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Hopf Bifurcation Analysis for a Stage-Structure Predator-Prey System with Monod-Haldane Type Response Function and Time Delay

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Abstract: This paper is concerned with stage-structure predator-prey system with time delay due to the gestation of the predator and Monod-Haldane response function. Firstly, the local stability of the interior equilibrium is investigated by analyzing the corresponding characteristic equations. Furthermore, the stability and direction of the Hopf bifurcation at the interior equilibrium are researched by using the normal form theory and center manifold theorem. We find that time delay has important influences on the stability and direction of Hopf bifurcation. When time delay is less than the bifurcation point, the interior equilibrium is asymptotically stable. When time delay is greater than the bifurcation point, the interior equilibrium is unstable. Finally, numerical simulations are given to verify the above results.

Keywords: Predator-prey system; Stage-structure; Time delay; Stability; Hopf bifurcation

1 Introduction

Predator-prey system, an important and popular prototype model appearing in various fields of mathematical biology, has been studied extensively in different forms. The first predator-prey model was introduced by Lotka and Volterra in 1925 and 1926, respectively [1, 2]. Further, the existence and direction of Hopf bifurcation at the interior equilibrium are studied [3–6]. In addition, there are many factors that affect dynamic properties of predator-prey system such as the time delay, stage-structure, and function response, etc, which can lead to complex dynamical behaviors. By analyzing the predator-prey model, the delay differential equation describes the richer and more complicated dynamic behavior, it can be more comprehensive and reflect the actual phenomena of the population precisely. So it is important to research the influence of delay about system dynamic properties. At present, predator-prey model has been gradually perfected the research of predator-prey with time delay by many authors [7–11]. By analyzing the stage structure, it makes the predator-prey model more complex and it is more practical to consider the predator-prey model with stage structure. The research of stage structure makes the system more detailed and more practical [12–16]. In predator-prey system model, the predator's ability is one of the important factors that effect the population dynamics properties. Functional response function embodies the predator's capturing ability. In 1965, on the basis of experiment, Holling put forward the following three different functional response function: Holling type-I, Holling type-II, Holling type-III [17]. With the expansion of the field of research, Andrews proposed a response function to model in micro-organisms [18]

$$\varphi_1(x) = \frac{ax}{h + \beta x + \frac{x^2}{i}}, \tag{1}$$

which is known as Monod-Haldane response function. Sokol and Howell suggested a simplified Monod-Haldane type or Holling type-IV response function [19]

$$\varphi_2(x) = \frac{ax}{h + \gamma x^2}. \tag{2}$$

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In [19], Eq (2) has proved to be practical in biology.

After that, many scholars combine different functional response functions with other situations. These studies make the biological model more consistent with the actual situations and can better predict the trend of population. Most of researchers only consider model with one or two effects about HollingI II III functional response functions.

In this paper, based on the above discussion, motivated by the work [20], we incorporate stage structure for the predator and the influence about the gestation time delay of the mature predator. We research the following stage-structure predator-prey model with Monod-Haldane type response function and time delay:

$$\begin{cases} \dot{x}(t) = \rho_1 x(t)(1 - \frac{x(t)}{k_1}) - \frac{\beta x(t) y_2(t)}{\phi + \gamma x^2(t)}, \\ \dot{y_1}(t) = \frac{c\beta x(t-\tau) y_2(t-\tau)}{\phi + \gamma x^2(t-\tau)} - (a + \delta_1) y_1(t), \\ \dot{y_2}(t) = a y_1(t) - \delta_2 y_2(t). \end{cases} \tag{3}$$

Where $x(t)$ represents the density of the prey at time t, according to stage-structure theory, we will consider two stages of predator population: immature stage and mature stage, $y_1(t)$ and $y_2(t)$ represent the density of the immature and the mature predator at time t respectively. In this article, we assume that the immature predators have no ability to attack prey species and they have no reproductive capability, they get their living resources from their parents, whereas the mature predators can attack prey species and obtain living resources and they can produce the next generation. The parameters $\rho_1, k_1, \beta, \phi, \gamma, c, a, \delta_1, \delta_2, \tau$ are positive constants in which ρ_1 represents the biotic potential, k_1 is the carrying capacity, β can be defined as the maximal prey species uptake of adult predators, ϕ is the half saturation constant , γ is known as the inverse measure of inhibitory effect, c represents the conversion coefficient from prey to predator and a is the transition rate on the evolution of the immature predators to mature predators, δ_1 is the nature death rate of immature predators and δ_2 is the nature death rate of mature predators, τ represents the time delay due to the gestation of the mature predators.

The initial conditions for system (3) take the form of

$$x(0) > 0, y_1(0) > 0, y_2(0) > 0. \tag{4}$$

According to the fundamental theories of functional differential equations [21], the system (3) has a unique solution $(x(t), y_1(t), y_2(t))$ satisfying initial conditions (4). It is easy to show that all solutions of system (3) corresponding to initial conditions (4) are defined on $[0, +\infty)$ and remain positive for all $\tau \geq 0$.

