin{aligned}
&\frac{\partial s_i}{\partial\theta j^{(m+1)}} = \alpha R_{ij} + \frac{P_{inj}}{\Delta t}[\alpha\chi_n^{(m+1)} + \omega\chi_n^{(m)}] + \\
&\frac{P_{inj}}{\Delta t}[\theta_n^{(m+1)} - \theta_n^{(m)}]\frac{d\chi_j^{(m+1)}}{d\theta_j^{(m+1)}} - \alpha P_{inj}[\alpha\chi_n^{(m+1)} + \omega\chi_n^{(m+1)}]\frac{dF_n^{(m+1)}}{d\theta_j} \\
&+\alpha P_{inj}[F_n^{(m+1)} + F_n^{(m)}]\frac{d\chi_j^{(m+1)}}{d\theta_j^{(m+1)}} - (\Phi_{inj})\{\varphi_n^{(m+1)}\}\frac{d\Theta_j^{(m+1)}}{d\theta_j^{(m+1)}}
\end{aligned} \tag{11a}$$

and

$$\frac{\partial s_i}{\partial\varphi_j^{(m+1)}} = \alpha[L_{ij} - \Phi_{inj}\Theta_n^{(m+1)}] \tag{11b}$$

We call this formulation mod-1. We move on to another level of approximation by approximating φ and $1/\varphi$ by linear piecewise interpolation functions within a generic element of the problem domain.

$$\varphi = \Omega_n\varphi_n = \left(1 - \frac{x-x_1}{l}\right)\varphi_1 + \left(\frac{x-x_1}{l}\right)\varphi_2$$

Similarly

$$1/\varphi(\theta) = \chi(\theta) \approx \Omega_n\chi_n$$

hence

$$-\frac{1}{\varphi(\theta)}\frac{d\varphi(\theta}{dx} = -(\Omega_n\chi_n)\frac{d(\Omega_n\varphi_n)}{ld\zeta} \tag{12}$$

Substituting equation (13) into equation (7) and discretizing the element integral equation yields

$$\Big[\alpha R_{ij} + \tfrac{P_{ijn}[\alpha\chi_n^{m+1}+\omega\chi_n^m]}{\Delta t} + P_{ijn}[\alpha\chi_n^{m+1} + \omega\chi_n^{(m)}]\alpha\psi^2\Big]\,\theta_j^{(m+1)} +$$
$$\Big[\alpha\{L_{ij} - \Lambda_{iknj}D_k^{(m+1)}\chi_n^{m+1}\}\Big]\,\varphi_j^{(m+1)} +$$
$$\Big[\omega R_{ij} - \tfrac{P_{ijn}[\alpha\chi_n^{(m+1)}+\omega\chi_n^{(m)}]0}{\Delta t} + P_{ijn}[\alpha\chi_n^{(m+1)} + \omega\chi_n^{(m)}]\omega\psi^2\Big]\,\theta_j^{(m)} + \tag{13}$$
$$\Big[\omega\{L_{ij} - \Lambda_{iknj}D_n^{(m)}\chi_n^{(m)}\}\Big]\,\varphi_j^{m} -$$
$$\Big[P_{inj}\{\alpha\chi_n^{(m+1)} + \omega\chi_n^{(m)}\}\{(\alpha(F)^{(m+1)} + \omega(F)^{(m)}\}\Big] \equiv s_i = 0 \quad i,k,j,n = 1,2$$

with a corresponding Jacobian given by:

$$\tfrac{\partial s_i}{\partial\theta_j^{(m+1)}} = \alpha R_{ij} + \tfrac{P_{inj}}{\Delta t}[\alpha\chi_n^{(m+1)} + \omega\chi_n^{(m)}] + \tfrac{P_{inj}}{\Delta t}[\theta_n^{(m+1)-\theta_n^{(m)}}]\tfrac{d\chi_j^{(m+1)}}{d\theta_j^{(m+1)}}$$
$$-\{\alpha\Lambda_{iknj}\}(\phi_k^{(m+1)})\left(\tfrac{d\chi_j^{(m+1)}}{d\theta_j}\right) - \{\alpha\Lambda_{iknj}\}(\chi_n^{(m+1)})\left(\tfrac{d\phi_j^{(m+1)}}{d\theta_j}\right) + \tag{14}$$
$$\alpha P_{inj}\Big[\{\alpha(\chi_n^{(m+1)})\} + \{\omega(\chi_n^{(m)})\}\Big]\tfrac{dF_j^{(m+1)}}{d\theta_j} + \alpha P_{inj}\Big[\{\alpha(F_n^{(m+1)})\} + \{\omega(F_n^{(m)})\}\Big]\tfrac{d\chi_j^{(m+1)}}{d\theta_j}$$

