

## Multiple Solutions for Nonhomogeneous Elliptic Problem Involving $P$ -Biharmonic Operator and Concave-Convex Nonlinearities

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**Abstract:** This article is devoted to the study of the existence and multiplicity of weak solutions for a  $p$ -biharmonic nonhomogeneous kirchhoff type problem involving convex and concave nonlinearities of the following form

$$\left( -M \int_{\Omega} (|\Delta u|^p + |u|^p) dx \right) (\Delta |\Delta u|^{p-2} \Delta u - |u|^{p-2} u) = \lambda f(x) |u|^{q-2} u + g(x) |u|^{r-2} u + h(x) \text{ in } x \in \Omega,$$

$$u(x) = \nabla u(x) = 0 \text{ on } x \in \partial\Omega,$$

where  $\Omega$  is the complement of a smooth bounded domain  $D$  in  $\mathbb{R}^N (N \geq 3)$ ,  $\lambda > 0$ ,  $M(s) = a + bs^k$ ,  $a, b > 0, k \geq 0$ ,  $f(x), g(x)$  and  $h(x)$  are continuous functions which may change sign on  $\Omega$ . The parameters  $p, q, r$  satisfy  $1 < q < p(k+1) < r < p^* = \frac{Np}{N-p}$ . Using variational approach and applying the Mountain Pass Theorem, the existence and multiplicity of solutions is obtained in the Sobolev space  $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ .

**Keywords:** Existence; Ekeland variational principle;  $P$ -Biharmonic; Kirchhoff-type equation; Nonlocal problems

### 1 Introduction

The purpose of this article is to investigate the existence and multiplicity of positive solutions for the following problem involving concave and convex nonlinearities:

$$\left( -M \int_{\Omega} (|\Delta u|^p + |u|^p) dx \right) (\Delta |\Delta u|^{p-2} \Delta u - |u|^{p-2} u) = \lambda f(x) |u|^{q-2} u + g(x) |u|^{r-2} u + h(x) \text{ in } x \in \Omega,$$

$$u(x) = \nabla u(x) = 0 \text{ on } x \in \partial\Omega, \tag{1}$$

where  $\Omega$  is the complement of a smooth bounded domain  $D$  in  $\mathbb{R}^N (N \geq 3)$  with boundary  $\partial\Omega$ ,  $\lambda > 0$ ,  $a, b > 0, k \geq 0$ ,  $f(x), g(x)$  and  $h(x)$  are continuous functions which may change sign on  $\Omega$ . The parameters  $p, q, r$  satisfy  $1 < q < p(k+1) < r < p^* = \frac{Np}{N-p}$  and the operator  $\Delta(|\Delta u|^{p-2} \Delta u)$  is the  $p$ -biharmonic with  $1 < p < N$ . Problem (1) is called a nonlocal one because of the presence of an integral in  $\Omega$  in (1), the equation is not a point wise identity and therefore it is called a nonlocal problem. The nonlocal property of the problem of interest generates a lot of difficulties, which makes the study of such a problem particularly interesting. The nonlocal differential equations are also known as Kirchhoff type equations have received specific attention in recent years, an important number of surveys dealing with this type of equations can model phenomena which arise from the study of electrorheological fluids or elastic mechanics, in numerous physical phenomena such as systems of particles in thermodynamical equilibrium, dielectric breakdown, image

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restoration and biological phenomena. We may refer to [2–4] as important references in this field. In 1983, Kirchhoff introduced the following general version of the Kirchhoff equation.

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in equation (2) have the following meanings:  $L$  is the length of the string,  $h$  is the area of cross-section,  $E$  is the Young modulus of the material,  $\rho$  is the mass density and  $P_0$  is the initial tension. The considered problem becomes an object of a large interest after J. L. Lions an abstract framework analysis in [6]. Recently, C. Y. Chen et al. in [1] have investigated the multiplicity of solutions to a class of Dirichlet boundary value problems by using Nehari manifold and fibering maps for the case  $p = 2$ . The study of Kirchhoff type equations has already been extended to the case involving  $p$ -Laplacian operator (or details, see [9, 13] ) and  $p(x)$ -Laplacian operator (see [10–12, 14, 15]). But no result has been established for non-homogeneous  $p$ -biharmonic Kirchhoff type problem involving convex and concave nonlinearities in the unbounded domain  $\Omega$ . Since nonhomogeneous term causes difficulty to obtain multiple solutions by Nehari manifold methods so we use variational method and Mountain Pass Theorem. The problem studied is a stationary version of the original Kirchhoff equation involving the  $p$ -biharmonic operator. The question of the existence of weak solutions is treated. By using Ekeland's variational principle and the Mountain Pass Theorem, the existence of multiple solutions for problem (1) are obtained. Note that the weak solutions of problem (1) are considered in the generalized Sobolev space