This paper is organized as follows. In Section 2, we discuss the local stability and analyze the corresponding characteristic equations about the existence of Hopf bifurcation at the interior equilibrium point. In Section 3, by applying the normal form theory and the center manifold theorem by Hassard, we obtain the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions of system (3). In Section 4, numerical simulations are given to verify the theoretical analysis. Finally, a brief conclusion is shown.

2 Local stability and the Hopf bifurcation

In this section, we shall discuss the local stability and the Hopf bifurcation at the interior equilibrium. It is easy to see that system (3) has a unique interior equilibrium $E^*(x^*, y_1^*, y_2^*)$, where $y_1^* = \frac{\delta_2}{a} y_2^*$, $y_2^* = \frac{\rho_1}{\beta}(\phi + r(x^*)^2)(1 - \frac{x^*}{k_1})$, x^* is derived from the following two functional equations $r(x^*)^2 - \frac{ac\beta}{\delta_2(a+\delta_1)} x^* + \phi = 0$. It is clear that x^* has two non negative real roots. y_2^* also can be written $y_2^* = \frac{c\rho_1 a x^*}{\delta_2(a+\delta_1)}(1 - \frac{x^*}{k_1})$. In Section 4, It is easy to prove that y_2^* has a unique non negative real root on the given parameter.

Next, in view of the practical significance of biological field, it is necessary to consider the local stability at the interior equilibrium $E^*(x^*, y_1^*, y_2^*)$ and the conditions of producing Hopf bifurcation.

We can obtain that linearized matrix at $E^*(x^*, y_1^*, y_2^*)$ of system (3), let $J_{(E^*)}$ represents linearized matrix.

$$J_{(E^*)} = \begin{pmatrix} -B_1 & 0 & B_2 \\ B_3 & -B_4 & B_5 \\ 0 & B_6 & -B_7 \end{pmatrix}, \tag{5}$$

when $B_1 = \frac{\rho_1}{k_1} x^* - \frac{2\beta r(x^*)^2 y_2^*}{(\phi + r(x^*)^2)^2}$, $B_2 = -\frac{\beta x^*}{\phi + r(x^*)^2}$, $B_3 = \frac{c\beta\phi y_2^* - c\beta r(x^*)^2 y_2^*}{(\phi + r(x^*)^2)^2} e^{-\lambda\tau}$, $B_4 = a + \delta_1$, $B_5 = \frac{c\beta x^*}{\phi + r(x^*)^2} e^{-\lambda\tau}$, $B_6 = a$, $B_7 = \delta_2$. The linearized characteristic equation of eq (5) is

$$\lambda^3 + \varphi_1 \lambda^2 + \varphi_2 \lambda + \varphi_3 + (\varphi_4 \lambda + \varphi_5) e^{-\lambda\tau} = 0, \tag{6}$$

when
$\varphi_1 = a + \delta_1 + \delta_2 + \frac{\rho_1}{k_1}x^* - \frac{2\beta r(x^*)^2 y_2^*}{(\phi+r(x^*)^2)^2}$,
$\varphi_2 = (a+\delta_1)\delta_2 + (\frac{\rho_1}{k_1}x^* - \frac{2\beta r(x^*)^2 y_2^*}{\phi+r(x^*)^2})(a+\delta_1+\delta_2)$,
$\varphi_3 = (\frac{\rho_1}{k_1}x^* - \frac{2\beta r(x^*)^2 y_2^*}{(\phi+r(x^*)^2)^2})(a+\delta_1)\delta_2$,
$\varphi_4 = -\frac{ac\beta x^*}{\phi+r(x^*)^2}$,
$\varphi_5 = -\frac{ac\rho_1\beta(x^*)^2}{k_1(\phi+r(x^*)^2)} + \frac{ac\beta^2 x^* y_2^*(\phi+r(x^*)^2)}{(\phi+r(x^*)^2)^3}$.
Because the system (3) has time delay, so we consider the following two cases:$\tau = 0 and \tau = 1$.
Case 1: $\tau = 0$.
Then the linearized characteristic eq (6) becomes

$$\lambda^3 + \eta_1\lambda^2 + \eta_2\lambda + \eta_3 = 0, \tag{7}$$

when $\eta_1 = (a+\delta_1+\delta_2) + x^*(\frac{\rho_1}{k_1} - \frac{2\beta r x^* y_2^*}{(\phi+r(x^*)^2)^2})$, $\eta_2 = (a+\delta_1+\delta_2)\delta_2(a+\delta_1)(\frac{\rho_1}{k_1}x^* - \frac{2\beta r(x^*)^2 y_2^*}{\phi+r(x^*)^2)^2}) - \frac{ac\beta x^*}{\phi+r(x^*)^2}$, $\eta_3 = (\delta_2(a+\delta_1) - \frac{ac\beta x^*}{\phi+r(x^*)^2})(\frac{\rho_1}{k_1}x^* - \frac{2\beta r(x^*)^2 y_2^*}{(\phi+r(x^*)^2)^2}) + \frac{ac\beta^2 x^* y_2^*(\phi-r(x^*)^2)}{(\phi+r(x^*)^2)^3}$.
We use H_1, H_2 and H_3 to represent the eigenvalues,

$$H_1 = \begin{vmatrix} \eta_1 \end{vmatrix} = \eta_1,$$

$$H_2 = \begin{vmatrix} \eta_1 & \eta_3 \\ 1 & \eta_2 \end{vmatrix} = \eta_1\eta_2 - \eta_3,$$

$$H_3 = \begin{vmatrix} \eta_1 & \eta_3 & 0 \\ 1 & \eta_2 & 0 \\ 0 & \eta_1 & \eta_3 \end{vmatrix} = \eta_3 H_2,$$

