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Numerical Method for Solving Fractional-Order PDEs by the Second Kind Chebyshev Wavelets

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Abstract: In this paper, a new numerical method for solving fractional-order PDEs is presented. The method is based upon the second kind Chebyshev wavelets approximation. We construct the second kind Chebyshev wavelets and derive the operational matrix of fractional-order integration which is utilized to transform fractional-order PDEs to a system of algebraic equations. Numerical examples show that the proposed method has good efficiency and precision.

Keywords: The second kind Chebyshev wavelets; Fractional-order PDEs; Operational matrix; The approximate solutions; Fractional calculus

1 Introduction

Fractional calculus is a branch of mathematics that deals with generalization of the well-known operations of differentiations to arbitrary orders. Mary papers on fractional calculus have been published for the real world applications in science and engineering such as viscoelasticity[1], bioengineering[2], biology[3] and more can be found in[4, 5]. Moreover, fractional partial differential equations are also widely used in the areas of signal processing[6], mechanics[7], econometrics[8], fluid dynamics[9] and electromagnetics[10]. As the analytical solutions of fractional partial differential equations are not easily to derive, the numerical solutions of these equations have received considerable attention.

In recent years, various numerical methods have been proposed for solving fractional partial differential equations. These methods include wavelets methods[11, 12], Chebyshev and Legendre polynomials methods[13], collocation method[14], adomian decomposition method[15], differential transformation method[16] and so on. Among these methods, the wavelet method is more attractive and wavelets have received considerable attention in different field. The interested reader is referred to the published paper by Abd-Elhameed, et al.[17]. In Ref.[18], the authors used the second kind Chebyshev wavelets to obtain the numerical solutions.

In this paper, we consider the following fractional-order partial differential equations(PDEs) with initial conditions given by

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial x^\alpha} + \dfrac{\partial^\beta u}{\partial t^\beta} = g(x,t),\ 0 \le x,t < 1,\ 1 < \alpha,\beta < 2 \\ u(0,t) = f(t,)\ u(x,0) = g(x),\ u'(0,t) = f_1(t),\ u'(x,0) = g_1(x) \end{cases}$$

The aim of this paper is to develop Chebyshev wavelets method with the operational matrix of integration to solve the above PDEs. The rest part of this paper is organized as follows: Section 2 introduces the basic definitions of fractional calculus. Section 3 illustrates the second kind Chebyshev wavelets and their properties. In Section 4, the proposed method is applied to approximate the solution of the problem via two numerical examples. A conclusion is drawn in Section 5.

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2 Preliminaries of the fractional calculus

In this section, we give some necessary definitions and mathematical preliminaries on fractional calculus which will be used further in this paper.

Definition 1 *The Riemann-Liouville fractional integral operator* J^{α} $(\alpha > 0)$ *of a function* $f(t)$*, is defined as*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0 \text{ and } \alpha \in \mathfrak{R}^+ \tag{1}$$

Some properties of the operator J^{α} are provided as follows:

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t), \quad (\alpha > 0,\ \beta > 0), \tag{2}$$

$$J^{\alpha} t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} t^{\alpha+\gamma}, \quad (\gamma > -1). \tag{3}$$

Definition 2 *The Caputo fractional derivative* ${}_0D_t^{\alpha}$ *of a function* $f(t)$ *is defined as*

$${}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{n-\alpha+1}} d\tau, \quad (n-1 < \alpha \leq n, n \in N). \tag{4}$$

Some properties of the Caputo fractional derivative are given as follows

$${}_0D_t^{\alpha} t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \quad 0 < \alpha < \beta+1,\ \beta > -1. \tag{5}$$

$$J^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad n-1 < \alpha \leq n \text{ and } n \in N. \tag{6}$$

3 The second kind Chebyshev wavelet operational matrix of fractional integration

3.1 The second kind Chebyshev wavelet and its properties

The second kind Chebyshev wavelet $\psi_{nm}(t) = \psi(k,n,m,t)$ have four arguments, $n = 1, 2, \ldots, 2^{k-1},\ k \in N^*$, m is the degree of the second kind Chebyshev polynomials, t is the normalized time. They are defined on the interval $[0,1]$ as

$$\psi_{nm}(t) = \begin{cases} 2^{k/2} \tilde{U}_m(2^k t - 2n + 1) \quad , & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0, & otherwise. \end{cases} \tag{7}$$

with

$$\tilde{U}_m = \sqrt{\frac{2}{\pi}} U_m(t), \quad m = 0, 1, 2, \ldots, M-1. \tag{8}$$

