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Consequences of Providing Alternative Food to Predator in an Exploited Prey Predator System Controlled by Optimal Taxation

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Abstract: In this paper we develop a simple prey-predator model where an additional food source exists and there is a dynamic correspondence between net economic revenue and the cost of harvesting of prey species in presence of taxation as a controlling parameter. In both exploited and unexploited system we investigate the existence of possible steady states and their local and global stability. We establish that, addition of alternative food source makes a positive impact on harvesting. As, modeled prey species is subjected to harvest, abundance of alternative prey species helps to sustain the predator population. The catchability coefficient is one of the main factor that plays a pivotal role for hopf bifurcation in the present model. Transversality conditions for Hopf bifurcation are also discussed. The main objective is to set up the whole dynamics in a way such that predator's growth is sustainable and also maximum profit is obtained from harvesting which is controlled by imposing tax. For this purpose, an optimal taxation policy is formulated and solved by Pontryagin's maximum principle. Some numerical simulations and graphical results are given for better understanding of species interaction.

Keywords: Harvesting; Alternative food source; Hopf bifurcation; Optimal taxation

1 Introduction

The applications of mathematical ecology create a great response among researchers for studying the development of frequently used biological and natural resources. Behind all these things there is a major factor, the economic achievement of the society. So, we have to keep in mind that to construct a system where economic profit comes from an environment, where all the valuable species has sustainable persistent development(other than any invasive or prey species). For this purpose, Scientist and researchers make a connection between mathematics and biology which starts a new area of research 'Mathematical Bio-economics'.

After the pioneering work of Lotka-Volterra many researchers enriched this field as per existing literature [1], [2], [3], [4]. Many times it has been observed in nature that instead of single spices, most of the times predator consume several species. As a consequence, complex food web evolves in nature [5]. Sometimes it has been observed that prey species has a tendency of migration, then an alternative food source must exist for sustaining the predator population. The consequences of the presence of alternative food source to the predator and its effect and utility on the prey predator dynamics becomes an important area of research in mathematical biology.

It is well accepted that, alternative prey creates a positive impact for sustaining predator population. In this context one mentionable work of Abrams and Routh [6] where a theoretical model is established as model of additional tropic levels. But, in this study we establish the effect of alternative prey in an exploited prey predator system where harvesting is controlled by imposing tax. Sometimes it may be happened that, as alternative prey consume natural resources for their growth, therefore due to less availability of resources, number of target prey diminished. Again, some authors like Baalen et al. [7], Rijn et al. [8] conclude that, by increasing the number of predators with the help of alternative food

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source ultimately reduces the focal prey population. But, according to, Harwood and Obryeki [9], Holt and Lowton [10], Wootton [11] addition of alternative food to the predator does not always increase the predation of modeled prey species. One relevant work is done by Spencer and Collie [12] where they established the fact considering intraspecific competition in predatory fish the existence of alternative food source can increase predator abundance when modeled prey abundance is low. Srinivasu et. al. [13] also considered the quality of food provided in a two dimensional prey predator system. In this continuation, significant impact of alternative food source in exploited prey-predator system has been studied extensively by Pahari et. al. ([14]) and Kar et. al. ([15]).

To maintain the ecological equilibrium there must be some regulatory system for exploitation of natural and biological resources. Here, we consider harvesting effort is applied on prey species only and taxation as controlling instrument. Harvesting plays a key role in the prey-predator dynamics. It has been shown many times by changing the applied harvesting strategies, long run population density are significantly changed. The motivation behind the harvesting of multi-species fisheries *i. e.* the objective and purpose are discussed in detail by Clark [16] and also draw the attention to the combined harvesting of ecologically interdependent species. Harvesting problem in a prey-predator system has been studied by, Kar and Matsuda [17], Kar et al. [18], Chaudhury [19], Chaudhury and Johnson [20], Dubey et. al. [21] etc. Dai and Tang [22], Myerscough et. al. [23], Xiao and Ruan [24] studied the behavior of an exploited prey predator system with constant rate of harvesting. Some relevant work on this topic has been done by Kar et. al. [25], [26]. They investigated a dynamic reaction system where harvesting is controlled by imposing tax and also established alternative food source played a key role, for certain value of alternative food source hopf bifurcation occurs. Dynamics of inshore offshore fishing model where impulsive diffusion occurs has been studied by Zhang et. al. [27]. It is obvious that, for the case of open access fishery, harvesting effort expand or contract according as the total perceived rent to the fisherman is positive or negative. Thus, our considered model indicates a correlation between fishing effort and the net economic revenue. So, depending on demand of harvesting biomass in society the total system change instantaneously. Model with this characteristics is known as dynamic reaction model.

The main purpose of this work is

(I) to formulate a proper taxation policy to achieve the best profit to the society with sustainable species development,

(II) to study the consequences of adding additional food source in the model and its effect in maintaining the dynamics of an exploited prey-predator system, which is an extension of traditional two species model, alternative prey species does not make it complex like three species food chain model. In existing literature we have found many researchers like Bake [28] and Gakkhar et. al. [29] studied the dynamics of three species food chain model.

In addition to the above literature survey, now we mention a particular example which also motivate us to consider an exploited prey predator system with alternative prey and taxation. International whaling commission has ordered a complete ban on killing whales. Whales predate on krills which are heavily harvested nowadays. Thus, to ensure the growth of whales the authority try to control the harvesting of krills (may be by applying high taxation). In addition, increase in the availability of crustaceans and squid, alternative food source of whales, may also be helpful for the growth of whales.

First we have shown the positivity and the boundedness of the system considered. Then we investigate sectionally starting from unexploited system and determine the possible steady states. Then, we consider the effect of harvesting and after finding the equilibrium points we determine the criteria for local and global stability and check the boundedness of the system. Then we develop the criteria for Hopf bifurcation due to the parameter which is the catchability coefficient. Then optimal taxation policy is formulated and solved by Pontryagin's Maximum Principle. Some numerical verification is given for better understanding of key findings and to illustrate most of the analytical result. Lastly some brief conclusion is given.

2 Model Formulation

Initially let us look the following model as,

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{\alpha Sxy}{1+x}-qEx \\ \frac{dy}{dt} &= \frac{\beta\alpha Sxy}{1+x}+(1-S)\,y-my^2 \end{aligned} \tag{1}$$

where x and y denotes prey and predator population size respectively at any time t. Prey species grow logistically with intrinsic growth rate r, k is the environmental carrying capacity of the prey population. Here, we take Holling-type-II response function to the predator. β ($\beta < 1$) is the conversion rate at which prey biomass converted to predator biomass. E denotes the harvesting effort applied to the prey and q is catchability coefficient. Density dependent mortality rate my^2 is considered for the predators as the nonlinear dependency gives the resultant effect of predation of any super predator species (if exist) and competition among predators. [30]. It is well accepted that a linear term my caused to unstable equilibrium followed by a limit cycle therefore the incorporation of linear term is not appropriate for many marine species. Whenever abundance of the predator species is high density dependent starts to play an important role towards the stability of the system considered. S is a constant fraction which is time-independent and incorporates alternative food source. Here we assume, predator consume S portion from focal prey species and $1 - S$ portion from alternative food sources. Therefore it is readily understood that, when $S = 0$, the two species become independent to each other and hence the system dynamics is neither relevant not revealing. Similarly, when $S = 1$; then the predator species will survive only on the predator species. Therefore, we have to consider $0 < S < 1$ for studying exploited prey predator system with alternative food source.

Here we follow the catch-per-unit-effort hypothesis which implies catch per unit effort is proportional to the stock level for the harvesting of the prey population. We consider tax τ as a control technique imposed by the regulatory agency for the harvesting of the prey species. Any subsidy to the fisherman may be considered as negative value of τ. The perceived rent of the fisherman is given by $E[q(p - \tau)x - c]$, where p is the price per unit harvested biomass of the prey population and c is the cost of unit harvesting effort. In an open access fishery, in a fully dynamic model the level of fishing effort increased or decreased according to the perceived rent to the fisherman is positive or negative. Therefore, the harvesting effort is a dynamic variable governed by the differential equation,

$$\frac{dE}{dt} = \gamma \left[q(p - \tau)x - c\right] E \tag{2}$$

Here, γ is the stiffness parameter measuring the intensity of the relationship between harvesting effort and perceived rent.