$$X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Through out this paper, we make the following assumptions:

$$(A1) \quad M(s) = a + bs^k, \quad a, b > 0, \quad 0 \leq k < \frac{p}{N-p}, \quad \forall s \geq 0$$

$$(A2) \quad f(x) \in L^{q_0}(\Omega) \cap L^\infty(\Omega), \quad g(x) \in L^{r_0}(\Omega) \cap L^\infty(\Omega), \quad h(x) \in L^\mu(\Omega) \quad \text{with } q_0 = \frac{p^*}{p^* - q}, \quad r_0 = \frac{p^*}{p^* - r} \quad \text{and} \\ \mu = \frac{p^*}{p^* - 1}. \quad \text{Also, } g(x) > 0 \text{ in non-empty open domain } \Omega_2 \subset \Omega.$$

$$(A3) \quad 1 < q < p < p(k + 1) < r < p^* = \frac{Np}{N-p}$$

Our main result is given as follows.

**Theorem 1** *Suppose that (A1)-(A3) hold true. Then there exist  $\lambda_0, m_0 > 0$  such that for all  $\lambda \in (0, \lambda_0)$ , problem (1) admits atleast two nontrivial weak solutions in  $X$  when  $\|h\|_\mu < m_0$ .*

## 2 Proof of Theorem 1

We discuss the existence of two non-trivial weak solutions of (1) by using the mountain pass theorem of Ambrosetti and Rabinowitz and Ekeland's variational principle. The corresponding Euler lagrange functional associated to (1) is given by

$$I(u) = \frac{1}{p} \hat{M} \int_\Omega \left( |\Delta u|^p + |u|^p \right) dx - \frac{\lambda}{q} \int_\Omega f(x) |u|^q dx - \frac{1}{r} \int_\Omega g(x) |u|^r dx - \int_\Omega h(x) u dx, \tag{3}$$

where  $\hat{M}(t) = \int_0^t M(s) ds$ . It is easy to verify that  $I \in C^1(X, \mathbb{R})$  with norm  $\|u\| = \left( \int_\Omega |\Delta u|^p + |u|^p dx \right)^{1/p}$  is

weakly lower semi-continuous with the derivative given by

$$\begin{aligned} \langle I'(u), v \rangle = & M \left( \int_{\Omega} (|\Delta u|^p + |u|^p) dx \right) \int_{\Omega} \left( |\Delta u|^{p-2} \Delta u \Delta v + |u|^{p-2} uv \right) dx \\ & - \lambda \int_{\Omega} f(x) |u|^{q-2} u dx - \int_{\Omega} g(x) |u|^{r-2} u dx - \int_{\Omega} h(x) v dx, \end{aligned} \quad (4)$$

for all  $u, v \in X$ . Thus, we notice that we can seek weak solutions of (3) as critical point of the energetic functional  $I$ .

Throughout the article, the letters  $c, c_1, C_i, i = 1, 2, \dots$  denote positive constants which may change from line to line.

**Lemma 2** Assume that (A1)- (A3) hold. There exists  $\Lambda_0 > 0$  such that for all  $\lambda \in (0, \Lambda_0)$ , there exist  $\rho > 0$  and  $\alpha > 0$  such that  $I(u) \geq \alpha > 0$  for any  $u \in X$  with  $\|u\| = \rho$ .

**Proof.** By the Holder's inequality, it follows from (A2) that

$$\left| \frac{\lambda}{q} \int_{\Omega} f(x) |u|^q dx \right| \leq \frac{\lambda}{q} \|f\|_{q_0} \|u\|_{p^*}^q \leq \frac{\lambda}{q} S^{-q/p} \|f\|_{q_0} \|u\|^q \quad (5)$$