$\eta_1\eta_2 - \eta_3 = (a+\delta_1+\delta_2)(\frac{\rho_1}{k_1}x^* - \frac{2\beta r(x^*)^2 y_2^*}{(\phi+r(x^*)^2)^2)}) + (a+\delta_1+\delta_2)(\delta_2(a+\delta_1) - \frac{ac\beta x^*}{\phi+r(x^*)^2}) + \frac{ac\beta^2 x^* y_2^*}{(\phi+r(x^*)^2)^3}(\phi - r(x^*)^2) + (a+\delta_1)(a+\delta_1+2\delta_2+\delta_2^2)(\frac{\rho_1}{k_1}x^* - \frac{2\beta r(x^*)^2 y_2^*}{(\phi+r(x^*)^2)^2})$.

By using well-known Routh-Hurwitz Theorem, the roots of the characteristic Eq (7) have negative real parts if $\eta_1 > 0, \eta_1\eta_2 - \eta_3 > 0, \eta_3 > 0$, that is
$(i) \quad \rho_1 > \frac{2k_1\beta r x^* y_2^*}{(\phi+r(x^*)^2)^2}$,
$(ii) \quad x^* < \sqrt{\frac{\phi}{r}}$,
$(iii) \quad c < \frac{\delta_2}{\beta}(1+\frac{\delta_1}{a})(rx^* + \frac{\phi}{x^*})$.

Theorem 1 *The interior equilibrium* $E^*(x^*, y_1^*, y_2^*)$ *is locally asymptotically stable if the condition* $(H_{11}) : \rho_1 > \frac{2k_1\beta r x^* y_2^*}{(\phi+r(x^*)^2)^2}, x^* < \sqrt{\frac{\phi}{r}}, c < \frac{\delta_2}{\beta}(1+\frac{\delta_1}{a})(rx^* + \frac{\phi}{x^*})$ *holds* .

Case 2:$\tau > 0$.
The linearized characteristic eq (6) is $\lambda^3 + \varphi_1\lambda^2 + \varphi_2\lambda + \varphi_3 + (\varphi_4\lambda + \varphi_5)e^{-\lambda\tau} = 0$, Let $i\omega(\omega > 0)$ be a root of eq (6), separating real and imaginary parts, we get

$$\begin{cases} \varphi_4\omega sin\omega\tau + \varphi_5 cos\omega\tau = \varphi_1\omega^2 - \varphi_3, \\ \varphi_4\omega cos\omega\tau - \varphi_5 sin\omega\tau = \omega^3 - \varphi_2\omega, \end{cases} \tag{8}$$

which follow that

$$\omega^6 + h_2\omega^4 + h_1\omega^2 + h_0 = 0, \tag{9}$$

where $h_2 = \varphi_1^2 - 2\varphi_2, h_1 = \varphi_2^2 - 2\varphi_1\varphi_3 - \varphi_4^2, h_0 = \varphi_3^2 - \varphi_5^2$.
For convenience, let $\omega_1^2 = v_1$, then eq (9) becomes

$$v_1^3 + h_2 v_1^2 + h_1 v_1 + h_0 = 0. \tag{10}$$

Denote

$$f(v_1) = v_1^3 + h_2 v_1^2 + h_1 v_1 + h_0. \tag{11}$$

Since $f(0) = h_0, \lim_{v_1 \to +\infty} f(v_1) = +\infty$, and from eq (10), we have

$$f'(v_1) = 3v_1^2 + 2h_2 v_1 + h_1. \tag{12}$$

After discussion about the roots of eq (10) is similar to that of Song and Wei [22], we have the following theorem.

Theorem 2 *For the polynomial eq (10), we have the following results:*

(i) if the condition $(H_{21}) : h_0 \geq 0, \Delta = h_0^2 - 3h_1 \leq 0$ holds, then eq (10) has no positive root;

(ii) if the condition $(H_{22}) : h_0 \geq 0, v_2^* = \frac{(-h_0+\sqrt{\Delta})}{3} > 0, f(v_2^*) \leq 0$ or the condition $(H_{23}) : h_0 < 0$ holds, then eq (10) has positive root.

Therefore, we suppose that eq (10) has positive roots. Without loss of generality, we assume that it has three positive roots which are denoted by v_{11}, v_{12}, v_{13}. Then eq (9) has three positive roots $\omega_{1k} = \sqrt{v_{1k}}, k = 1, 2, 3$.