Therefore, finally the equations of the model become,

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{k}\right) - \frac{\alpha Sxy}{1+x} - qEx \\ \frac{dy}{dt} &= \frac{\beta\alpha Sxy}{1+x} + (1 - S)\, y - my^2 \\ \frac{dE}{dt} &= \gamma\left[q(p - \tau)x - c\right] E \end{aligned} \tag{3}$$

where, $x = x_0 \geq 0$, $y = y_0 \geq 0$, $E = E_0 \geq 0$.

3 Positivity and Boundedness

In this section we are trying to establish the conditions under which the system (3) are positive and bounded.

Theorem 1 *All possible solutions of the system (3) are positive.*

Proof. From the first equation of (3) we can write,

$$\frac{dx}{x} = \left\{ r(1 - \frac{x}{k}) - \frac{\alpha Sy}{1+x} - qE \right\} dt$$

which on integration gives,

$$x(t) = x_0 e^{\int \phi(x,y)dt} > 0 \quad \forall\ t$$

where,

$$\phi(x, y) = \left\{ r(1 - \frac{x}{k}) - \frac{\alpha Sy}{1+x} - qE \right\}$$

Again, from second equation of (3) we can write,

$$\frac{dy}{y} = \left\{ \frac{\beta\alpha Sx}{1+x} + 1 - S - my \right\} dt$$

which on integration gives,

$$y(t) = y_0 e^{\int \psi(x,y)dt} > 0 \quad \forall \ t$$

where,

$$\psi(x,y) = \left\{ \frac{\beta\alpha S x}{1+x} + 1 - S - my \right\}$$

Again from the third equation of (3) we can write,

$$\frac{dE}{E} = \{(p-\tau)qx - c\}\, dt$$

Integrating we get,

$$E(t) = E_0 e^{\int \eta(x)dt} > 0 \quad \forall \ t$$

where,

$$\eta(x) = \{(p-\tau)qx - c\}$$

Hence, it can be concluded that all the solutions of the system (3) are positive. ■

In the next theorem, we try to find some conditions under which the solutions of the system (3) is bounded.

Theorem 2 *If $S < 1$ then the solutions of the system (3) are bounded above.*

Proof.

From the first equation of (3) we can write that,

$$x(t) \leq k \quad \forall \ t$$

From the second equation of (3) we can write,

$$y(t) \leq \frac{1-S}{m} \quad \forall \ t$$

Again from the first equation we get,

$$r - \frac{rx(t)}{k} - qE(t) \geq 0$$

Again, from above we can write,

$$E(t) \leq \frac{r}{q} \quad \forall \ t$$

Therefore, all the solutions of the system (3) are bounded if $S < 1$. ■

Note: Here we use the term S to incorporate additional food source. Whenever $S < 1$ there is always a presence of alternative food source. So, in order to obtain a bounded positive solution of the system (3) harvesting effort must be less than bio-technical productivity and predators must get an origin of alternative food source simultaneously.

4 Case 1: Unexploited System

In this case we consider, $E = 0$. Then the system of equations (3) becomes,

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{k}\right) - \frac{\alpha S x y}{1+x} \\ \frac{dy}{dt} &= \frac{\beta\alpha S x y}{1+x} + (1-S)\,y - my^2 \end{aligned} \tag{4}$$

The system (4) has four equilibrium points $P_0\,(0,0)$, $P_1\,(k,0)$, $P_2\left(0, \frac{1-S}{m}\right)$ and $P_3\,(\hat{x},\hat{y})$, where,

$$\hat{y} = \frac{1}{m}\left[\frac{\beta\alpha S\hat{x}}{1+\hat{x}} + 1 - S\right] \tag{5}$$

where $\hat{x}$ satisfies the equation,

$$x^3 + (2-k)\,x^2 + \left((1-2k) + \frac{\alpha S k}{rm}\,(\beta\alpha S + 1 - S)\right)x + \frac{\alpha S k}{rm}\,(1-S) - k = 0 \tag{6}$$

To analyze the dynamic behavior of the system (4) we have to find the eigen value of the equilibrium points mentioned above.

For the point $P_0\,(0,0)$, the eigen values are given by, $r, 1-S$. As previously we assume $0 < S < 1$, both the eigen value is positive. Hence P_0 is unstable.

For the point $P_1\,(k,0)$, the eigen values are given by, $-r$, $\frac{\alpha\beta S k}{1+k} + 1 - S$, therefore P_1 is also unstable.

For the point $P_2\,\left(0, \frac{1-S}{m}\right)$ the eigen values are given by, $-m, r - \frac{\alpha S(1-S)}{rm}$. If, $\frac{\alpha}{rm} > \frac{1}{S(1-S)}$, then P_2 becomes stable.

The cubic equation (6) has at least one positive real root if, $\frac{\alpha}{rm} < \frac{1}{S(1-S)}$. Therefore the system (4) has a unique interior equilibrium point $P_2\,(\hat{x},\hat{y})$ if and only if $\frac{\alpha}{rm} < \frac{1}{S(1-S)}$.

For the point $P_3\,(\hat{x},\hat{y})$ the variational matrix of the system (4) is,

$$J = \begin{bmatrix} r - \frac{2r\hat{x}}{k} - \frac{\alpha S\hat{y}}{(1+\hat{x})^2} & \frac{\alpha S\hat{x}}{1+\hat{x}} \\ \frac{\beta\alpha\hat{y}}{(1+\hat{x})^2} & -m\hat{y} \end{bmatrix}$$

We obtain,

$$Tr(J) = r - \frac{2r\hat{x}}{k} - \frac{\alpha S\hat{y}}{(1+\hat{x})^2} - m\hat{y} \tag{7}$$

and

$$Det(J) = -\left[r - \frac{2r\hat{x}}{k} - \frac{\alpha S\hat{y}}{(1+\hat{x})^2}\right] m\hat{y} - \frac{\alpha^2\beta S^2\hat{x}\hat{y}}{(1+\hat{x})^3} \tag{8}$$

Now, $Tr(J) < 0$ if, $r\left(1 - \frac{2\hat{x}}{k}\right) < \left(\frac{\alpha S}{(1+\hat{x})^2} + m\right)\hat{y}$ and $Det(J) > 0$ if, $r\left(1 - \frac{2\hat{x}}{k}\right) < \frac{\alpha S}{(1+\hat{x})^2}\left[\hat{y} - \frac{\alpha\beta S\hat{x}}{m(1+\hat{x})}\right]$

Thus if, $r\left(1 - \frac{2\hat{x}}{k}\right)$ <min[$\left(\frac{\alpha S}{(1+\hat{x})^2} + m\right)\hat{y}$, $\frac{\alpha S}{(1+\hat{x})^2}\left(\hat{y} - \frac{\alpha\beta S\hat{x}}{m(1+\hat{x})}\right)$]

Then, $P_3(\hat{x},\hat{y})$ is locally asymptotically stable.

Now, we will investigate the global stability of $P_3(\hat{x},\hat{y})$.

Theorem 3 *If α <min$\left[\frac{rm}{S(1-S)}, \frac{S}{2m}\right]$, then the interior equilibrium point$P_3(\hat{x},\hat{y})$ is globally asymptotically stable.*

Proof. Let us define, $H(x,y) = \frac{1}{xy}$. Clearly, $H > 0$ whenever $x > 0, y > 0$

Let,

$$f_1(x,y) = rx\left(1 - \frac{x}{k}\right) - \frac{\alpha Sxy}{1+x}$$
$$f_2(x,y) = \frac{\beta\alpha Sxy}{1+x} + (1-S)\,y - my^2$$

Now,

$$\begin{aligned} &\nabla(x,y) \\ &= \frac{\partial}{\partial x}(f_1 H) + \frac{\partial}{\partial y}(f_2 H) \\ &= -\frac{r}{ky} + \frac{\alpha S}{(1+x)^2} - \frac{m}{x} \\ &< 0 \end{aligned}$$

If, $\frac{\alpha S}{(1+x)^2} < \frac{m}{x}$. *i.e.*, $m\left(x^2+1\right) + (2m - \alpha S)\,x > 0$, *i.e.*, $\alpha < \frac{S}{2m}$. Again $P_3(x,y)$ exists if, $\alpha < \frac{rm}{S(1-S)}$.