and

$$\frac{\partial s_i}{\partial\varphi_j^{(m+1)}} = \alpha\Big[L_{ij} - \Lambda_{iknj}D_k^{(m+1)}\chi_n^{(m+1)}\Big] \tag{15}$$

We call equation (15) model-2.

Computation is initiated by estimating the unknown dependent variables:$\{\theta_j^{(m+1,k)}, \varphi_j^{(m+1,k)}\}^T$ and updating them according to $\{\theta_j^{(m+1,k)} + \Delta\theta_j^{(m+1,k+1)} + \varphi_j^{(m+1,k)} + \Delta\theta_j^{(m+1,k+1)}\}$ where the incremental values: $\{\Delta\theta_j^{(m+1)}, \Delta\varphi_j^{(m+1)}\}$ are obtained by solving the matrix equation:

$$[J_{ij}^{(m+1)}]\begin{Bmatrix}\Delta\theta_j^{(m+1)}\\ \Delta\varphi_j^{(m+1)}\end{Bmatrix} = -s_i^{(m+1)} \tag{16}$$

where J_{ij} is the Jacobian matrix

3 Numerical Calculations

Transport processes in nonlinear materials are used to validate and compare the current models. The first example represents water-table displacements in two-stream-unconfined-aquifer systems due to a recharge from infiltration [29]. While the second is representative of transient diffusion with nonlinear reaction in a multicomponent mixture.

3.1 Example 1

Unconfined saturated groundwater is defined by the following nonlinear partial differential equation

$$S_y\left(\boldsymbol{\frac{\partial h}{\partial t}}\right) = \frac{\partial}{\partial x}\left(h\tfrac{\partial h}{\partial x}\right) + I(x,t) - Q(x,t) \tag{17}$$

where h is the hydraulic head, Q is the specific flux prescribed on the flux boundary, I is the net rate of recharge that reaches the free surface , and S_y is the specific yield of the porous media , x and t are the space and time coordinates. The initial height of water is calculated from equation (12) of [27] for zero infiltration, while the boundary conditions are given by $h_1 = 30cm$, $h_2 = 10cm$. Other parameters needed for the solution are given by $L = 100cm, S_y = 0.15, I_0 = 0.025cmmin^{-1}, \Delta t = 10min, \Delta x = 10cm$. Carbrera and Matthey [27] applied a finite element Galerkin residual method and the Newton-Raphson method to solve equation (17) with initial and boundary conditions together with the given problem parameters. The height of the water table at different times was calculated using model(1) and model (2) of this study. Tables 1 and 2 show the behavior of the free surface as predicted by both numerical and analytical solutions at an intermediate time and close to the final equilibrium conditions. The results show that the numerical solutions tend towards the steady state solution in way that is validated by the analytic results. The formulation developed herein can be relied on to produce faithful results.

Table 1: Model validation for transient conditions (display of hydraulic heads at t= 60 min.)

x(cm)	analytical solution	FE model[29]	GEM Model(1)	GEM Model(2)	FE[29] *error	GEM(1) *error	GEM(2) *error
0.0	30.00	30.00	30.00	30.00	–	–	–
10.0	31.38	31.30	31.23	31.22	0.25	0.48	0.51
20.0	31.97	31.85	31.72	31.70	0.38	0.78	0.84
30.0	31.90	31.80	31.61	31.58	0.31	0.90	0.90
40.0	31.22	31.19	30.95	30.92	0.10	0.86	0.96
50.0	29.94	30.05	29.79	29.75	0.37	0.50	0.63
60.0	28.09	28.36	28.09	28.05	0.96	0.00	0.14
70.0	25.44	26.01	25.75	25.72	2.24	1.20	1.10
80.0	21.96	22.79	22.56	22.55	3.78	2.50	2.60
90.0	17.24	18.14	17.99	17.99	5.22	4.40	4.40