The corresponding critical value of time delay τ_{1k} is

$$\tau_{1k}^{(j)} = \frac{1}{\omega_{1k}} arccos \frac{\varphi_4 \omega_{1k}^4 + (\varphi_1\varphi_5 - \varphi_4\varphi_2)\omega_{1k}^2 - \varphi_3\varphi_5}{\varphi_4^2 \omega_{1k}^2 + \varphi_5^2} + \frac{2j\pi}{\omega_{1k}}, \quad k = 1, 2, 3; j = 0, 1, 2... \tag{13}$$

Thus, $\pm i\omega_{1k}$ is a pair of purely imaginary roots of eq (6) with $\tau_{10} = \min_{k\in(1,2,3)} \{\tau_{1k}^{(0)}\}, \omega_{10} = \omega_{1k_0}$.

According to the Hopf bifurcation theorem [20], we need to verify the transversality condition differentiating eq (6) with respect to τ, and noticing that λ is a function of τ , we can obtain

$$(\frac{d\lambda}{d\tau})^{-1} = -\frac{3\lambda^2 + 2\varphi_1\lambda + \varphi_2}{\lambda(\lambda^3 + \varphi_1\lambda^2 + \varphi_2\lambda + \varphi_3)} + \frac{\varphi_4}{\lambda(\varphi_4\lambda + \varphi_5)} - \frac{\tau}{\lambda}, \tag{14}$$

which leads to

$$\begin{cases} Re(\frac{d\lambda}{d\tau})^{-1} & = Re(-\frac{3\lambda^2+2\varphi_1\lambda+\varphi_2}{\lambda(\lambda^3+\varphi_1\lambda^2+\varphi_2\lambda+\varphi_3)})_{\lambda=i\omega_{10}} + Re(\frac{\varphi_4}{\lambda(\varphi_4\lambda+\varphi_5)})_{\lambda=i\omega_{10}} \\ & = \frac{3\omega_{10}^4+2(\varphi_1^2-2\varphi_2)\omega_{10}^2+\varphi_2^2-2\varphi_1\varphi_3}{(\omega_{10}^3-\varphi_2\omega_{10})^2+(\varphi_3-\varphi_1\omega_{10}^2)^2} - \frac{\varphi_4^2}{\varphi_4^2\omega^2+\varphi_5^2}. \end{cases} \tag{15}$$

From eq (8), we have

$$(\omega_{10}^3 - \varphi_2\omega_{10})^2 + (\varphi_3 - \varphi_1\omega_{10}^2)^2 = \varphi_4^2\omega^2 + \varphi_5^2. \tag{16}$$

Noting that$\{\frac{d(Re\lambda)}{d\tau}\}_{\lambda=i\omega_{10}}$ and $\{Re(\frac{d\lambda}{d\tau})^{-1}\}_{\lambda=i\omega_{10}}$ have the same sign, then

$$sign\{\frac{d(Re\lambda)}{d\tau}\}_{\lambda=i\omega_{10}} = sign\{Re(\frac{d\lambda}{d\tau})^{-1}\}_{\lambda=i\omega_{10}} = \frac{3(\omega_{10}^2)^2 + 2h_2\omega_{10}^2 + 2h_1}{\varphi_4^2\omega_{10}^2 + \varphi_5^2} = \frac{f'(\omega_{10}^2)}{\varphi_4^2\omega_{10}^2 + \varphi_5^2} \neq 0. \tag{17}$$

Therefore, $\{\frac{d(Re\lambda)}{d\tau}\}_{\lambda=i\omega_{10}} \neq 0$ if the condition $(H_{24}) : f'(\omega_{10}^2) \neq 0$ holds. By the analysis of above, we have the following results.

Theorem 3 *For system (3),*

(1) if (H_{21}) holds, the interior equilibrium $E^*(x^*, y_1^*, y_2^*)$ is asymptotically stable for all $\tau > 0$.

(2) if $(H_{22}) or (H_{23}) and (H_{24})$ holds, the interior equilibrium $E^*(x^*, y_1^*, y_2^*)$ is asymptotically stable for all $\tau \in [0, \tau_{10})$ and unstable for $\tau > \tau_{10}$. Furthermore, the system (3) undergoes a Hopf bifurcation at the interior equilibrium $E^*(x^*, y_1^*, y_2^*)$ when $\omega = \omega_{10}, \tau = \tau_{10}$.

3 Stability of bifurcated periodic solutions

It is valuable to research the stability and direction of periodic solutions on the interior equilibrium E^*. In section 2 , we have concluded the conditions that a family of periodic solutions bifurcated from the interior equilibrium E^* of system (3) . In this section, by using the normal form theory and the center manifold theorem introduced by Hassard [10]. We will derive explicit formulas on determining the properties of the Hopf bifurcation at τ_{10} . Let $\bar{x}(t)=x(t)-x^*, \bar{y_1}(t)=y_1(t)-y_1^*, \bar{y_2}(t)=y_2(t)-y_2^*$. Then system (3) becomes