Hence, if α <min$\left[\frac{rm}{S(1-S)}, \frac{S}{2m}\right]$ then, $\nabla(x,y) < 0$

We have shown in the previous section that $P_3(x,y)$ is locally asymptotically stable. Now, by Benedixson-Dulac criterion $P_3(x,y)$ is globally asymptotically stable if $\alpha < min\left[\frac{rm}{S(1-S)}, \frac{S}{2m}\right]$. This completes the proof. ■

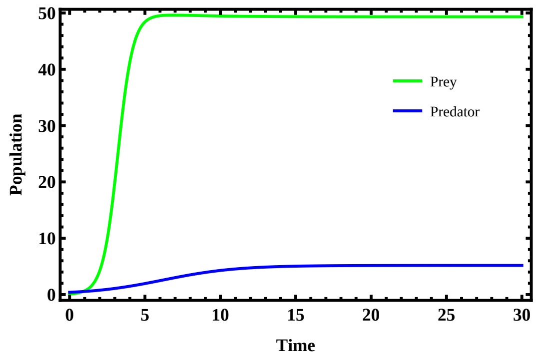

Figure 1: The prey-predator trajectory for the model system (4) for r = 2, k = 50, α =0.4, β =0.25, S =0.65, m = 0.08 depicts

5 Case 2: Effect of Harvesting without assuming it as a dynamic variable

In this section we consider a simple exploited prey predator model, where harvesting is not changing instantaneously with time, instead of a dynamic reaction model. So, now we look into the previously considered model system 1 which has four equilibrium points, which are

(i)$D_0(0,0)$, (ii)$D_1(0,\frac{1-S}{m})$, (iii)$D_2(k(1-\frac{E}{\frac{r}{q}}),0)$, (iv)$D_3(\hat{x},\hat{y})$, where,

$$\hat{y}=\frac{1}{m}\left[\frac{\beta\alpha S\hat{x}}{1+\hat{x}}+1-S\right]$$

and $\hat{x}$ satisfies the equation,

$$x^3+\left(2-k+\frac{Eqk}{r}\right)x^2+\left(1-2k+\frac{\alpha Sk}{rm}\left(\beta\alpha S+1-S\right)+\frac{2Eqk}{r}\right)x+\frac{\alpha Sk}{rm}\left(1-S\right)-k+\frac{kE}{\frac{r}{q}}=0 \tag{9}$$

These equilibrium points D_1 and D_2 exist whenever, $S<1$ and $E<\frac{r}{q}$. $\frac{r}{q}$ is well understood as biotechnical productivity, *i.e.* the equilibrium point D_2 exists if biotechnical productivity is greater than harvesting effort. From (9) the cubic equation has at least one positive real root if,

$$\frac{\alpha}{rm}<\frac{1-\frac{E}{\frac{r}{q}}}{S\left(1-S\right)}. \tag{10}$$

Therefore the system (1) has a unique interior equilibrium point $D_3\left(\hat{x},\hat{y}\right)$ if and only if (10) holds. As harvesting effort is less than biotechnical productivity, comparing with model system (4), inclusion of harvesting on focal prey species increases the existence region of interior equilibrium point in parametric space. The Fig. (2) and (4) are the phase plane diagram for the model system (4) and (1) respectively. Green dots represent the possible equilibrium and their stability are shown by the direction of the vector field. In Fig. (2)[a]-[b] we observe prey and predator equilibrium density decreases as S is increases from 0.05 to 0.45, both the equilibrium density for the prey predator population decreases. But further increase in S in Fig. (2)[c], results in sudden increase of the prey population, but predator is still decreasing. The same observation is obtained from Fig.(4[a]-[c]) as well. Thus, one can observe inclusion of harvesting effort only decrease the prey equilibrium density as usual but does not alter the dynamics of the system. From the Fig. (1) and (3) we observe equilibrium density of the prey population decreases, this happens due to applied harvesting effort. But, predator equilibrium density remains unchanged *i. e.* not affected by harvesting. This happens due to the presence of alternative food source.

6 Case 3: Effect of Harvesting assuming it as dynamic variable

The system (3) has six possible equilibrium points:

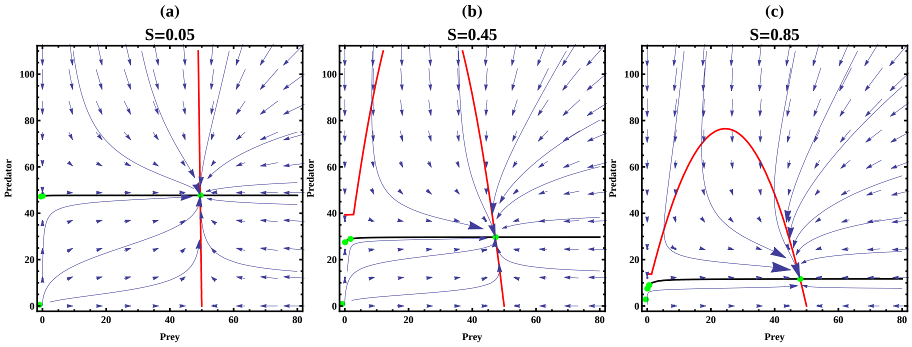

Figure 2: The phase plane diagram for the model system (4) for r = 2, k = 50, α =0.4, β =0.25, m = 0.08 depicts

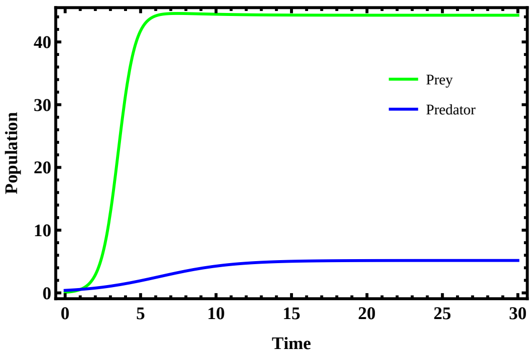

Figure 3: The equilibrium density of prey and predator population for the system (1) for r = 2, k = 50, α =0.4, β =0.25, S =0.65, q = 0.02, m = 0.08, E = 10 demonstrates

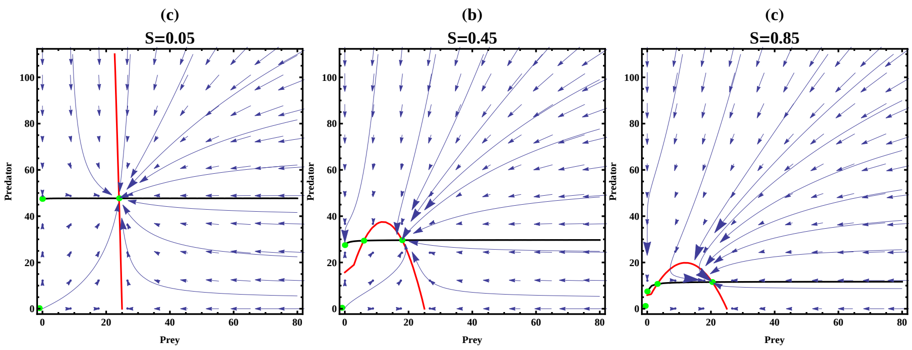

Figure 4: The phase plane diagram for the model system (1) for r = 2, k = 50, α =0.4, β =0.25, m = 0.08, q = 0.02, m = 0.08, E = 10 depicts