Here M is a fixed positive integer, $U_m(t)$ are the second kind Chebyshev polynomials with the weight function $w(t) = \sqrt{1-t^2}$ and satisfy the following recursive formula

$$U_0(t) = 1,\ U_1(t) = 2t,\ U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \quad m = 1, 2, \ldots$$

A function $f(t)$ defined over [0,1) may be expanded as

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) = C^T \Psi(t), \tag{9}$$

where

$$c_{n,m} = \langle f(t), \psi_{n,m}(t)\rangle_{w_n} = \int_0^1 w_n(t) f(t) \psi_{n,m}(t) dt, \tag{10}$$

and the weight function $w_n(t) = w(2^k t - 2n + 1)$. Moreover, C and $\Psi(t)$ are $\hat{m} = (2^{k-1}M)$ column vectors given by

$$C = \left[c_{1,0}, c_{1,1}, \cdots, c_{1,M-1}, c_{2,0}, c_{2,1}, \cdots, c_{2,M-1}, \cdots, c_{2^{k-1},0}, \cdots, c_{2^{k-1},M-1}\right]^T, \tag{11}$$

$$\Psi(t) = \left[\psi_{1,0}, \psi_{1,1}, \cdots, \psi_{1,M-1}, \psi_{2,0}, \psi_{2,1}, \cdots, \psi_{2,M-1}, \cdots, \psi_{2^{k-1},0}, \cdots, \psi_{2^{k-1},M-1}\right]^T. \tag{12}$$

Taking the collocation points as follows

$$t_i = \frac{2i-1}{2^k M},\ i = 1, 2, \ldots, 2^{k-1}M,\ \hat{m} = 2^{k-1}M.$$

We define the second kind Chebyshev wavelet matrix $\Phi_{\hat{m}\times\hat{m}}$ as

$$\Phi_{\hat{m}\times\hat{m}} = \left[\Psi(\frac{1}{2\hat{m}}), \Psi(\frac{3}{2\hat{m}}), \ldots, \Psi(\frac{2\hat{m}-1}{2\hat{m}})\right],$$

An arbitrary function of two variables $u(x,t)$ defined over $[0,1)\times[0,1)$, may be expanded into Chebyshev wavelets basis as:

$$u(x,t) \simeq \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} u_{ij} \psi_i(x) \psi_j(t) = \Psi^T(x) U \Psi(t), \tag{13}$$

where $U = [u_{ij}]_{\hat{m}\times\hat{m}}$ and $u_{ij} = (\psi_i(x), (u(x,t), \psi_j(t)))$.

The following theorem discusses the convergence and accuracy estimation of the proposed method.

Theorem 1 *[18] Let $f(t)$ be a second-order derivative square-integrable function defined over $[0,1)$ with bounded second-order derivative, satisfy $\left|f''(t)\right| \le B$ for some constants B, then*
(a) $f(t)$ can be expanded as an infinite sum of the second kind Chebyshev wavelets and the series converge to $f(t)$ uniformly, that is

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t),$$

where $c_{n,m} = \langle f(t), \psi_{n,m}(t)\rangle_{L^2_\omega[0,1)}$.

(b) $\sigma_{f,k,M} < \frac{\sqrt{\pi}B}{2^3}\left(\sum_{n=2^{k-1}+1}^{\infty} \frac{1}{n^5} \sum_{m=M}^{\infty} \frac{1}{(M-1)^4}\right)^{\frac{1}{2}}$,
where $\sigma_{f,k,M} = \left(\int_0^1 \left|f(t) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t)\right|^2 w_n(t) dt\right)^{\frac{1}{2}}$.