$$\begin{cases}\dot{\bar{x}}(t)=a_{11}\bar{x}(t)+a_{12}\bar{y_2}(t)+a_{13}\bar{x}^2(t)+a_{14}\bar{x}^3(t)+a_{15}\bar{x}(t)\bar{y_2}(t)+a_{16}\bar{x}^2(t)\bar{y_2}(t),\\ \dot{\bar{y_1}}(t)=a_{21}\bar{x}(t-\tau)+a_{22}\bar{y_2}(t-\tau)+a_{23}\bar{x}^2(t-\tau)-ca_{14}\bar{x}^3(t-\tau)\\ \qquad -ca_{15}\bar{x}(t-\tau)\bar{y_2}(t-\tau)-ca_{16}\bar{x}^2(t-\tau)\bar{y_2}(t-\tau)+a_{24}\bar{y_1}(t),\\ \dot{\bar{y_2}}(t)=a\bar{y_1}(t)-\delta_2\bar{y_2}(t),\end{cases} \tag{18}$$

where $a_{11}=c_1-\frac{\beta y_2^*}{c_2^3}c_3, a_{12}=-\frac{(a+\delta_1)\delta_2}{ac}, a_{13}=-\frac{\rho_1}{k_1}+\frac{\beta r x^* y_2^*}{c_2^3}(c_2+2x^*)+\frac{2\beta r x^* c_3}{c_2^4}, a_{14}=\beta r(\frac{y_2^*(\phi-3r(x^*)^2)}{c_2^3}-4r(x^*)^2(\frac{y_2^*}{c_2^4}+\frac{c_3}{c_2^5})), a_{15}=-\frac{\beta(\phi-r(x^*)^2)}{c_2^2}, a_{16}=\frac{\beta r x^*}{c_2^2}+\frac{2\beta r x^*}{c_2^3}(\phi-r(x^*)^2), a_{21}=\frac{c\beta y_2^*}{c_2^2}(\phi-r(x^*)^2), a_{22}=\frac{c\beta x^*}{c_2}, a_{23}=-\frac{c\beta r x^* y_2^*}{c_2^3}(c_2+2x^*)-\frac{2c\beta r x^* c_3}{c_2^4}, a_{24}=-(a+\delta_1), c_1=\rho_1-2\frac{\rho_1}{k_1}x^*-\frac{\beta x^* y_2^*}{c_2^2}, c_2=\phi+r(x^*)^2, c_3=y_2^*(\phi^2-r^2(x^*)^4-\phi x^*-r(x^*)^3)$.

Let $t=s\tau, \bar{x}(s\tau)=\hat{x}(s), \bar{y_1}(s\tau)=\hat{y_1}(s), \bar{y_2}(s\tau)=\hat{y_2}(s), \tau=\tau_{10}+\mu, \mu\in R, \tau_{10}$ is defined by (13). We denote $x=\hat{x}, y_1=\hat{y_1}, y_2=\hat{y_2}$ and $t=s$. It is easy to obtain that $\mu=0$ is a Hopf bifurcation value of system (3). Then system (18) becomes the following FDE in $C=C([-1,0],R^3)$:

$$\begin{aligned}\dot{x}(t) &= (\tau_{10}+\mu)(a_{11}x(t)+a_{12}y_2(t)+a_{13}x^2(t)+a_{14}x^3(t)+a_{15}x(t)y_2(t)+a_{16}x^2(t)y_2(t)),\\ \dot{y_1}(t) &= (\tau_{10}+\mu)(a_{21}x(t-1)+a_{22}y_2(t-1)+a_{23}x^2(t-1)-ca_{14}x^3(t-1)\\ &\quad -ca_{15}x(t-1)y_2(t-1)-ca_{16}x^2(t-1)y_2(t-1)+a_{24}y_1(t)),\\ \dot{y_2}(t) &= (\tau_{10}+\mu)(ay_1(t)-\delta_2 y_2(t)).\end{aligned}$$

We show the matrix form of system (18)

$$\dot{u}(t)=L_\mu u_t+f(\mu,u_t), \tag{19}$$

where $u(t)=(x(t),y_1(t),y_2(t))^T\in R^3$ and $\phi(t)=(\phi_1(t),\phi_2(t),\phi_3(t))^T\in C([-\tau,0],R^3)$, and $L_\mu: C\to R^3, f: R\times C\to R^3$ are given by

$$L_\mu\phi=D_1\phi(0)+D_2\phi(-1), \tag{20}$$

and

$$f(\mu,\phi)=(\tau_{10}+\mu)M, \tag{21}$$

where

$$D_1=(\tau_{10}+\mu)\begin{pmatrix}a_{11}&0&a_{12}\\0&a_{24}&0\\0&a&-\delta_2\end{pmatrix}, D_2=(\tau_{10}+\mu)\begin{pmatrix}0&0&0\\a_{21}&0&a_{22}\\0&0&0\end{pmatrix},$$

$$M=\begin{pmatrix}a_{13}\phi_1^2(0)+a_{14}\phi_1^3(0)+a_{15}\phi_1(0)\phi_3(0)+a_{16}\phi_1^2(0)\phi_3(0)\\ a_{23}\phi_1^2(-1)-ca_{14}\phi_1^3(-1)-ca_{15}\phi_1(-1)\phi_3(-1)-ca_{16}\phi_1^2(-1)\phi_3(-1)\\ 0\end{pmatrix}. \tag{22}$$