Similarly,

$$\left| \frac{1}{r} \int_{\Omega} g(x) |u|^r dx \right| \leq \frac{1}{r} \|g\|_{r_0} \|u\|_{p^*}^r \leq \frac{S^{-r/p}}{r} \|g\|_{r_0} \|u\|^r \quad (6)$$

and

$$\int_{\Omega} |h(x)| |u| dx \leq \|h\|_{\mu} \|u\|_{p^*} \leq S^{-1/p} \|h\|_{\mu} \|u\| \leq \epsilon \|u\|^p + C_{\epsilon} \|h\|^{p'} \quad (7)$$

with  $\epsilon > 0, C_{\epsilon} > 0$

Thus,

$$\begin{aligned} I(u) & \geq \frac{a}{p} \|u\|^p + \frac{b}{p(k+1)} \|u\|^{p(k+1)} - \frac{\lambda}{q} S^{-q/p} \|f\|_{q_0} \|u\|^q - \frac{S^{-r/p}}{r} \|g\|_{r_0} \|u\|^r - \epsilon \|u\|^p - C_{\epsilon} \|h\|^{p'}, \\ I(u) & \geq \frac{a}{p} \|u\|^p + \frac{b}{p(k+1)} \|u\|^{p(k+1)} - \lambda C_1 \|u\|^q - C_2 \|u\|^r - \epsilon \|u\|^p - C_{\epsilon} \|h\|^{p'}, \\ I(u) & \geq \frac{b}{p(k+1)} \|u\|^{p(k+1)} - \lambda C_1 \|u\|^q - C_2 \|u\|^r - C_{\epsilon} \|h\|^{p'}, \end{aligned} \quad (8)$$

with  $0 < \epsilon \leq \frac{a}{p}, p' = \frac{p}{p-1}$

Let us define

$$h(t) = \lambda C_1 t^{q-p(k+1)} - C_2 t^{r-p(k+1)}, t > 0 \quad (9)$$

Since the function  $h(t) \rightarrow +\infty$  whenever  $t \rightarrow 0^+$  or  $t \rightarrow +\infty$ . So that the function  $h(t)$  minimizes at some point  $t_1 > 0$ . In order to find  $t_1$ , we obtain the  $h'(t)$  at point  $t_1$ . so that

$$t_1 = \frac{\lambda C_1 (p(k+1) - q)^{1/(r-q)}}{C_2 (r - p(k+1))} \equiv \lambda^{1/(r-q)} C_3^{1/(r-q)} > 0. \quad (10)$$

Futhermore  $h(t_1) < b/p(k + 1)$  implies that

$$h(t_1) = C_1(r - q)(r - m)^{-1}C_3^{(q-m)(r-q)} < b/p(k + 1). \tag{11}$$

Therefore, it follows from (7) and (10) that there exist  $\wedge_0, m_0, \rho, \alpha > 0$  suficiently small such that  $I(u) \geq \alpha$  with  $\lambda \in (0, \wedge_0)$ , and  $\|h\|_\mu \leq m_0$ . ■

**Lemma 3** *There exists  $\phi \in X$   $\phi \neq 0$  such that  $\lim_{t \rightarrow +\infty} I(t\phi) \rightarrow -\infty$ .*

**Proof.** Let  $\phi \in C_0^\infty(\Omega_0), \phi \geq 0$  and there exist  $x \in \Omega_0^c = \Omega \setminus \overline{\Omega_0^c}$  such that for any  $x \in \Omega_0^c$  we have  $\phi(x) = 0$

$$I(t\phi) = \frac{at^p}{p}\|\phi\|^p + \frac{bt^{p(k+1)}}{p(k+1)}\|\phi\|^{p(k+1)} - \frac{\lambda t^q}{q} \int_\Omega f(x)|\phi|^q dx - \frac{t^r}{r} \int_\Omega g(x)|\phi|^r dx - t \int_\Omega h(x)\phi dx$$

where  $\Omega_0 \subset \Omega_2$  is a bounded domain. Since  $q < p(k + 1) < r$ , we have  $I(t\phi) \rightarrow -\infty$  as  $t \rightarrow +\infty$ . Therefore, there exist  $t_1 \in X$  large enough, such that  $I(t\phi) < 0$ .■

By Lemmas 1 and 2 and the mountain pass theorem of Ambrosetti and Rabinowitz [8], we deduce the existence of a sequence  $(u_n)$  such that

$$I(u_n) \rightarrow c \text{ and } I'(u_n) \rightarrow 0 \text{ in } X^* \tag{12}$$

We prove that  $(u_n)$  is bounded in  $X$ . Assume for the sake of contradiction, if necessary to a subsequence, still denote by  $(u_n), \|u_n\| \rightarrow \infty$  and  $\|u_n\| > 1$  for all  $n$ .

