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The Construction Method of Interpolation Polynomial by Fractal Interpolation

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Abstract: The differentiability of the fractal interpolation function has been widely researched by many researchers. Because of non-smooth of irregular curves, so it is very difficult to study the differentiability of fractal interpolation by the conventional method. In this paper, we study a special class of fractal interpolation functions with two-parameters which can influence the form of fractal interpolation functions. The dimension of the image generated by fractal interpolation function is between one and two dimensions. Theoretical and numerical analysis are done on the polynomial construction of such two-parameter fractal interpolation functions. We show that both parameters are weak linear correlation, the generated fractal interpolation function is quadratic polynomial. It is also shown that the vertical scaling factor is constant while two displacement parameters exhibit a linear relationship, the fractal interpolation function is cubic polynomial. Through the study of the construction of fractal interpolation polynomial, it is of great significance to further complete the theoretical system of fractal interpolation, so as to better guide the application of fractal interpolation in practical work.

Keywords: Fractal interpolation function; smooth; polynomial; curve

1 Introduction

In the 70s, the earliest concept of fractal which is extracted from no rules of the nature was originally proposed by Mandelbort [1]. Hence, that is meaningful that though using fractal fit into forms presented in nature. Several studies on the fractal interpolation function have also been proposed in resent years [3-5]. The American mathematician MF Barnsley [2] introduced the fractal interpolation function in which a new thinking and new method of synthesis data that can be seen as a kind of extension of polynomial interpolation and spline interpolation. He constructed the hyperbolic Iterated Function systems according to using cyclic iteration method for iterating the given interpolation points into countless points which are in turn attached to a line. The image of continuous function, which is consisted of interpolation points, is called the attractor of iterated function systems. And Barnsley introduced some real-valued interpolation functions, defined on a compact interval in R, which appear well suited for approximating naturally occurring functions which display some sort of self-similarity under magnification. The functions are analogous to splines and polynomial interpolations in that their graphs are set to go through a finite number of prescribed points. These functions differ from classical interpolations in that they obey a functional relation related to self-similarity on smaller scales Let

$$x_0 < x_1 < \cdots < x_N.$$

Let L_i be the affine map satisfying

$$L_i(x_0) = x_{i-1}, L_i(x_N) = x_i, i = 1, 2, 3, \cdots .$$

Let $y_0, y_1, y_2, \cdots, y_N \epsilon \Re$. Let $-1 < a_i < 1, i = 1, 2, 3, \cdots$. Let $x = [x_0, x_N]$. Let $F_i : X \times \Re \to \Re$ satisfy

$$|F_i(x, y_1), F_i(x, y_2)| \le a_i|y_1 - y_2|, x\epsilon X, y_1, y_2\epsilon\Re, F_i(x_0, y_0) = y_{i-1}, F_i(x_N, y_N) = y_i, i = 1, 2, 3, \cdots .$$

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The fractal interpolation function, abbreviated FIF, associated with $\{(L_i(X), F_i(x,y))\}_{i=1}^{N}$ is the unique function

$$f \to R$$

satisfying

$$f(L_i(x)) = F_i(x, f(x)), i = 1, 2, 3, \cdots x\epsilon f.$$

G.Donovan [3] considered that the fractal interpolation function based on affine transformation is generated by iteration function system with two given interpolation points. Furthermore, it is proved that fractal interpolation function is Ho1der continuous. Qian [4] used the fractal interpolation function to study that the maximum of a kind of the fractal interpolation function. Ke [5] studied the differentiability of non-uniform fractal interpolation. The property analysis of the dimensions of the fractal interpolation function is discussed in [6-8]. In [9,10], Sha Zhen and Barnsley investigated approximating and Holder properties of FIFs, but the image has too many discontinuous points, so it is hard to analyze. In this paper, we mainly study a class of smooth fractal interpolation function.