$(i)E_0(0,0,0)$, $(ii)E_1(k,0,0)$, $(iii)E_2(0,\frac{1-S}{m},0)$, $(iv)E_3(\frac{c}{(p-\tau)q},0,\frac{r}{q}(1-\frac{c}{(p-\tau)qk}))$, $(v)E_4(\hat{x},\hat{y},0)$, where, $\hat{y}=\frac{r}{k\alpha S}(k-\hat{x})(1+\hat{x})$ and $\hat{x}$ is the unique positive root of the equation,

$$mrx^3+(2-k)\,mrx^2+(mr(1-2k)+k\alpha S(1-S)+\alpha\beta S)\,x+k\alpha S(1-S)-mrk=0$$

$(vi)E_5(x^*,y^*,E^*)$ where,

$$x^*=\frac{c}{(p-\tau)q}$$

$$y^*=\frac{1}{m}(1-S+\frac{\beta\alpha Sc}{(p-\tau)q+c})$$

$$E^*=\frac{1}{q}\left[r(1-\frac{c}{(p-\tau)qk})-\frac{\alpha S(p-\tau)q}{m((p-\tau)q+c)^2}((1-S)((p-\tau)q+c)+\beta\alpha Sc)\right] \tag{11}$$

Now we try to find the different conditions under which the above steady states exist.

The points $E_0(0,0,0)$ and $E_1(k,0,0)$ will always exist. The points E_2 and E_4 exist if $0<S<1$ and $\alpha<\frac{rm}{S(1-S)}$ hold. Now, in case of taxation, we assume obviously, $p>\tau>0$. E_3 exists if

$$\tau<p-\frac{c}{qk} \tag{12}$$

which is required for a fishery to sustain afterwards. The interior equilibrium point E_5 exists if,

$$r>\frac{rc}{(p-\tau)qk}+\frac{\alpha S(p-\tau)q}{m((p-\tau)q+c)^2}$$

After some calculations we obtain,

$$(c+\rho)^2(c-\rho k)mr+S\rho^2k\alpha<0 \tag{13}$$

where, $\rho=p-\tau$. All the terms in (13) except $c-\rho k$ is positive, *i. e.* the fact is $c-\rho k$ becomes negative enough to make the left hand side of (13) negative.

Let us assume, $c-\rho k=-b$, where, b is positive. Therefore, from (13), we obtain,

$$(\frac{c}{\rho}+1)^2>\frac{Sk\alpha}{bmr} \tag{14}$$

Using (12) and (14) we obtain,

$$\rho>max[\frac{c}{qk},\frac{c}{\sqrt{\frac{Sk\alpha}{bmr}}-1}] \tag{15}$$

i. e., $p-\tau>max[\frac{c}{qk},\frac{c}{\sqrt{\frac{Sk\alpha}{bmr}}-1}]$.

One can verify that it is very difficult to find any suitable range of tax analytically. So, we attempt to find it numerically. In Fig. (5) we have plotted harvesting effort as a function of tax and it depicts that at $\tau=5$ harvesting effort vanishes.

Before, analyzing the stability of the model we show that the solution of the model is bounded in a finite region containing $(x(0),y(0),E(0))$.

Theorem 4 *All the solutions of the system (3) starts in R^3 are uniformly bounded.*

Proof. Let, $x(t),y(t),E(t)$ be any solution of the system with positive initial conditions. We define the function,

$$W=x+\frac{y}{\beta}+\frac{E}{\gamma(p-\tau)} \tag{16}$$

Therefore, time derivative is found to be,

$$\begin{aligned}\frac{dW}{dt}=&\frac{dx}{dt}+\frac{1}{\beta}\frac{dy}{dt}+\frac{1}{\gamma(p-\tau)}\frac{dE}{dt}\\ &=rx(1-\frac{x}{k})+\frac{1-S}{\beta}y-\frac{m}{\beta}y^2-\frac{c}{p-\tau}E\end{aligned}$$

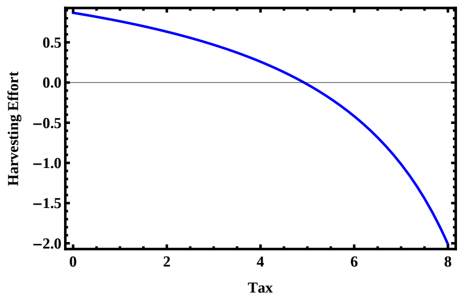

Figure 5: The above curves represents how harvesting effort varies with tax for the system (3) for r = 2, k = 50, α =0.4, β =0.25, S =0.65, q = 0.02, m = 0.08, p= 11, c = 5, γ = 0.71

Now,

$$\frac{dW}{dt} + vw = rx(1-\frac{x}{k}) + \frac{1-S}{\beta}y - \frac{m}{\beta}y^2 - \frac{c}{p-\tau}E + vx + \frac{vy}{\beta} + \frac{vE}{\gamma(p-\tau)}$$

Then taking $c = v$, we have

$$\frac{dW}{dt} + vW = rx(1-\frac{x}{k}) + \frac{1-S}{\beta}y - \frac{m}{\beta}y^2 + vx + \frac{vy}{\beta} \leq \frac{k}{2r}(r+v)^2 + \frac{1}{2m}(1-S+v)^2$$

Thus we obtain,

$$\frac{dW}{dt} + vW \leq \Omega \tag{17}$$

where,

$$\Omega = \frac{k}{2r}(r+v)^2 + \frac{1}{2m}(1-S+v)^2$$

Applying the theory of differential inequality(Birkoff and Rota) [31] we obtain,

$$0 \leq W(x,y,E) \leq \frac{\Omega}{v}\left(1-e^{-vt}\right) + W\left(x(0),y(0),E(0)\right)e^{-vt} \tag{18}$$

which upon letting $t \to \infty$ yields,

$$0 \leq W \leq \frac{\Omega}{v} \tag{19}$$

Therefore, all the solutions of the system (3) that starts in R_3^+ are confined in the region $D = (x,y,E) \in R_3^+ : 0 \leq W \leq \frac{\Omega}{v} + \epsilon$, for any $\epsilon > 0$. ■

7 Local Stability Analysis

For the point $(0,0,0)$ the eigen values are given by, $r, 1-S, -\gamma c$. Therefore, it is unstable.

For the point $(k,0,0)$ the eigen values are given by, $-r$, $\frac{\beta\alpha Sk}{1+k} + 1 - S, \gamma[(p-\tau)qk - c]$. For existence we have, $1-S > 0$. So, it is unstable.

For the point $(0, \frac{1-S}{m}, \frac{1}{q}(r - \frac{\alpha S(1-S)}{m}))$ the eigen values are given by, $0, -(1-S), -\gamma c$. It is also unstable.

For the point $(\frac{c}{(p-\tau)q}, 0, \frac{r}{q}(1-\frac{c}{(p-\tau)qk}))$ the characteristic equation is given by,

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where,

$$a_1 = \frac{rc}{(p-\tau)qk}$$

$$a_2 = \frac{\gamma rc}{k(p-\tau)}((p-\tau)qk - c)$$

$$a_3 = \frac{\alpha Sc\gamma r}{k(c+(p-\tau)q)}(1 - S + \frac{\beta\alpha Sc}{c+(p-\tau)q})((p-\tau)qk - c)$$

Then, clearly, $a_i > 0, (i = 1, 2, 3)$. and, $a_1a_2 - a_3 > 0$ if, $\frac{rc}{k(p-\tau)}\left(1 - \frac{c}{(p-\tau)qk}\right) > \frac{\alpha S((p-\tau)qk-c)}{c+(p-\tau)q}(1 - S + \frac{\beta\alpha Sc}{c+(p-\tau)q})$

For the point $(\hat{x}, \hat{y}, 0)$, one eigen value of the variational matrix is given by, $(p-\tau)q\hat{x} - c$, which is negative if, $\tau > p - \frac{c}{q\hat{x}}$.