For  $n$  large enough, we have

$$c + 1 + \|u_n\| \geq I(u_n) - \frac{1}{r}\langle I'(u_n), u_n \rangle = \left(\frac{1}{p} - \frac{1}{r}\right)a\|u_n\|^p + \left(\frac{1}{p(k+1)} - \frac{1}{r}\right)b\|u_n\|^{p(k+1)} + \lambda\left(\frac{1}{r} - \frac{1}{q}\right)C_1\|u_n\|^q + \left(1 - \frac{1}{r}\right)C_3\|u_n\|c + 1 + \|u_n\| \geq \frac{ar - ap}{pr}\|u_n\|^p. \tag{13}$$

Dividing the above inequality by  $\|u_n\|^p$  and passing to the limit as  $n \rightarrow \infty$  we obtain a contradiction. It follows that by  $\|u_n\|$  is bounded in  $X$ . This information implies that there exists a subsequence, still denoted by  $(u_n)$ , and  $u_1 \in X$  such that  $u_n \rightharpoonup u_1$  in  $X$ .

Let  $B_\rho$  is an open ball centered at the origin and of radius  $\rho$ . Since  $f(x) \in L^{q_0}(\Omega)$ , for every  $\epsilon > 0$ , there exists  $\rho_0 > 0$  such that

$$\int_{\Omega_\rho^c} |f(x)|^{q_0} < \epsilon, \text{ for } \rho \geq \rho_0 \tag{14}$$

where  $\Omega_\rho$  is the complement of a smooth bounded domain  $D$  in open ball  $B_\rho$  and  $\Omega_\rho^c = \Omega \setminus \Omega_\rho$ . Let  $\rho$  be so large that  $D \subset B_\rho$ . Using Sobolev compact embedding theorem in the bounded domain  $\Omega_\rho$ ,  $(u_n)$  has a subsequence, still denoted by  $(u_n)$ , and  $u_1$  such that  $u_n \rightarrow u_1$  in  $L^q(\Omega_\rho)$ .

Since  $f(x)$  is bounded in  $\Omega_\rho$  and  $(u_n)$  is bounded in  $L^{p^*}(\Omega)$  then by Holder's inequality, we have

$$\int_\Omega |f(x)||u_n - u_1|^q dx \leq \|f(x)\|_\infty \int_{\Omega_\rho} |u_n - u_1|^q dx + \left(\int_{\Omega_\rho^c} |f(x)|^{q_0} dx\right)^{1/q_0} \left(\int_{\Omega_\rho^c} |u_n - u_1|^{p^*} dx\right)^{q/p^*} \rightarrow 0 \text{ as } n \rightarrow \infty \tag{15}$$

So we deduce that using Holder's inequality

$$\int_\Omega |f(x)||u_n|^{q-2}|u_n||u_n - u_1| dx \leq \left(\int_\Omega |f||u_n - u_1|^q dx\right)^{1/q} \left(\int_\Omega |f||u_n|^q\right)^{(q-1)/q} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{16}$$

Similarly, we can derive that

$$\int_{\Omega} |g(x)| |u_n|^{r-2} |u_n| |u_n - u_1| dx \rightarrow 0, \int_{\Omega} |h(x)| |u_n - u_1| dx \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (17)$$

Also using the fact  $I'(u_n) \rightarrow 0$  in  $X^*$  implies that

$$\begin{aligned} \langle I'(u_n), u_n - u_1 \rangle &= (a + b \|u_n\|^{pk}) \int_{\Omega} |\Delta u_n|^{p-2} \Delta u_n \Delta(u_n - u_1) dx - \lambda \int_{\Omega} f(x) |u_n|^{q-2} u_n (u_n - u_1) dx \\ &\quad - \int_{\Omega} g(x) |u_n|^{r-2} u_n (u_n - u_1) dx - \int_{\Omega} h(x) (u_n - u_1) dx \rightarrow 0. \end{aligned} \quad (18)$$

Then it follows from (15) and (16)-(17) that

$$(a + b \|u_n\|^{pk}) \int_{\Omega} |\Delta u_n|^{p-2} \Delta u_n \Delta(u_n - u_1) dx \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (19)$$

Futhermore, the fact  $u_n \rightarrow u_1$  in  $X$  implies

$$(a + b \|u_n\|^{pk}) \int_{\Omega} |\Delta u_1|^{p-2} \Delta u_1 \Delta(u_n - u_1) dx \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (20)$$

From (18)- (19) , we have

$$(a + b \|u_n\|^{pk}) \int_{\Omega} (|\Delta u_n|^{p-2} \Delta u_n - |\Delta u_1|^{p-2} \Delta u_1) \Delta(u_n - u_1) dx \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (21)$$

So we deduce that  $u_n \rightarrow u_1$  in  $X$ . Then by relation (11) we have

$$I(u_1) = c \geq 0 \text{ and } I'(u_1) = 0, \quad (22)$$

that is,  $u_1$  is a non-trivial weak solution of (1) . Now we seek for the solution  $u_2$ .