By the Riesz representation theorem, there exist a 3×3 matrix function $\eta(\theta,\mu):[-1,0]\to R^3$ whose parts each have bounded variation and such that for all $\phi\in C([-1,0],R^3)$,

$$L_\mu(\phi)=\int_{-1}^0 d_\eta(\theta,\mu)\phi(\theta). \tag{23}$$

In fact, we can choose

$$\eta(\theta,\mu)=D_1\delta(\theta)+D_2\delta(\theta+1), \tag{24}$$

where δ represents the Dirac delta function. For $\phi\in C^1([-1,0],R^3)$ we define

$$A(\mu)\phi(\theta)=\begin{cases}\frac{d\phi(\theta)}{d\theta}, & -1\le\theta<0,\\ \int_{-1}^0 d\eta(s,\mu)\phi(s), & \theta=0,\end{cases}$$

and

$$R(\mu)\phi(\theta)=\begin{cases}0, & -1\le\theta<0,\\ f(\mu,\theta), & \theta=0.\end{cases}$$

Hence, eq (19) is equivalent to the operator equation $\dot{u_t}=A(\mu)u_t+R(\mu)u_t$, where $u_t(t)=u(t+\theta)$ for $\theta\in[-1,0]$.

For $\psi\in C([-1,0],(R^3)^*)$, where $(R^3)^*$ is the 3-dimensional space of row vectors, we define the adjoint A^* of A ,

$$A^*(\mu)\psi(s)=\begin{cases}-\frac{d\psi(s)}{ds}, & 0<s\le 1,\\ \int_{-1}^{0}d\eta(s,\mu)\psi(-s), & s=0,\end{cases}$$

$$\langle\phi,\psi\rangle=\bar{\psi}^T(0)\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{\psi}^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi. \tag{25}$$

It is easy to verify that $\langle\psi,A(0)\phi\rangle=\langle A^*(0)\psi,\phi\rangle$. By the discussion in Section 2, we know that $\pm i\omega_{10}\tau_{10}$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*.

Assume that $q(\theta)=(1,q_1,q_2)^Te^{i\omega_{10}\tau_{10}\theta}$ is the eigenvalues of $A(0)$ corresponding to $i\omega_{10}\tau_{10}$. Then $A(0)q(0)=i\omega_{10}\tau_{10}q(\theta)$. From the definition of $A(0)$ and eq (20), eq (23) and eq (24), for $q(-1)=q(0)e^{-i\omega_{10}\tau_{10}}$, we have

$$\begin{pmatrix}a_{11}&0&a_{12}\\0&a_{24}&0\\0&a&-\delta_2\end{pmatrix}\begin{pmatrix}1\\q_1(0)\\q_2(0)\end{pmatrix}=i\omega_{10}\tau_{10}\begin{pmatrix}1\\q_1(0)\\q_2(0)\end{pmatrix},$$

then we obtain $q_1=\frac{(\delta_2+i\omega_{10})(i\omega_{10}-a_{11})}{aa_{12}}$,$q_2=\frac{-a_{11}+i\omega_{10}}{a_{12}}$.

Similarly, we can calculate the eigenvector $q^*(s)=D(1,q_1^*,q_2^*)^Te^{i\omega_{10}\tau_{10}}$ of A corresponding to $-i\omega_{10}\tau_{10}$ where $q_1^*=-\frac{(a_{11}+i\omega_{10})}{a_{21}e^{i\omega_{10}\tau_{10}}}$, $q_2^*=\frac{(i\omega_{10}+a_{24})(a_{11}+i\omega_{10})}{aa_{21}e^{i\omega_{10}\tau_{10}}}$. We normalize q and q^* by the condition $\langle q^*(s),q(\theta)\rangle=1$. Clearly, $\langle q^*(s),\overline{q}(\theta)\rangle=0$. In order to ensure that $\langle q^*(s),q(\theta)\rangle=1$, we need to determine the value of D. By eq (25) we can choose $\bar{D}=(1+q_1\bar{q_1}^*+q_2\bar{q_2}^*+(a_{21}+a_{22}q_2)q_1^*\tau_{10}e^{-i\omega_{10}\tau_{10}})^{-1}$.

In the remainder of this section, we use the same notation given in [10] and use a computation process which is similar to that in [23], we can obtain the coefficients that will be used for determining the important qualities:

$$\begin{cases}g_{20}=2\tau_{10}\bar{D}(k_{11}q_1^*+k_{21}q_2^*), & g_{11}=\tau_{10}\bar{D}(k_{12}q_1^*+k_{22}q_2^*),\\ g_{02}=2\tau_{10}\bar{D}(k_{13}\bar{q_1}^*+k_{23}\bar{q_2}^*), & g_{21}=2\tau_{10}\bar{D}(k_{14}q_1^*+k_{24}q_2^*),\end{cases}$$

where $k_{11}=a_{13}+a_{15}q_2$, $k_{12}=2a_{13}+a_{15}(\bar{q_2}+q_2)$, $k_{13}=a_{13}+a_{15}\bar{q_2}$,