2 Two-parameter fractal interpolation function

Fractal interpolation function has good smoothness under certain conditions. Hence, it is meaningful to discuss the differentiability of the fractal interpolation function which is generated by iteration function system. The iteration function system can be written as:

$$\begin{cases} L_i(x) = a_i(x) + e_i, \\ F_i(x,y) = d_i y + r_i x^2 + c_i x + f_i, \end{cases} \tag{1}$$

where d_i, $r_i (i = 1, 2, \cdots, N)$ are parameters.

Set a series of data $\{(x_i, y_i) \in I \times R; i = 1, 2, \cdots, N\}$, where $I = [x_0, x_N] \subset R$, $N \geq 2$ and $r_1, r_2, \cdots, r_N \in R$. The above assuming should satisfy the following conditions:

$$L_i(x_o) = x_{i-1}, L_i(x_N) = x_i.i = 1, 2, \cdots, N, \tag{2}$$

$$|L_i(u_1) - L_i(u_2)| \leq l_i|u_1 - u_2|, \forall u_1, u_2 \in I, \tag{3}$$

$$F_i(x_o, y_o) = y_{i-1}, F_i(x_N, y_N) = y_i, \tag{4}$$

where $0 < |a_i| < 1, |d_i| < 1$.

Inserting Eqs. (2) to Eqs. (4) into Eqs. (1), we obtain four dimensional equations:

$$\begin{cases} a_i x_0 + e_i = x_{i-1}, \\ a_i x_N + e_i = x_i, \\ d_i y_0 + r_i x_0^2 + c_i x_0 + f_i = y_{i-1}, \\ d_i y_N + r_i x_N^2 + c_i x_N + f_i = y_i. \end{cases} \tag{5}$$

Solving the Eqs. (5), we obtain the solutions of the equations:

$$\begin{cases} a_i = (x_i - x_{i-1})/(x_N - x_0), \\ e_i = (x_N x_{i-1} - x_0 x_i)/(x_N - x_0), \\ c_i = -(x_N + x_0)r_i + (y_i - y_{i-1})/(x_N - x_0) - d_i(y_i - y_{i-1})/(x_N - x_0), \\ f_i = x_0 x_N r_i + (x_N y_{i-1} - x_0 y_1)/(x_N - x_0) + d_i(x_0 y_N - x_N y_0)/(x_N + x_0). \end{cases} \tag{6}$$

According to the fractal interpolation function generated by the above iteration function system, we obtain the theorem of constructing quadratic polynomial and cubic polynomial.

Theorem 1 *Set interpolation points: $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ and take them into iteration function system (1). If the parameter d_i, r_i are to satisfy the equality:*

$$s_1 d_i + r_i = a_i^2 s_1, i = 1, 2,$$

where

$$s_1 = \frac{y_2 - y_0}{(x_2 - x_1)(x_2 - x_0)} - \frac{y_1 - y_0}{(x_1 - x_0)(x_2 - x_1)}, a_i = \frac{(x_i - x_{i-1})}{x_N - x_0}, i = 1, 2,$$

then the generated fractal interpolation function is quadratic polynomial.

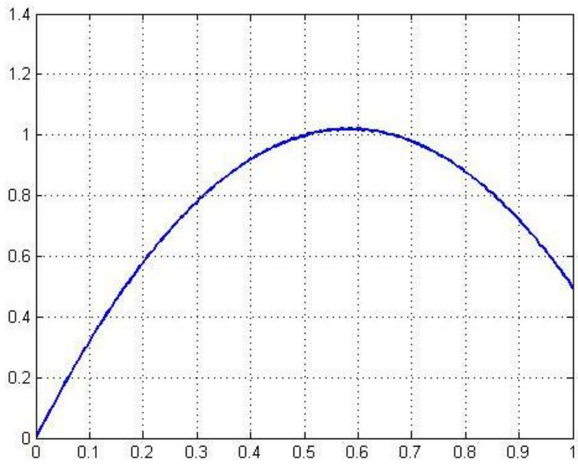

Figure 1: The stable region of $r_1 = 1/4, r_2 = 3/4$

Proof. If the generated fractal interpolation function is quadratic polynomial, let

$$f(x) = s_1x^2 + s_2x + s_3, s_1 \neq 0. \tag{7}$$

By using fixed point theorem, let

$$f(a_ix + e_i) = d_iy + r_ix^2x + c_ix + f_i, i = 1, 2. \tag{8}$$

Taking the Eqs. (7) into Eqs. (8), we have

$$\begin{cases} a_i^2s_1 = d_is_1 + r_i, \\ 2a_ie_is_1 + a_is_2 = d_is_2 + c_i, \\ e_i^2s_1 + e_is_2 + s_3 = d_is_3 + f_i. \end{cases} \tag{9}$$