The other two eigenvalues are given by the quadratic equation,

$$\lambda^2 + s\lambda + t = 0$$

where,

$$s = m\hat{y} - r + \frac{2r\hat{x}}{k} + \frac{\alpha S\hat{y}}{(1+\hat{x})^2}$$

$$t = \frac{\alpha^2 S^2 \beta\hat{x}\hat{y}}{(1+\hat{x})^3} - m\hat{y}(r - \frac{2r\hat{x}}{k} - \frac{\alpha S\hat{y}}{(1+\hat{x})^2})$$

The sign of the real part of the eigenvalues are determined by s. Now, $s > 0$ if $2\hat{x} > k$. Therefore, the equilibrium point $(\hat{x}, \hat{y}, 0)$ is locally asymptotically stable if, $2\hat{x} > k$ and $\tau > p - \frac{c}{q\hat{x}}$. Now, for the interior equilibrium point $E_5(x^*, y^*, E^*)$ the characteristic equation of the corresponding variational matrix is given by,

$$\lambda^3 + v_1\lambda^2 + v_2\lambda + v_3 = 0 \tag{20}$$

where,

$$v_1 = \frac{rx^*}{k} + \frac{my^*}{k} - \frac{\alpha Sx^*y^*}{k(1+x^*)^2} \tag{21}$$

$$v_2 = mrx^*y^* + \frac{S\alpha x^*}{(1+x^*)^2}(\frac{S\alpha\beta}{1+x^*} - my^{*2}) + E^*q^2x^*\gamma(p-\tau) \tag{22}$$

$$v_3 = mq^2x^*y^*E^*\gamma(p-\tau) \tag{23}$$

The Routh-Hurwitz criterion gives a set of necessary and sufficient conditions so that all the roots of the characteristic equation have negative real parts. For the above cubic equation these criteria are $v_1 > 0$, $v_3 > 0$ and $v_1v_2 - v_3 > 0$. We find that if,

$$rx^* + my^* > \frac{S\alpha x^*y^*}{(1+x^*)^2} \tag{24}$$

then $v_1 > 0$, and in case of taxation, we assume $p > \tau$ then $v_3 > 0$.

Now,

$$\begin{aligned} v_1v_2 - v_3 = &\frac{1}{(1+x^*)^5}[mx^*y^*(1+x^*)(r(1+x^*)^2 - y^*\alpha S)^2 + \\ &(r(1+x^*)^2 - y^*\alpha S)(m^2y^{*2}(1+x^*)^3 + x^*(\alpha^2\beta S^2 \\ &+ q^2\gamma E^*(1+x^*)^3(p-\tau))) + m(1+x^*)^2y^*(\alpha^2\beta S^2 \\ &- (1+x^*)^3(k-1)q^2\gamma E^*(p-\tau))] \end{aligned}$$

Hence, $v_1v_2 - v_3 > 0$ if,

$$r(1+x^*)^2 > y^*\alpha S$$

and,

$$\alpha^2\beta S^2 > (1+x^*)^3(k-1)q^2\gamma E^*(p-\tau) \tag{25}$$

Therefore, by the Routh Hurwitz criterion, we say that, (24) and (25) are the sufficient conditions for the local asymptotic stability of the nontrivial steady state $E_5(x^*, y^*, E^*)$. Thus we have the following theorem.

Theorem 5 *If the carrying capacity* $k > 1$ *and* $r(1+x^*)^2 > y^*\alpha S$*, the interior equilibrium point* $E_5(x^*, y^*, E^*)$ *is locally asymptotically stable if* $rx^* + my^* > \frac{S\alpha x^* y^*}{(1+x^*)^2}$ *and* $\alpha^2\beta S^2 > (1+x^*)^3(k-1)q^2\gamma E^*(p-\tau)$ *holds together.*

From the point of ecological management, globally stable equilibrium point is more desirable in order to make a sustainable harvesting strategy.

8 Global stability analysis

The question of global stability in population biology is a very interesting mathematical problem. Usually, a biologists believes that a unique positive, locally asymptotically stable equilibrium in an ecological system is global stable. It is well accepted that, analysis of the global stability of the equilibrium point is essential for the better understanding of stability of ecological systems.

Let us consider the Lyapunov function,

$$\begin{aligned} V(x,y,E) =& p_0\left[x-\hat{x}-\hat{x}ln\frac{x}{\hat{x}}\right]+p_1\left[y-\hat{y}-\hat{y}ln\frac{y}{\hat{y}}\right]+ \\ & p_2\left[E-\hat{E}-\hat{E}ln\frac{E}{\hat{E}}\right] \end{aligned}$$

on $D = (x, y, E) : x > 0, y > 0, E > 0$, where p_0, p_1, p_2 are positive constants to be determined in subsequent steps. It can be easily verified that the function V is zero at the equilibrium $(\hat{x}, \hat{y}, \hat{E})$ and positive on D.

The time derivative of V along the trajectories of (3) is,

$$\begin{aligned} \frac{dV}{dt} =& p_0\frac{x-\hat{x}}{x}\frac{dx}{dt}+p_1\frac{y-\hat{y}}{y}\frac{dy}{dt}+p_2\frac{E-\hat{E}}{E}\frac{dE}{dt} \\ =& p_0\left[x-\hat{x}\right]\left[r(1-\frac{x}{k}-\frac{\alpha S y}{1+x}-qE)\right] \\ &+p_1\left[y-\hat{y}\right]\left[\frac{\beta\alpha S x}{1+x}+(1-S)-my\right] \\ &+p_2\gamma\left[E-\hat{E}\right]\left[(p-\tau)qx-c\right] \end{aligned}$$

A little manipulation yields

$$\begin{aligned} \frac{dV}{dt} =& -p_0\frac{r}{k}(x-\hat{x})^2+p_0\frac{\alpha S(x-\hat{x})^2\hat{y}}{(1+x)(1+\hat{x})} \\ &-p_0\alpha S\frac{(x-\hat{x})(y-\hat{y})}{1+x}-p_0 q(x-\hat{x})(E-\hat{E}) \\ &+p_1\alpha\beta S\frac{(x-\hat{x})(y-\hat{y})}{(1+x)(1+\hat{x})} \\ &-p_1 m(y-\hat{y})^2+p_2\gamma q(p-\tau)(x-\hat{x})(E-\hat{E}) \end{aligned}$$

If we choose $p_0 = 1$, $p_1 = \frac{1+\hat{x}}{\beta}$, $p_2 = \frac{1}{\gamma(p-\tau)}$ then we have,

$$\begin{aligned} \frac{dV}{dt} =& -\frac{r}{k}\left[x-\hat{x}\right]^2+\frac{\alpha S\hat{y}}{(1+x)(1+\hat{x})}\left[x-\hat{x}\right]^2- \\ & \frac{m(1+\hat{x})}{\beta}\left[y-\hat{y}\right]^2 \\ =& -\left[\frac{r}{k}-\frac{\alpha S\hat{y}}{(1+x)(1+\hat{x})}\right]\left[x-\hat{x}\right]^2-\frac{m(1+\hat{x})}{\beta}\left[y-\hat{y}\right]^2 \end{aligned} \tag{26}$$

The coefficient of $(x-\hat{x})^2$ in (26) is,

$$\begin{aligned} R(x) =& -\left[\frac{r}{k}-\frac{\alpha S\hat{y}}{(1+x)(1+\hat{x})}\right] \\ &\leq -\left[\frac{r}{k}-\frac{\alpha S\hat{y}}{(1+\hat{x})}\right] \\ &= -R_0, (say) \end{aligned}$$

where, $R_0=\left[\frac{r}{k}-\frac{\alpha S\hat{y}}{(1+\hat{x})}\right]$ Thus if, $R_0>0$ then, $\frac{dV}{dt}<0$. Now, $R_0>0$ gives,

$$\frac{r}{k}>\frac{\alpha S\hat{y}}{(1+\hat{x})} \tag{27}$$

Therefore, if (27) holds then the equilibrium point E_5 is globally asymptotically stable.

Therefore, we have the following theorem.