$*error = (devation/analyt.solution)\%$

Table 2: Model validation for transient conditions (display of hydraulic heads at t= 240 min.)

x(cm)	analytical solution	FE model[27]	GEM Model(1)	GEM Model(2)	FE[27] *error	GEM(1) *error	GEM(2) *error
0.0	30.00	30.00	30.00	30.00	–	–	–
10.0	32.33	32.29	32.30	32.29	0.12	0.12	0.12
20.0	33.76	33.70	33.71	33.70	0.18	0.15	0.18
30.0	34.42	34.35	34.35	34.34	0.20	0.20	0.23
40.0	34.35	34.27	34.26	34.25	0.23	0.26	0.29
50.0	33.54	33.46	33.45	33.43	0.24	0.27	0.33
60.0	31.94	31.87	31.84	31.83	0.22	0.31	0.34
70.0	29.41	29.38	29.32	29.31	0.10	0.31	0.34
80.0	25.69	25.75	25.61	25.60	0.23	0.31	0.35
90.0	20.12	20.37	20.07	20.06	1.24	0.25	0.30

3.2 Example 2

We consider the case of transient multiple differential equation of the form : $\partial\theta/\partial t = \partial^2\theta/\partial x^2 - f_k(\theta_1, \theta_2...\theta_M$ with k representing the equation number index; $k = 1$ to M is the total number of equations. The forcing function in this case is nonlinear and represents the function of all dependent variables $\theta_1, \theta_2, ...\theta_m$. This type of formulation represents transient mass diffusion with nonlinear reaction in a membrane reactor. A mathematical description of such a system is given by the balance partial differential equations [30]:

$$\frac{\partial\theta_1}{\partial t} = \frac{\partial^2\theta_1}{\partial x^2} - k_1\theta_1^2 \tag{18}$$

$$\frac{\partial\theta_2}{\partial t} = \frac{\partial^2\theta_2}{\partial x^2} - k_2\theta_2^2 \tag{19}$$

$$\text{at} \quad t = 0, \theta_1 = \theta_2 = 0$$

$$\text{at} \quad x = 0, d\theta_1/dx = d\theta_2/dx = 0$$

$$\text{at} \quad x = 1, \theta_1 = 1, \theta_2 = 0$$

A steady state version of these equations was solved by [28]. An equivalent integral representation of the transient nonlinear system of equations after applying the two-level time discretization is given by

$$\begin{aligned} &\alpha(R_{ij})\theta_j^{(m+1)} + \alpha(L_{ij})\varphi_j^{(m+1)} + \{1-\alpha\}(R_{ij})\theta_j^{(m)} + \{1-\alpha\}(L_{ij})\varphi_j^{(m+1)} \\ &T_{ij}\left[\tfrac{1}{\Delta t}(\theta_j^{(m+1)} - \theta_j^{(m)})\right] + \alpha f\left[(\theta^2)_j^{(m+1)}\right] + \{1-\alpha\} f\left[(\theta^2)_j^{(m)}\right] \equiv s_i = 0 \end{aligned} \tag{20}$$

with the accompanying Jacobian expressed as:

$$\begin{aligned} &\tfrac{\partial s_i}{\partial \theta_j^{(m+1)}} = \alpha R_{ij} + \tfrac{T_{ij}}{\Delta t} + \alpha(2\theta_j^{(m+1)}) \\ &\tfrac{\partial s_i}{\partial \theta_j^{(m+1)}} = \alpha L_{ij} \end{aligned} \tag{21}$$

Tables 3 and 4 display the results at an intermediate time and at steady state to allow for comparison with [28]. The closeness of the computed profiles with those of [28] at steady state not only validates the numerical formulation but also confirms that the it displays stability in its approach to steady state.