3.2 Operational matrix of the fractional integration

On the interval $[0,1)$, we defined a $\hat{m}-$ set of block-pulse functions (BPFs) as

$$b_i(t) = \begin{cases} 1, & i/\hat{m} \le t < i + 1/\hat{m}, \\ 0, & otherwise, \end{cases} \quad i = 0, 1, 2, \ldots, \hat{m}-1. \tag{14}$$

The functions $\{b_i(t)\}$ are disjoint and orthogonal

$$b_i(t) b_j(t) = \begin{cases} 0, & i \ne j, \\ b_i(t), & i = j. \end{cases} \tag{15}$$

$$\int_0^1 b_i(s)b_j(s)ds = \begin{cases} 0, & i \neq j, \\ 1/m, & i = j. \end{cases} \tag{16}$$

Similarly, the second kind Chebyshev wavelet may be expanded into an $\hat{m}-$ term block-pulse functions as

$$\Psi(t) = \Phi_{\hat{m}\times\hat{m}} B_{\hat{m}}(t). \tag{17}$$

Kilicman has given the block-pulse operational matrix of the fractional integration F^α as follows:

$$(I^\alpha B_{\hat{m}})(t) \approx F^\alpha B_{\hat{m}}(t), \tag{18}$$

where

$$B_{\hat{m}}(t) = [b_0(t), b_1(t), \ldots, b_{\hat{m}-1}(t)]^T,$$

and

$$F^\alpha = \frac{1}{\hat{m}^\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{\hat{m}-1} \\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{\hat{m}-2} \\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{\hat{m}-3} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{19}$$

Next, we derive the second kind Chebyshev wavelet operational matrix of the fractional integration. Let

$$(I^\alpha \Psi)(t) = P^\alpha_{\hat{m}\times\hat{m}}\Psi(t), \tag{20}$$

where $P^\alpha_{\hat{m}\times\hat{m}}$ is called the second kind Chebyshev wavelet operational matrix of the fractional integration and it can be given by

$$P^\alpha_{\hat{m}\times\hat{m}} = \Phi_{\hat{m}\times\hat{m}} F^\alpha \Phi^{-1}_{\hat{m}\times\hat{m}}. \tag{21}$$

More details can see in [18].

4 Numerical simulations

In this section, we use the second kind Chebyshev wavelet operational matrices of fractional-order integration to analysis the fractional PDEs.

Example 1 *We consider the following fractional-order PDEs*

$$\frac{\partial^{3/2}u}{\partial x^{3/2}} + \frac{\partial^{3/2}u}{\partial t^{3/2}} = g(x,t), \quad 0 \leq x, t < 1, \tag{22}$$

such that $\frac{\partial u}{\partial x}\big|_{t=0} = \frac{\partial u}{\partial t}\big|_{x=0} = u(x,0) = u(0,t) = 0, g(x,t) = 4x^{1/2}t^{1/2}(x^{3/2}+t^{3/2})/\sqrt{\pi}$, *the analytical solution of the problem is* $u(x,t) = x^2t^2$. *Let* $\frac{\partial^4 u}{\partial t^2 \partial x^2} \approx \Psi^T(x)U\Psi(t)$, *then*

$$\frac{\partial^2 u}{\partial x^2} = \Psi(x)^T U P^2_{\hat{m}\times\hat{m}}\Psi(t) + \frac{\partial^2 u}{\partial x^2}\bigg|_{t=0} + t\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right)\bigg|_{t=0} = \Psi(x)^T U P^2_{\hat{m}\times\hat{m}}\Psi(t), \tag{23}$$

$$\frac{\partial^2 u}{\partial t^2} = \Psi(x)^T (P^2_{\hat{m}\times\hat{m}})^T U\Psi(t) + \frac{\partial^2 u}{\partial t^2}\bigg|_{x=0} + x\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial t^2}\right)\bigg|_{x=0} = \Psi(x)^T (P^2_{\hat{m}\times\hat{m}})^T U\Psi(t). \tag{24}$$

Therefore

$$u(x,t) = \Psi(x)^T (P^2_{\hat{m}\times\hat{m}})^T U P^2_{\hat{m}\times\hat{m}}\Psi(t). \tag{25}$$