Theorem 6 *If $\frac{r}{k}>\frac{\alpha S\hat{y}}{(1+\hat{x})}$ holds, the interior equilibrium point $E_5(\hat{x},\hat{y},\hat{E})$ is globally asymptotically stable.*

9 Optimal harvesting policy

The main objective of the regulatory agency is to maximize the total discounted net revenues that the society obtained from the fishery. Therefore our objective is to maximize the present value J of a continuous time stream of revenues given by,

$$J=\int_0^\infty e^{-\delta t}[pqx-c]Edt$$

where, δ is the instantaneous annual rate of discount, c is the fishing cost per unit effort and p is the price per unit harvested biomass and q is the catchability coefficient. Here we use Pontryagin's maximum principle to solve this optimization problem. We take τ as control variable and wish to determine a tax policy $\tau=\tau(t)$ which maximizes J subject to the system (3).

The Hamiltonian of this problem is given by,

$$\begin{aligned} H =& e^{-\delta t}[pqx-c]E+\lambda_1\left[rx(1-\frac{x}{k})-\frac{\alpha Sxy}{1+x}-qEx\right]+ \\ &\lambda_2\left[\frac{\beta\alpha Sxy}{1+x}+(1-S)y-my^2+\right] \\ &\lambda_3\gamma\left[(p-\tau)qx-c\right]E \end{aligned}$$

where $\lambda_1,\lambda_2,\lambda_3$ are adjoint variables, here we maximize the Hamiltonian for τ. Assuming that the control constraints are not binding (*i.e* the optimal solution does not occur at $\tau=\tau_{min}$ or $\tau=\tau_{max}$), we have singular control given by, $\frac{\partial H}{\partial\tau}=0$.

Now, $\frac{\partial H}{\partial\tau}=0$ gives

$$\lambda_3 qEx=0$$

We use a singular control and for this we take,

$$\lambda_3=0 \tag{28}$$

Now the adjoint equations are given by,

$$\frac{d\lambda_1}{dt}=-\frac{\partial H}{\partial x}=-e^{-\delta t}pqE-\lambda_1[r-\frac{2rx}{k}-\frac{1}{(1+x)^2}-qE]-\lambda_2\frac{1}{(1+x)^2}-\lambda_3\gamma(p-\tau)qE \tag{29}$$

$$\frac{d\lambda_2}{dt}=-\frac{\partial H}{\partial y}=\lambda_1\frac{\alpha Sx}{1+x}-\lambda_2\left[\frac{\beta\alpha Sx}{1+x}+1-S-2my\right] \tag{30}$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial E} = -e^{-\delta t}\left[pqx - c\right] + \lambda_1 qx - \lambda_3\gamma\left[(p-\tau)qx - c\right] \tag{31}$$

Using (28) from (31) we obtain,

$$\lambda_1 qx = e^{-\delta t}(p - \frac{c}{qx}) \tag{32}$$

Here, we deal with optimal equilibrium solution, so x, y, E are treated to be constants in the subsequent steps.

Using (32) in (30) we obtain,

$$\begin{aligned}\frac{d\lambda_2}{dt} &= e^{-\delta t}(p - \frac{c}{qx})\frac{\alpha Sx}{1+x} - \lambda_2\left[\frac{\beta\alpha Sx}{1+x} + 1 - S - 2my\right]\\ &= P_0 e^{-\delta t} + Q_0\lambda_2\end{aligned}$$

where,

$$P_0 = (p - \frac{c}{qx})\frac{\alpha Sx}{1+x}$$

$$Q_0 = -\left[\frac{\beta\alpha Sx}{1+x} + 1 - S - 2my\right]$$

Therefore,

$$\lambda_2 = -\frac{P_0}{Q_0+\delta}e^{-\delta t} + Me^{Q_0 t}$$

where M is a constant. The shadow price $\lambda_2 e^{-\delta t}$ is bounded as $t \to \infty$, iff $M = 0$.

Therefore,

$$\lambda_2 = \frac{\frac{\alpha Sx}{1+x}(p - \frac{c}{qx})}{\frac{\beta\alpha Sx}{1+x} + 1 - S - 2my - \delta}e^{-\delta t} \tag{33}$$

Using (32), (33) into (29) we obtain,

$$pqE + \left[(p - \frac{c}{qx})(r - \frac{2rx}{k} - \frac{1}{(1+x)^2} - qE)\right] + \frac{\alpha Sx\left(p - \frac{c}{qx}\right)}{\left(\beta\alpha Sx + (1+x)(1 - S - 2my - \delta)\right)(1+x)^2} = 0 \tag{34}$$

Now for the optimal equilibrium solution we have from (3),

$$\begin{aligned} r\left(1 - \frac{x}{k}\right) - \frac{\alpha Sy}{1+x} - qE = 0\\ \frac{\beta\alpha Sx}{1+x} + (1 - S) - my = 0\\ \gamma\left[q(p-\tau)x - c\right] = 0\end{aligned} \tag{35}$$

$$x = \frac{c}{(p-\tau)q} = \frac{c}{Tq} \tag{36}$$

$$y = \frac{1}{m}\left[\frac{\beta\alpha Sc}{c+Tq} + 1 - S\right] \tag{37}$$

$$E = \frac{1}{q}\left[r(1 - \frac{c}{Tqk}) - \frac{\alpha STq}{m}(\frac{\beta\alpha Sc}{(c+Tq)^2} + \frac{1-S}{c+Tq})\right] \tag{38}$$

where, $p - \tau = T$, Using (36), (37), (38) in (34), we obtain,

$$A_1T^5 + A_2T^4 + A_3T^3 + A_4T^2 + A_5T + A_6 = 0 \tag{39}$$

where,

$$A_1 = kq^4(\alpha S(1-S) - m(1+\delta))D$$

$$A_2 = kmpq^4(1-r+\delta)D - ckmq^3\alpha S(1+\beta+\beta\delta)+ ckq^3((2\alpha(1-S)S+\alpha^2\beta S^2)D+\alpha^2\beta S^2(1-S))- cmq^3(k+r+3k\delta)D$$

$$A_3 = ckmpq^3((1-3r+3\delta)D+\alpha S(1+\beta)-\alpha\beta S(r-\delta))- c^2mq^2(3(r+k\delta)D+\alpha\beta S(r+2k\delta))+ c^2kq^2((\alpha(1-S)S+\alpha^2\beta S^2(1-S))D+\alpha^2\beta S^2(1-S+\alpha\beta S))$$

$$A_4 = c^2kmpq^2((\delta-r)(3D+2\alpha\beta S)-c^3mq((3r+k\delta)D+\alpha\beta S(k\delta+2r)) \; 2c^2mqr\alpha\beta S(pq-c)$$

$$A_5 = c^3mpqr((6-k)D+\alpha\beta S(4-k))+c^3m(kpq\delta-c^3r)(D+\alpha\beta S)$$

$$A_6 = 2c^4mpr(D+\alpha\beta S)$$

where

$$D=(1-S+\delta)$$

Let, $T^*(=p-\tau^*)$ be a solution of the equation (39) for which the corresponding tax τ^* is in the admissible set of controls. Then, the corresponding optimal equilibrium solution $(x(\tau^*), y(\tau^*), E(\tau^*))$ can be obtained from the equations (36), (37), (38) respectively.

10 Bifurcation Analysis

Every parameter in prey predator models has its own contribution to the stability of the system as well as equilibrium density of the species. It has been found many times that a little change in some parameter may cause a abrupt change in the behavior of the system, or the system can looses its stability as a certain parameter passes through its critical value. This critical value for which such transition occurs is called Bifurcation point. Assuming, q as the bifurcation parameter we analyze the bifurcation of the model system (3).