Taking the interpolation points: $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ into the Eqs. (7), we can obtain the dynamical system:

$$\begin{cases} x_0^2s_1 + s_2x_0 + s_3 = y_0, \\ x_1^2s_1 + s_2x_1 + s_3 = y_1, \\ x_2^2s_1 + s_2x_2 + s_3 = y_2. \end{cases} \tag{10}$$

From (9) and (10), by calculating we get

$$s_1d_i + r_i = a_i^2s_1, i = 1, 2,$$

where

$$s_1 = \frac{y_2 - y_0}{(x_2 - x_1)(x_2 - x_0)} - \frac{y_1 - y_0}{(x_1 - x_0)(x_2 - x_1)}, a_i = \frac{(x_i - x_{i-1})}{x_N - x_0}, i = 1, 2.$$

Then the proof is completed. ■

Theorem 2 *Set interpolation points:* $(x_0, y_0), (x_1, y_1), (x_2, y_2)$*, and take them into the iteration system (1). If the parameter* d_i, r_i *are to satisfy the following equalities:*

$$(1)d_i = a_i^3,$$

$$(2)a_1(x_1 + x_2 - 2x_0)r_2 + a_2(2x_2 - x_0 - x_1)r_1 = 3a_1^2a_2^2\left(\frac{y_2 - y_0}{x_2 - x_1} - \frac{(y_1 - y_0)(x_2 - x_0)}{(x_2 - x_1)(x_1 - x_0)}\right),$$

where

$$a_i = \frac{x_i - x_{i-1}}{x_N - x_0}, i = 1, 2,$$

the generated fractal interpolation function is cubic polynomial.

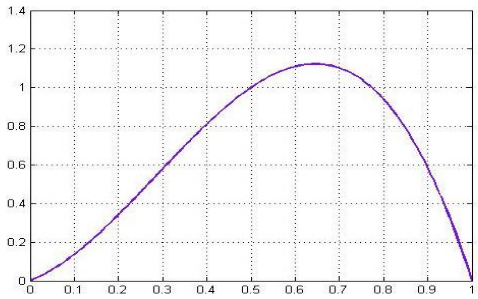

Figure 2: The stable region of $r_1 = 2/3, r_2 = -5/3$.

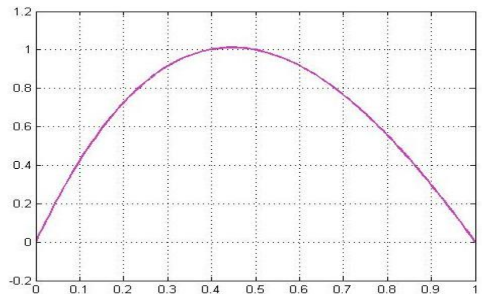

Figure 3: The stable region of $r_1 = -5/6, r_2 = -1/6$.

Proof. If the generated fractal interpolation function is cubic polynomial, let

$$f(x) = s_1x^3 + s_2x^2 + s_3x + s_4, s_1 \neq 0, \tag{11}$$

according to the fixed point theorem, let

$$f(a_ix + e_i) = d_iy + r_ix^2x + c_ix + f_i, i = 1, 2. \tag{12}$$

Taking the Eqs. (11) into Eqs. (12), then the system (12) is taken its form as:

$$\begin{cases} a_i^3s_1 = d_is_1, \\ 3a_i^2e_is_1 + a_i^2s_2 = d_is_2 + r_i, \\ 3a_ie_i^2 + 2a_1e_1s_2 + a_1a_3 = d_is_3 + c_i, \\ e_i^3s_1 + e_i^2s_2 + e_is_3 + s_4 = d_is_4 + f_i. \end{cases} \tag{13}$$