$k_{14}=a_{13}(\omega_{20}^{(2)}(0)+2\omega_{11}^{(2)}(0))+3a_{14}+a_{16}(\bar{q_2}+2q_2)+a_{15}(\frac{1}{2}\bar{q_2}\omega_{20}^{(2)}(0)+q_2\omega_{11}^{(2)}(0)+\frac{1}{2}\omega_{20}^{(3)}(0)+\omega_{11}^{(3)}(0))$,

$k_{21}=(a_{23}-ca_{15}q_2)e^{-2i\omega_{10}\tau_{10}}$, $k_{22}=(2a_{23}-ca_{15}(\bar{q_2}+q_2))$, $k_{23}=(a_{23}-ca_{15}\bar{q_2})e^{-2i\omega_{10}\tau_{10}}$,

$k_{24}=a_{23}(\omega_{20}^{(2)}(-1)+2\omega_{11}^{(2)}(-1))e^{-i\omega_{10}\tau_{10}}-3ca_{14}e^{-3i\omega_{10}\tau_{10}}-ca_{16}(\bar{q_2}+2q_2)e^{-3i\omega_{10}\tau_{10}}-ca_{15}(\frac{1}{2}\bar{q_2}\omega_{20}^{(2)}(-1)$
$+q_2\omega_{11}^{(2)}(-1)+\frac{1}{2}\omega_{20}^{(3)}(-1)+\omega_{11}^{(3)}(-1))e^{-i\omega_{10}\tau_{10}}$,

$$\begin{cases}\omega_{20}(\theta)=\frac{ig_{20}}{\omega_{10}\tau_{10}}q(0)e^{i\omega_{10}\tau_{10}\theta}+\frac{i\bar{g}_{02}}{3\omega_{10}\tau_{10}}\bar{q}(0)e^{-i\omega_{10}\tau_{10}\theta}+E_1e^{2i\omega_{10}\tau_{10}\theta},\\ \omega_{11}(\theta)=-\frac{ig_{11}}{\omega_{10}\tau_{10}}q(0)e^{i\omega_{10}\tau_{10}\theta}+\frac{i\bar{g}_{11}}{\omega_{10}\tau_{10}}\bar{q}(0)e^{-i\omega_{10}\tau_{10}\theta}+E_2.\end{cases}$$

E_1 and E_2 are constant vector, moreover E_1 and E_2 satisfy the following equations:

$$\begin{pmatrix}2i\omega_{10}-a_{11}&0&-a_{12}\\-a_{21}e^{-2i\omega_{10}\tau_{10}}&2iw_{10}-a_{24}&-a_{22}e^{-2i\omega_{10}\tau_{10}}\\0&-a&2i\omega_{10}+\delta_2\end{pmatrix}E_1=2\begin{pmatrix}k_{11}\\k_{21}\\0\end{pmatrix},$$

$$\begin{pmatrix}-a_{11}&0&-a_{12}\\-a_{21}&-a_{24}&-a_{22}\\0&-a&\delta_2\end{pmatrix}E_2=\begin{pmatrix}k_{12}\\k_{22}\\0\end{pmatrix}.$$

Therefore, each g_{ij} an be determined by the parameters and delay in system (3). In the following, we use the same notation as in [23], we can compute the following quantities:

$$\begin{cases} C_1(0) = \frac{i}{2\omega_{10}\tau_{10}}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\frac{d\lambda(\tau_{10})}{d\tau}\}}, \\ \beta_2 = 2Re\{C_1(0)\}, \\ T_2 = \frac{Im\{C_1(0)\}+\mu_2 Im\{\frac{d\lambda(\tau_{10})}{d\tau}\}}{\omega_{10}\tau_{10}}, k = 0, 1, 2.... \end{cases} \tag{26}$$

By the result of Hassard in [10] we have the following theorem.

Theorem 4 *For system (3), based on Eq (26), we can get the following results:*

(i)the sign of μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0(\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the the bifurcating periodic solutions exist for $\tau > \tau_{10}(\tau < \tau_{10})$;

(ii)the sign of β_2 determines the stability of the bifurcating periodic solutions; the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0(\beta_2 > 0)$;

(iii) the sign of T_2 determines the period of the bifurcating periodic solutions: the period is increasing (decreasing) if $T_2 > 0(T_2 < 0)$.

4 Numerical simulation

In this section, we will show some numerical simulations by using Matlab to illustrate the analytical results, we obtain the corresponding waveform and the phase plots of system (3).

In system (3), we let $\rho_1 = 2, k_1 = 2, \beta = \frac{3}{4}, r = 0.01, c = \frac{1}{2}, \phi = 1, a = 0.9, \delta_1 = 0.5, \delta_2 = 0.1$. Then, we have the following particular example of system (3)

$$\begin{cases} \dot{x}(t) = 2x(t)(1 - \frac{x(t)}{2}) - \frac{\frac{3}{4}x(t)y_2(t)}{1+0.01x^2(t)}, \\ \dot{y_1}(t) = \frac{\frac{3}{8}x(t-\tau)y_2(t-\tau)}{1+0.01x^2(t-\tau)} - 1.4y_1(t), \\ \dot{y_2}(t) = 0.9y_1(t) - 0.1y_2(t), \end{cases} \tag{27}$$

Through the above set of parameter values, we can get the interior equilibrium $E^*(0.4155, 02351, 2.1161)$. For interior equilibrium E^* we can get $\omega_{10} = 0.2294, \tau_{10} = 4.8670$. From $Theorem3$, we can obtain E^* is asymptotically stable when $\tau \in [0, \tau_{10})$, when the time delay τ passes through the critical value τ_{10}, E^* will lose its stability and a Hopf bifurcation occurs. The corresponding waveform and the phase plots are depicted in $Fig.1$ and 2.