The characteristic equation (20) has two purely imaginary roots if and only if,

$$v_1v_2 - v_3 = 0 \tag{40}$$

This happens for a unique value of q (say q^*) at which we have a Hopf Bifurcation. Thus, in the neighborhood of q^* the equation (20) can't have any real roots. For $q = q^*$, we have,

$$(\lambda^2+v_2)(\lambda+v_1)=0$$

Then the roots are in general of the form:

$$\lambda_1(q^*) = z_1(q^*)+iz_2(q^*)\lambda_2(q^*) = z_1(q^*)-iz_2(q^*)\lambda_3(q^*) = -v_1(q^*)$$

To apply Hopf bifurcation theorem as stated in Liu's criterion [32–34] we need to verify the transversality condition,

$$\left[\frac{dz_1}{dq}\right]_{q=q^*} \neq 0$$

Substituting $\lambda_1(q^*) = z_1(q^*)+iz_2(q^*)$ in equation (20) and differentiating the resultant equation w. r. t. q, and setting $z_1(q^*)=0$ and $z_2(q^*)=\sqrt{v_2}$, we get the transversality condition at $q=q^*$ as,

$$\left[\frac{dz_1}{dq}\right]_{q=q^*} = \left[-\frac{v_1v_2^{'}+v_1^{'}v_2-v_3^{'}}{2(v_2^2+v_1^2v_2)}\right]_{q=q^*}$$

To establish the Hopf bifurcation we just need to show that,

$$v_1v_2^{'}+v_1^{'}v_2-v_3^{'} \neq 0$$

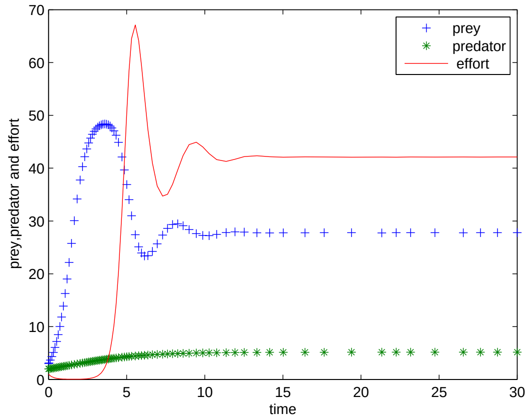

Figure 6: The above curves depicts the trajectory of prey predator and harvesting effort of the system (3) for r = 2, k = 50, α =0.4, β =0.25, S =0.65, q = 0.02, m = 0.08, p= 11, c = 5, γ = 0.71, τ = 2

where, $v_1^{'} = \frac{dv_1}{dq}$,$v_2^{'} = \frac{dv_2}{dq}$, $v_3^{'} = \frac{dv_3}{dq}$.

Here, $v_1^{'} = \frac{dv_1}{dq} = 0$, $v_2^{'} = \frac{dv_2}{dq} = 2q\gamma x^* E^*(p-\tau)$, $v_3^{'} = \frac{dv_3}{dq} = 2mqx^*y^*E^*\gamma(p-\tau)$.

Therefore,

$$v_1^{'}v_2 + v_1v_2^{'} - v_3^{'} \quad = 2x^*E^*q\gamma(p-\tau)\left[\frac{rx^*+my^*}{k} - \frac{\alpha Sx^*y^*}{k(1+x^*)^2} - my^*\right] \tag{41}$$

Hence, we have the following theorem.

Theorem 7 *If the point $P(x^*, y^*, E^*)$ exists with $rx^* + my^* > \frac{\alpha Sx^*y^*}{(1+x^*)^2} + mky^*$, then a simple Hopf bifurcation occurs at the unique positive value $q = q^*$.*

11 Numerical Simulation and Discussion

Here, we represent some numerical results with graphical representations in order to discuss the inherent characteristics of the model considered. As provision of alternative food source is one of the important consideration in our model, so we try to understand how the total system changes with different values of S, the intake rate of predator from focal prey.

Here, we set the following set of parameters: r = 2, k = 50, α =0.4, β =0.25, S =0.65, q = 0.02, m = 0.08, p= 11, c = 5, γ = 0.71, τ = 2 in appropriate units. Then we obtain the Interior equilibrium at $(27.7778, 5.1593, 42.1138)$, which is found to be stable from figure (6) as in time series plot three curves converges in its equilibrium density for the model system (3).

In Figs.(7), (8), (9) we have shown the variation of harvesting effort, predator and prey with taxation. It occurs naturally if a fisherman have to pay a more tax then he reduces fishing effort.

The Figs. (10) and (11) represents how harvesting effort, predator population and prey responses with the most influential parameter S, the predator's intake rate of focal prey. As stated earlier, predator population decreases with increase of S, but the noticeable fact is the prey and harvesting effort equilibrium density remains almost same at high and low value of S as shown in Figs. (10) and (11). This is due to the fact that in reality predator population depend on alternative prey more than focal prey as combined effect of harvesting effort on prey and Holling type-(II) response function for predation of focal prey.

Table (1) shows how optimal tax and corresponding focal prey, predator and harvesting effort changes with different value of S. It has been observed that, with increasing value of S, optimal level of focal prey is decreasing, as a consequence optimal harvesting effort decreases and optimal tax decreases initially. Due to reduction of optimal harvesting effort, the optimal focal prey increased with increasing tax. But predator population decreases throughout as S increases. It has been shown earlier that, the same characteristic is obtained whenever we consider both the exploited and unexploited

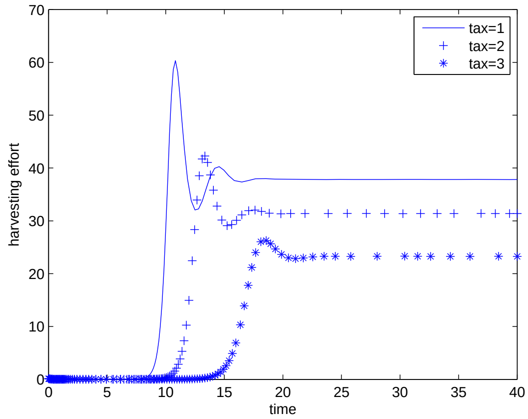

Figure 7: The figure demonstrates how harvesting effort changes for different tax level of the system (3) for r = 2, k = 50, α =0.4, β =0.25, S =0.65, q = 0.02, m = 0.08, p= 11, c = 5, γ = 0.71

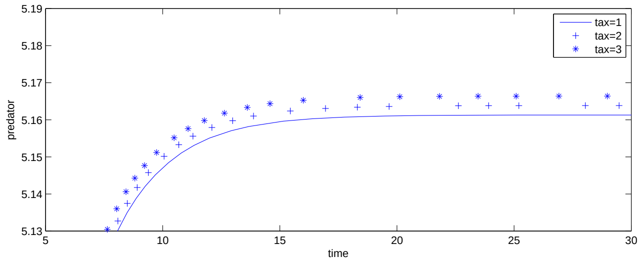

Figure 8: This shows how predator population varies with different tax level of the system (3) for r = 2, k = 50, α =0.4, β =0.25, S =0.65, q = 0.02, m = 0.08, p= 11, c = 5, γ = 0.71

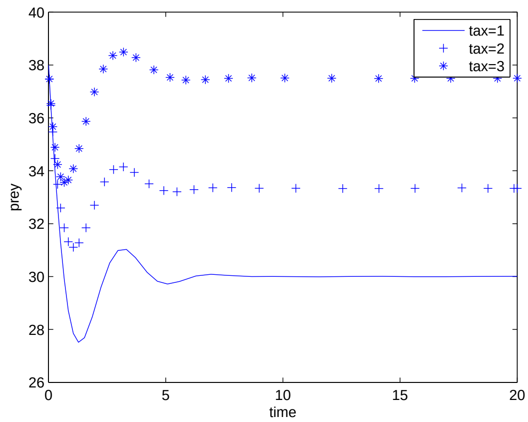

Figure 9: The figure depicts the variation of the prey population against time for different tax level of the system (3) for r = 2, k = 50, α =0.4, β =0.25, S =0.65, q = 0.02, m = 0.08, p= 11, c = 5, γ = 0.71