Table 3: Display of concentrations and gradients at t= 0.4)

x	Scalar θ_1	Gradient $\partial_{\theta_1}/\partial x$	Scalar θ_2	Gradient $\partial_{\theta_2}/\partial x$
0.0	0.1552	0.0000	0.1881	0.0000
0.1	0.1584	0.6583-01	0.1903	0.4426-01
0.2	0.1685	0.1368	0.1969	0.8689-01
0.3	0.1862	0.2191	0.2076	0.1253
0.4	0.2130	0.3218	0.2217	0.1544
0.5	0.2517	0.4591	0.2378	0.1635
0.6	0.3068	0.6549	0.2530	0.1296
0.7	0.3863	0.9605	0.2608	-0.5318-01
0.8	0.5049	0.1616+01	0.2477	-0.3267
0.9	0.6898	0.4805+01	0.1833	-0.1049+01

Table 4: Display of concentrations and gradients at t= 15.0)

x	Scalar θ_1	Gradient $\partial_{\theta_1}/\partial x$	Scalar θ_2	Gradient $\partial_{\theta_2}/\partial x$
0.0	0.1589,0.1611[30]	0.0000, 0.0000[30]	0.2029,0.2039[30]	0.0000
0.1	0.1621	0.6411	0.2049	0.3956-01
0.2	0.1719,0.1744[30]	0.1335,0.1369[30]	0.2108,0.2117[30]	0.07788-1,0.07638[30]
0.3	0.1892	0.2147	0.2203	0.1127
0.4	0.2156,0.2191[30]	0.3167,0.3243[30]	0.2330,0.2334[30]	0.1390,0.1344[30]
0.5	0.2537	0.4538	0.2475	0.1462
0.6	0.3083,0.3138[30]	0.65513,0.6642[30]	0.2609,0.2599[30]	0.1115,0.1002[30]
0.7	0.3872	0.9641	0.2668	-0.1436-01
0.8	0.5049,0.5130[30]	0.1623+01,1.464[30]	0.2515,.2481[30]	-0.3370,-.3573[30]
0.9	0.6893	0.4793+01,4.005[30]	0.1848	-1.073,-2.777[30]

4 Conclusions

This paper addresses the effectiveness of adopting modified boundary integral formulations for nonlinear transient problems. Domain integrals resulting from nonlinearity and body- force terms are handled efficiently. Adopting a boundary-driven BEM approach would have resulted in complicated numerical procedures that have often rendered BEM less competitive than other domain-based numerical techniques. The choice of simple benchmark solutions in order to enhance the availability of this technique is deliberate. This does not however compromise the ability of the method to offer t a unified approach in handling different types of nonlinearities, inhomogeneities, and body force terms straightforwardly without much complications (Onyejekwe[24,25,27]).

References

[1] Rizzo, F. J., An integral equation approach to boundary value problems of classical elastostatics, Quart. Appld. Math. 25 (1967) 83-95

[2] Kupradze, V. D., Potential methods in the theory of elasticity, Daniel Davey and Co. NY 1965

[3] Rizzo, F. J. and Shippy, D. J. An advanced boundary integral method for three dimensional elasticity , Int. Jnl. Num. Mthd. Engnr. 11 (1977) 1753-1768

[4] Brebbia, C. A. The boundary element method for engineers, Pentech Press, London 1978

[5] Kupradze, V. D. and Gegelia, T.G., Three dimensional problems of the mathematical theory of elasticity and thermoelasticity, North-Holland , Amsterdam, 1979

[6] Brebbia, C. A. and Dominguez, J., Boundary elements: An introductory course CMP/McGraw Hill Book Co., N.Y., N.Y., 1989

[7] Nardini, D. and Brebbia, C. A. A new approach to free vibration analysis using boundary elements , in Boundary element methods in engineering (ed. C.A. Brebbia) Springer-Verlag, Berlin and New York , 1982

[8] Wrobel, L. C., Brebbia, C. A. and Nardini, D. The dual reciprocity boundary element formulation for transient heat conduction. In Proc. Of the Vth Int. Conf. on FEM in water resources. Computational mechanics publications, Southamton and Boston, Springer, Verlag, Berlin 1986.