Then we have

$$\frac{\partial^{3/2}u}{\partial x^{3/2}} = I_x^{1/2}\left(\frac{\partial^2 u}{\partial x^2}\right) \approx I_x^{1/2}\left(\Psi(x)^T U(P^2_{\hat{m}\times\hat{m}})\Psi(t)\right) = \Psi(x)^T (P^{1/2}_{\hat{m}\times\hat{m}})^T U P^2_{\hat{m}\times\hat{m}}\Psi(t), \tag{26}$$

$$\frac{\partial^{3/2}u}{\partial t^{3/2}} = I_t^{1/2}\left(\frac{\partial^2 u}{\partial t^2}\right) \approx I_t^{1/2}(\Psi(x)^T (P^2_{\hat{m}\times\hat{m}})^T U\Psi(t)) = \Psi(x)^T (P^2_{\hat{m}\times\hat{m}})^T U P^{1/2}_{\hat{m}\times\hat{m}}\Psi(t), \tag{27}$$

Similarly, $g(x,t)$ may be expanded by the second kind Chebyshev wavelet basis as

$$g(x,t) \approx \Psi(x)^T G\Psi(t), \tag{28}$$

where $G = [g_{ij}]_{\hat{m}\times\hat{m}}$. Substituting Eqs.(26)-(28) into Eq.(22) we have

$$\Psi(x)^T (P^{1/2}_{\hat{m}\times\hat{m}})^T U P^2_{\hat{m}\times\hat{m}}\Psi(t) + \Psi(x)^T (P^2_{\hat{m}\times\hat{m}})^T U P^{1/2}_{\hat{m}\times\hat{m}}\Psi(t) = \Psi(x)^T G\Psi(t). \tag{29}$$

Dispersing Eq. (29) by the points (x_i, t_j) $i, j = 1, 2, \ldots, \hat{m}$, we obtain

$$(P^{1/2}_{\hat{m}\times\hat{m}})^T U P^2_{\hat{m}\times\hat{m}} + (P^2_{\hat{m}\times\hat{m}})^T U P^{1/2}_{\hat{m}\times\hat{m}} = G. \tag{30}$$

Eq. (30) can be also written as

$$U P^{3/2}_{\hat{m}\times\hat{m}} + (P^{3/2}_{\hat{m}\times\hat{m}})^T U = (P^{-1/2}_{\hat{m}\times\hat{m}})^T G P^{-1/2}_{\hat{m}\times\hat{m}}. \tag{31}$$

Eq. (31) is a Sylvester equation which is solved by using Matlab software. Solving it, we can get the unknown function. The approximate and analytical solutions when $k = 3 \quad (M = 3, M = 4, M = 5)$ are listed in Table 1. From Example 1, it can be concluded that the approximate solutions are in agreement with the analytical solutions well as k, M grows.

Table 1: The approximate and analytical solutions with some $M(k = 3)$

(x,t)	$Anal.Sol.$	$M = 3$	$M = 4$	$M = 5$
$(0,0)$	0	7.1258e-7	4.1265e-8	6.8431e-9
$(0.1,0.1)$	0.00010000	0.00010300	0.00010036	0.00010003
$(0.2,0.2)$	0.00160000	0.00160381	0.00160040	0.00160004
$(0.3,0.3)$	0.00810000	0.00810451	0.00810048	0.00810006
$(0.4,0.4)$	0.02560000	0.02560516	0.02560055	0.02560007
$(0.5,0.5)$	0.06250000	0.06250575	0.06250062	0.06250008
$(0.6,0.6)$	0.12960000	0.12960631	0.12960069	0.12960009
$(0.7,0.7)$	0.24010000	0.24010683	0.24010077	0.24010010
$(0.8,0.8)$	0.40960000	0.40960733	0.40960081	0.40960012
$(0.9,0.9)$	0.65610000	0.65610780	0.65610086	0.65610013

Example 2 *Consider the following fractional-order PDEs*

$$\frac{\partial^{4/3}u}{\partial x^{4/3}} + \frac{\partial^{5/3}u}{\partial t^{5/3}} = \frac{2}{\Gamma(2/3)}x^{2/3} + t^2 + x^2 + \frac{2}{\Gamma(1/3)}t^{1/3}, \quad 0 \le x, t < 1, \tag{32}$$

with initial conditions:

$$\left.\frac{\partial u}{\partial t}\right|_{x=0} = 2t, \left.\frac{\partial u}{\partial x}\right|_{t=0} = 2x, u(x,0) = x^2, u(0,t) = t^2.$$