5 Conclusions

In this paper, we have incorporated Monod-Haldane response function and time delay into a stage structured predator-prey system. By analyzing the corresponding characteristic equations, the sufficient conditions for the local stability of the interior equilibrium E^* are obtain. By choosing time delay as the bifurcation parameters, the existence and direction of Hopf bifurcation of the interior equilibrium E^* are derived. We can obtain that the interior equilibrium E^* is asymptotically stable when $\tau \in [0, \tau_{10})$; when the time delay τ passes through the critical value τ_{10} , the interior equilibrium E^* will lose its stability and a Hopf bifurcation occurs. Finally, numerical simulations are given to verity the theoretical analysis.

Furthermore, we can consider that time delay will replace into double time delays that include the feedback time delay of the prey and the gestation of the predator, and the stage-structure for prey and predator. How will the dynamic behavior of the system change? These effects and practical significance of these situations deserve our attention, and we will leave it to research for the future work.

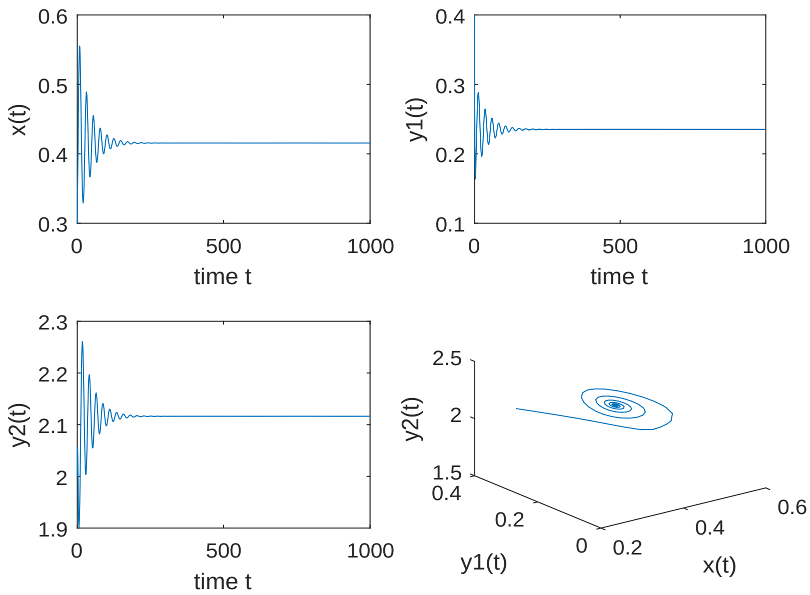

Figure 1: The figure shows that the interior equilibrium $E^*(0.4155, 02351, 2.1161)$ is asymptotically stable for $\tau = 3 < \tau_{10} = 4.8670$.

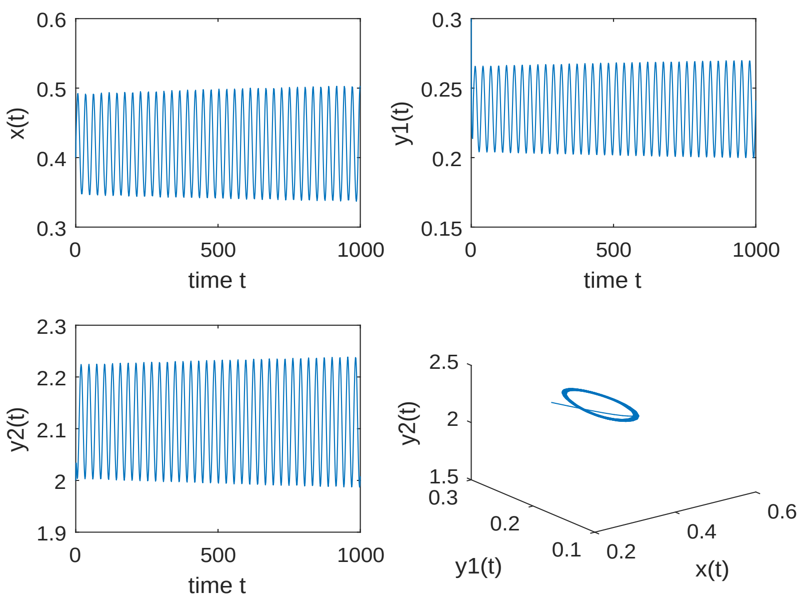

Figure 2: Numerical simulation shows that Hopf bifurcation occurs from $E^*(0.4155, 02351, 2.1161)$ for $\tau = 4.9 > \tau_{10} = 4.8670$.

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