From (13), we have

$$d_i = a_i^3.$$

Taking the interpolation points: $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ into $f(x)$, we get the equations:

$$\begin{cases} x_0^3s_1 + s_2x_2 + s_3x_0 + s_4 = y_0, \\ x_1^3s_1 + s_2x_1^2 + s_3x_1 + s_4 = y_1, \\ x_2^3s_1 + s_2x_2^2 + s_3x_1 + s_4 = y_2. \end{cases} \tag{14}$$

By combining successive elimination technologies, we rewrite (14) as follows:

$$(x_0 + x_1 + x_2)s_1 + s_2 = \frac{y_2 - y_0}{(x_2 - x_0)(x_2 - x_1)} - \frac{y_1 - y_0}{(x_1 - x_0)(x_2 - x_1)}. \tag{15}$$

Then taking the Eqs. (15) into the system (13), we obtain

$$a_1(x_1 + x_2 - 2x_0)r_2 + a_2(2x_2 - x_0 - x_1)r_1 = 3a_1^2a_2^2\left(\frac{y_2 - y_0}{x_2 - x_1} - \frac{(y_1 - y_0)(x_2 - x_0)}{(x_2 - x_1)(x_1 - x_0)}\right).$$

Hence, the claim easily follows. ■

3 Examples

Now we give the following examples which we will use to verify the feasibility of the Theorem 1 and Theorem 2.

Example 3 *Setting interpolation points:* $A(0, 0), P_0(1/2, 1), B(1, 1/2)$*, the following two affine transformation can be determined. We suppose that* $d_1 = 1/3$ *and* $d_2 = 1/2$*, then*

$$\omega_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 5/6 - r_1 & 1/3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ r_1x^2 \end{pmatrix} = \begin{pmatrix} x/2 \\ \frac{1}{3}y + r_1x^2 + (5/6 - r_1)x \end{pmatrix},$$

$$\omega_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -3/4-r_2 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ r_2x^2 \end{pmatrix} = \begin{pmatrix} 1/2+x/2 \\ \frac{1}{2}y+r_2x^2-(3/4+r_2)x+1 \end{pmatrix}.$$

According to Theorem 1, we have $r_1 = 1/4, r_2 = 3/4$. Though numerical simulations, we obtain the image of the quadratic function as shown in Figure 1.

Example 4 *Setting interpolation points: $A(0,0), P_0(1/2,1), B(1,0)$, $d_1 = d_2 = \frac{1}{8}$, $r_1 + r_2 = -1$ are determined by Theorem 2. Then the system (1) can be modeled as:*

$$\omega_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 1-r_1 & 1/8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ r_1x^2 \end{pmatrix} = \begin{pmatrix} x/2 \\ \frac{1}{8}y+r_1x^2+(1-r_1)x \end{pmatrix},$$

$$\omega_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1-r_2 & 1/8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ r_2x^2+1 \end{pmatrix} = \begin{pmatrix} 1/2+x/2 \\ \frac{1}{8}y+r_2x^2-(1+r_2)x+1 \end{pmatrix}.$$

Fig.2 is done for $r_1 = 2/3, r_2 = -5/3$. The parameter r_1 is fixed as $-5/6$ while the value of r_2 is $-1/6$, we get the graph of the cubic function as shown in Figure 3.

4 Conclusion

In this paper, a class of fractal interpolation functions was constructed by using a two-parameter iterated function system. By theory analysis, we obtained the conditions for the polynomial construction of such two-parameter fractal interpolation functions. Numerical evidences was provided to show that the influence of the two-parameter on the form of fractal interpolation functions. It was observed that if the two-parameter have weak linear relationship, the fractal interpolation functions can be represented by quadratic polynomial. By numerical simulations, we also shown that when the vertical scaling factor was constant, meanwhile two displacement parameters keep a linear relationship, the generated fractal interpolation functions was cubic polynomial.

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