The analytical solution of the system is $u(x,t) = x^2 + t^2$. Let $\frac{\partial^4 u}{\partial t^2 \partial x^2} \approx \Psi^T(x)U\Psi(t)$, then

$$\frac{\partial^2 u}{\partial x^2} = \Psi(x)^T U P^2_{\hat{m}\times\hat{m}}\Psi(t) + \left.\frac{\partial^2 u}{\partial x^2}\right|_{t=0} + t\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right)\Big|_{t=0} = \Psi(x)^T U P^2_{\hat{m}\times\hat{m}}\Psi(t) + 2, \tag{33}$$

$$\frac{\partial^2 u}{\partial t^2} = \Psi(x)^T (P^2_{\hat{m}\times\hat{m}})^T U\Psi(t) + \frac{\partial^2 u}{\partial t^2}\bigg|_{x=0} + x\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial t^2}\right)\bigg|_{x=0} = \Psi(x)^T (P^2_{\hat{m}\times\hat{m}})^T U\Psi(t) + 2. \tag{34}$$

Hence

$$u(x,t) \approx \Psi(x)^T (P^2_{\hat{m}\times\hat{m}})^T U P^2_{\hat{m}\times\hat{m}}\Psi(t) + x^2 + 2xt + t^2. \tag{35}$$

Then we have

$$\frac{\partial^{4/3} u}{\partial x^{4/3}} = I_x^{2/3}\left(\frac{\partial^2 u}{\partial x^2}\right) \approx I_x^{2/3}\left(\Psi(x)^T U P^2_{\hat{m}\times\hat{m}}\Psi(t) + 2\right) = \Psi(x)^T (P^{2/3}_{\hat{m}\times\hat{m}})^T U P^2_{\hat{m}\times\hat{m}}\Psi(t) + \frac{3}{\Gamma(2/3)}x^{2/3}, \tag{36}$$

and

$$\frac{\partial^{5/3} u}{\partial t^{5/3}} = I_t^{1/3}\left(\frac{\partial^2 u}{\partial t^2}\right) \approx I_t^{1/3}\left(\Psi(x)^T (P^2_{\hat{m}\times\hat{m}})^T U P^{1/3}_{\hat{m}\times\hat{m}}\Psi(t) + 2\right) = \Psi(x)^T (P^2_{\hat{m}\times\hat{m}})^T U P^{1/3}_{\hat{m}\times\hat{m}}\Psi(t) + \frac{6}{\Gamma(1/3)}t^{1/3}. \tag{37}$$

Substituting Eqs. (36)- (37) into Eq. (32), we have

$$\Psi(x)^T (P^{2/3}_{\hat{m}\times\hat{m}})^T U P^2_{\hat{m}\times\hat{m}}\Psi(t) + \Psi(x)^T (P^2_{\hat{m}\times\hat{m}})^T U P^{1/3}_{\hat{m}\times\hat{m}}\Psi(t) = f(x,t), \tag{38}$$

where $f(x,t) = t^2 + x^2 - \frac{1}{\Gamma(2/3)}x^{2/3} - \frac{4}{\Gamma(1/3)}t^{1/3}$.
Similarly, $f(x,t)$ *can be expressed as follows*

$$f(x,t) \approx \Psi(x)^T F\Psi(t) \tag{39}$$

where $F = [f_{ij}]_{\hat{m}\times\hat{m}}$. *Dispersing Eqs. (38) -(39) by the points* (x_i, t_j) $\quad i,j = 1,2,\ldots,\hat{m}$, *we have*

$$UP^{5/3}_{\hat{m}\times\hat{m}} + (P^{4/3}_{\hat{m}\times\hat{m}})^T U = (P^{-2/3}_{\hat{m}\times\hat{m}})^T F P^{-1/3}_{\hat{m}\times\hat{m}}, \tag{40}$$