[9] Wrobel, L. C. and Brebbia , C. A., Boundary elements for non-linear heat conduction problems. Commun. In Applied Num. Mthds., 4 (1988) 617-622

[10] Sladek, J., Sladek, V. and Atluri, S. N. , Local boundary integral equation (LBIE) method for solving problems of elasticity with nonhomogeneous material properties , Comput. Mech. 24 (2000) 456-462

[11] Gao, X. W., The radial integration method for evaluation of domain integrals with boundary-only discretization, Engineering Analysis with Boundary Elements 26 (2002) 905-916

[12] Gao, X. W., An effective boundary element algorithm for 2D and 3D elastoplastic problems, Int. Jnl. Solid Structures 37 (2000) 4987-5008

[13] Gao, X. W., Boundary element programming in mechanics, Cambridge: Cambridge University Press ; 2002

[14] Gao, X. W., Boundary only integral equations in boundary element analysis, Proc. Of the international conference on boundary element techniques, 16-18 July, 2001, Rutgers University, NJ USA

[15] AL-Jawary M.A. and Wrobel L.C., Numerical solution of two-dimensional Helmholtz equation with variable coefficents by the radial integration boundary integral and integro-differential equation methods , Int. Jnl. Com. Math. 89 (2012) 1463-1487

[16] AL-Jawary, M. A. and Wrobel, L.C., Radial integration boundary integral and integro-differential equation methods for two-dimensional heat conduction problems with variable coefficents , Eng. Anal. Bdry. Elem. 36 (2012) 685-695

[17] Mikhailov, S. E., Localized boundary-domain integral formulations for problems with variable coefficients, Eng. Anal. Boundry. Elem. 26 (2002) 681-690

[18] Zienkiewicz, O., Kelly, D. and Bettes, P., The coupling of the finite element method and boundary solution procedures , Int. Jnl. Num. Mthds. Engnr. 11 (1977) 355-375

[19] Ruberg, T., Non-conforming FEM/BEM coupling in time domain, vol.3 of Computation in engineering Science , Verlag der Technischen Universitat Graz, 2008

[20] Hsiao, G., The coupling of boundary and finite element methods , Zeit fur angewandte Mathematik und Mechanic 70 (1990) T493-503

[21] Steinbach, O., Mixed approximations for boundary elements , SIAM Jnl. Numer. Anal , 38 (2002) 401-413

[22] Esorf, O. and Hagen, C., iterative coupling of FEM and BEM in transient electrodynamics , Engnr. Anal. Boundary Elements 29 (2005) 775-787

[23] Taigbenu, A.E., The Green element method ., Kluwer Dordrecht 1999 .

[24] Onyejekwe, O. O., A Direct implementation of a modified boundary integral formulation for the extended Fisher-Kolmogorov equation, Jnl. Appld. Math. Phys. 3, (2015) 1262-1269

[25] Onyejekwe, O. O. A hybrid numerical treatment of nonlinear reaction-diffusion equations with memory: A prototypical Fisher-Kolmogorov-Petrovski-Piskunov equation, IJANS (to appear)

[26] Onyejekwe, O. O., Unsteady formulations for stagnation point flow towards a stretching sheet with prescribed surface heat flux and viscous dissipation, International Jnl. Fluid Mech. Thermal Sci. 3(2017) 16-24 [27] O.O. Onyejekwe, Green element procedures accompanied by nonlinear reaction; International Journal of Thermal Sciences 42(2003) 813-820

[27] Onyejekwe, O. O. An element-driven boundary integral treatment for nonlinearity : A coupled two-dimensional Burgers equation test case, Int. Jnl. Fluid mech. And Therm. Sci. 3(2017) 1-15

[28] Onyejekwe, O. O. Magnetohydrodynamic boundary layer slip flow and heat transfer over a flat plate with heat generation / absorption and viscous dissipation, IOSR Journal of Applied Phys. 9(2017) 100-110

[29] Guillermo, C. and Gabriel, M., Space-time finite-element model for two stream-unconfined Caquifer systems, 73 (1984) 299-314

[30] Ramachandran, P. A. Boundary element methods in transport phenomena, Computational mechanics Publications, Elsevier Applied Science, London, N.Y. 1994