**Lemma 4** *There exists  $\phi_1 \in X, \phi_1 \neq 0$  such that  $I(t\phi_1) < 0$  for  $t > 0$  small enough.*

**Proof.** Let  $\phi_1 \in C_0^\infty(\Omega)$ , then for small  $t > 0$

$$I(t\phi_1) = \frac{at^p}{p} \|\phi_1\|^p + \frac{bt^{p(k+1)}}{p(k+1)} \|\phi_1\|^{p(k+1)} - \frac{\lambda t^q}{q} \int_{\Omega} f(x) |\phi_1|^q dx - \frac{t^r}{r} \int_{\Omega} g(x) |\phi_1|^r dx - t \int_{\Omega} h(x) \phi_1 dx < 0.$$

By Lemma 1 it follows that on the boundary of the ball centered at the origin and of radius  $\rho$  in  $X$ , we have

$$\inf_{u \in \partial B_\rho} I > 0. \quad (23)$$

Moreover, for  $u \in B_\rho \subset X$ , we have

$$-\infty < c_1 = \inf_{B_\rho} I < 0.$$

We let now  $0 < \epsilon < \inf_{u \in \partial B_\rho} I - \inf_{u \in B_\rho} I$  as  $\epsilon \downarrow 0$ . Applying Ekeland's variational principle [7] to the functional  $I: \overline{B_\rho} \rightarrow \mathbb{R}$ , we find  $u \in \overline{B_\rho}$  such that

$$I(u_\epsilon) < c_1 + \epsilon \quad \text{and} \quad I(u_\epsilon) < I(u) + \epsilon \|u - u_\epsilon\|, \quad u \neq u_\epsilon. \quad (24)$$

Then it follows that

$$I(u_\epsilon) < c_1 + \epsilon \leq \inf_{B_\rho} I + \epsilon < \inf_{\partial B_\rho} I,$$

we deduce that  $u_\epsilon \in B_\rho$ . Now, we define the functional  $K: B_\rho \rightarrow \mathbb{R}$  by  $K(u) = I(u) + \epsilon \|u - u_\epsilon\|$ . It is clear that  $u_\epsilon$  is a minimum point of  $K$ . Moreover

$$\frac{K(u_\epsilon + tv) - K(u_\epsilon)}{t} \geq 0, \text{ for small } t > 0 \text{ and } \forall v \in B_{\rho_1}. \tag{25}$$

Hence,

$$\frac{I(u_\epsilon + tv) - I(u_\epsilon)}{t} + \epsilon \|v\| \geq 0, \tag{26}$$

Letting  $t \rightarrow 0$  it follows that

$$\langle I'(u_\epsilon), v \rangle + \epsilon \|v\| \geq 0 \quad \forall v \in B_\rho. \tag{27}$$

Replacing  $v$  in (26) by  $-v$ , we get

$$-\langle I'(u_\epsilon), v \rangle + \epsilon \|v\| \geq 0 \quad \forall v \in B_\rho. \tag{28}$$

So that  $\|I'(u_\epsilon)\| \leq \epsilon$ . We deduce that there exists a sequence  $(v_n) \subset B_\rho$  such that

$$I(v_n) \rightarrow c_1 \text{ and } I'(v_n) \rightarrow 0. \tag{29}$$

It is clear that  $(v_n)$  is bounded in  $X$ . Thus, there exists  $u_2 \in X$  such that, up to a subsequence,  $(v_n)$  converges weakly to  $u_2$  in  $X$ . Actually, with similar arguments as those used in the proof that the sequence  $u_n \rightarrow u_1$  in  $X$  we can show that  $v_n \rightarrow u_2$  in  $X$ . Thus, by relation (28),

$$I(u_2) = c_1 < 0 \text{ and } I'(u_2) = 0, \tag{30}$$

i.e.,  $u_2$  is a non-trivial weak solution of (1). Finally, since

$$I(u_1) = c > 0 > c_1 = I(u_2), \tag{31}$$

we see that  $u_1 \neq u_2$ . Thus, problem (1) has two non-trivial weak solutions. Then the proof of Theorem 1 is complete. ■

### 3 Conclusions

In this article, we have considered the study the existence and multiplicity of weak solutions for non-homogenous  $p$ -biharmonic Kirchhoff type problem involving convex and concave nonlinearities. From the results, we conclude that posed problem admits atleast two nontrivial weak solutions in  $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ . Since nonhomogeneous term causes difficulty to get multiple solutions by Nehari manifold methods so we proposed variational approach using Mountain Pass Theorem. The outputs in this paper lay the foundation for developing variational approach for more general and complicated nonlocal problems both stationary and evolutionary.

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