Eq. (40) is a Sylvester equation. Solving the system, we can obtain the unknown solutions. The graph of the analytical solution is shown in Figure 1. The graphs of the approximate solutions for $(k=3, k=4, k=5)$, $M=4$ *are shown in Figures 2-4.*

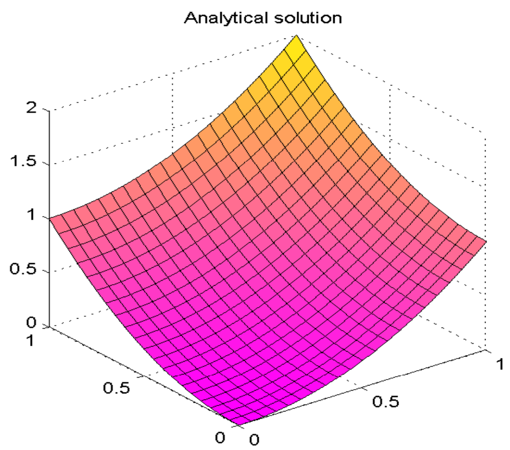

Figure 1: The analytical solution

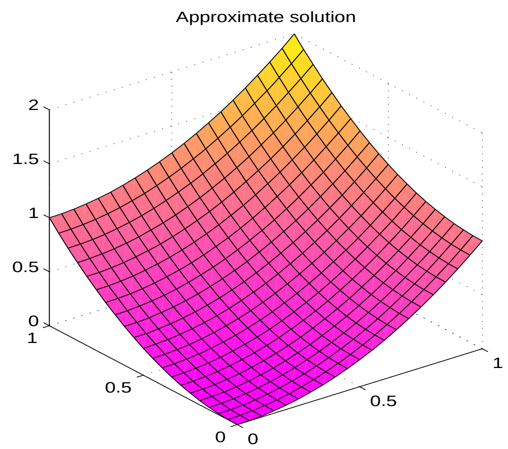

Figure 2: The approximate solution when $k=3$

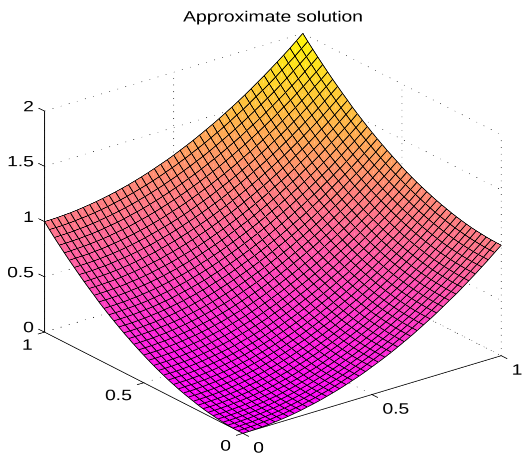

Figure 3: The approximate solution when $k = 4$

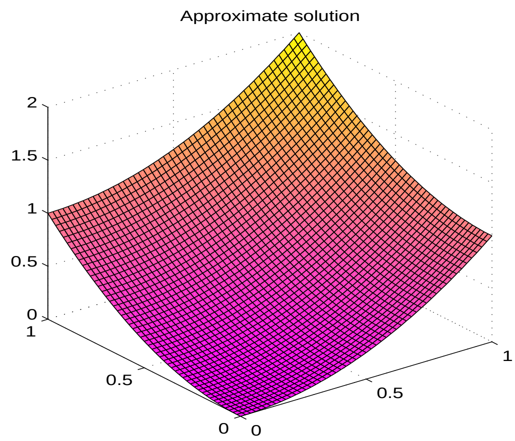

Figure 4: The approximate solution when $k = 5$

5 Conclusions

This paper has presented a numerical technique for approximating solutions of fractional-order PDEs by combining the second kind Chebyshev wavelet with its operational matrix of fractional-order integration. In the proposed method a small number of grid points guarantees the necessary accuracy. The main advantage of wavelet method for solving the equations is that the dispersing the coefficients matrix of algebraic equations is sparse. The solution is convenient, even though the size of increment may be large. Several examples are given to demonstrate the powerfulness of the proposed method.

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