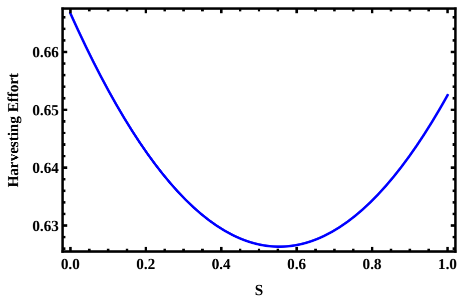

Figure 10: The figure demonstrates how harvesting effort changes with S of the system (3) for r = 2, k = 50, α =0.4, β =0.25, q = 0.02, m = 0.08, p= 11, c = 5, γ = 0.71, τ = 2

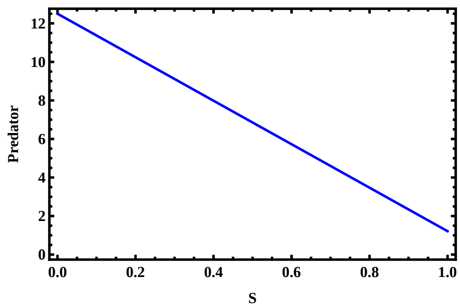

Figure 11: The figure shows how predator population changes with S for the system (3) for r = 2, k = 50, α =0.4, β =0.25, q = 0.02, m = 0.08, p= 11, c = 5, γ = 0.71, τ = 2

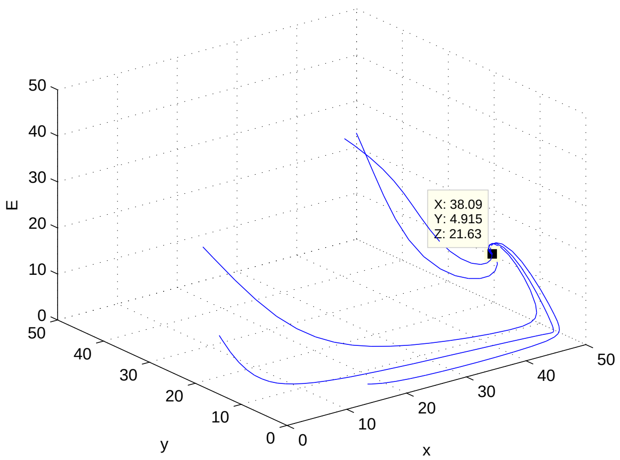

Figure 12: That the phase portrait corresponding the optimal tax τ^* = 3.17352 is stable for the system (3) for r = 2, k = 50, α =0.4, β =0.25, S =0.65, q = 0.02, m = 0.08, p= 11, c = 5, γ = 0.71 shows

Table 1: Table for optimal tax, prey predator and effort corresponding to different values of S

S	τ	x	y	E
0.1	3.2071	38.4964	11.2695	22.4365
0.2	3.1909	38.4167	10.0391	22.1478
0.3	3.1791	38.3589	8.8085	21.9395
0.4	3.1718	38.3231	7.5779	21.8121
0.5	3.1691	38.3098	6.3475	21.7657
0.6	3.1709	38.3188	5.1169	21.8007
0.7	3.1773	38.3501	3.8864	21.9171
0.8	3.1881	38.4031	2.6559	22.1154
0.9	3.2029	38.4758	1.4254	22.3983

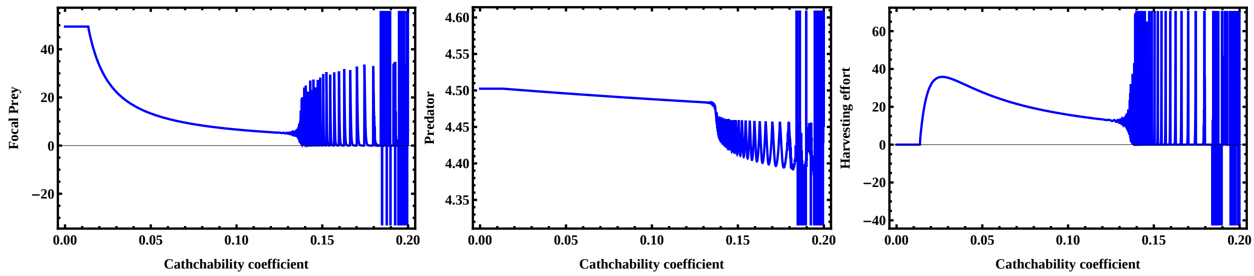

Figure 13: The above curves depicts how prey predator and Harvesting Effort varies with q, the catchability coefficient of the system (3) for r = 2, k = 50, α =0.4, β =0.25, S =0.65, m = 0.08, p= 11, c = 5, γ = 0.71, τ = 2

two dimensional system. The interaction of predator to the focal prey and alternative prey is independent of the applied harvesting strategy on focal prey species.

Now we attempt to visualize the effect of density dependent mortality rate on the dynamics of model system (3). So, for this purpose let us contract an another table.

Remark 8 *Behind this interactions, the possible considerations are, (i) From table (2) it is clear that, as predator's density dependent mortality rate increases, the equilibrium density of prey remains constant with increasing density of harvesting effort. It may be explained as, decreasing predator population and increasing harvesting effort may leads to obtain the predator population as bycatch product. Whenever predators come in the neighborhood of focal prey to attack it, predators are not able to escape from harvesting effort and consequently bycatch phenomena occurs and predator population decreases.*

(ii) There may be some disturbances (as for example noise) due to application of harvesting effort on focal prey species. i. e. these are the probable biological factor behind the fact that high intake of focal prey by predator makes a negative impact on the growth of predator population.

For numerical simulation we have choose, r = 2, k = 50, α =0.4, β =0.25, q = 0.02, m = 0.08, p= 11, c = 5, γ = 0.71, δ = 0.01 in appropriate units. Solving the equation (39) with this set of parameters we obtain the resulting value of T = 7.82648, for this we obtain the optimal value for τ as τ^*= 3.1781. For this optimal tax we obtain the the optimal equilibrium solution as (38.3314, 4.5017, 21.8493), which is found to be stable from the figure (12).

Table 2: Table for equilibrium density for prey predator and effort corresponding to different values of m, density dependent mortality rate of predator and all other parameters are kept unchanged

m	x	y	E
0.02	35.2941	20.6605	22.0115
0.08	35.2941	5.1651	27.5617
0.15	35.2941	2.7547	28.4251
0.25	35.2941	1.6528	28.8197

Using the set of parameters r = 2, k = 50, α =0.4, β =0.25, S =0.65, m = 0.08, p= 11, c = 5, γ = 0.71, τ = 2, in equation (40) we obtain the critical value of q, (q^*, say) as 0.119, as q passes through its critical value the system losses its stability. Thus, in figure (13) we have found how the trajectory of prey, predator and harvesting effort varies abruptly with q, the catchability coefficient.

12 Conclusion

In the present paper we consider an exploited prey predator system with Holling type-II functional response and harvesting on prey species only. An alternative food source is provided to the predator and the total system is controlled by imposing tax. But, it differs in some aspects from previous studies. Here we have established that, harvesting of an economically valuable prey species become easier if predators are not facing any difficulties to predate from a species or several species which is less economically important. If we create an abundance of such species which are easily accessable to the predator and less economic as well, then it is possible to obtain a long run sustainable environment for the prey species which is subject to harvest along with sustainability of all other species in the eco-system. It is interesting to observe that, addition of alternative prey makes the system (3) more realistic and above that taxation is used as controlling parameter. This work analyze the combined effect of the presence of alternative prey to the predator and taxation on harvested biomass of the focal prey population. It is evident that, catchability coefficient plays an important role in the dynamics of the proposed system. The catchability coefficient can cause a stable equilibrium to unstable and a simple hopf bifurcation occurs as it passes through its critical value. Imposing tax per unit biomass of the landed prey has been adopted here as a mechanism to stop indiscriminate killing of the prey. We have given here a finite range of tax and regulatory agency must follow this parametric condition at the time of imposing tax. We obtain a stable limit cycle when interior equilibrium is unstable. The total dynamics based on the quantity of prey biomass available for harvesting and the amount of the prey caught by the predator. It has been shown that, it is possible to control the system such that a sustainable required state can be achieved using the taxation as control parameter.

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