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Bifurcation Analysis and Chaos Control in the Resource-Economy-Pollution Dynamic System with Delayed Feedback

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Abstract: Chaos control can be used to eliminate chaotic behavior and to stabilize the chaotic system at one of the system's equilibrium points. A Resource-Economy-Pollution dynamical system (REP) has been proposed by considering the close relationship between resource, pollution, and economic growth. The delayed feedback control are used to the REP system. The linear stability and the existence of Hopf bifurcation of the REP system are investigated. We calculate the interval values of the time delay τ at which this system is stable(unstable). By establishing appropriate time delay τ and feedback strength k ranges, the REP system can be controlled to be stable. Numerical simulations indicate that delayed feedback control plays an effective role in control of chaos of the REP system.

Keywords: Hopf bifurcation; Delayed feedback control; The REP system

1 Introduction

Chaos has been extensively studied in the field of mathematics, physics, astronomy, chemistry and engineering communities in the last four decades [1-4]. Recently the trend of analyzing and understanding chaos has been extended to control and utilize chaos. The main goal of chaos control is to eliminate chaotic behavior and to stabilize the chaotic system at one of the system's equilibrium points [5].

Delayed feedback control method has been receiving considerable attention recently since it was proposed by Pyragas [6-7]. It provides an effective method for feedback control of chaos. The idea of this method is to inject an appropriate continuous controlling signal into the system which is proportional to the difference between the present state of a given system $Y(t)$ and its delayed value $Y(t-\tau)$. By choosing proper time delay, this difference is practically zero as the system evolves close to the desired steady state or periodic orbit which means stabilization [8]. The advantage of this method is that the delayed control has no need for a reference system. Since it generates the control force from the information of the system itself. Moreover, this method has been used to control chaos for a continuous dynamical system. Tian et al. proposed a dynamical delayed output-feedback (DDOF) control strategy for stabilizing unstable periodic orbits (UPOs) of chaotic systems [9]. Chen et al. applied a time-dependent delayed feedback control algorithm to control chaos in the TCP-RED system [10]. Salarieh et al. presented chaos control of a tapping mode atomic force microscope (AFM) model via delayed feedback method [11]. Guo et al. investigated Lorenz system used delayed feedback control [12]. Guan et al. apply distributed delay as a self-controlling feedback to implement continuous control of a new butterfly-shaped chaotic system [13]. Vasegh et al. concerned with bifurcation and chaos control in scalar delayed differential equations with delay parameter τ [14]. Yang et al. investigated chaos control for circuit system by delayed feedback method [15].

Since the 1980s, China has achieved rapid economic growth, followed by strong growth in energy consumption and thus pollution emissions [16], China's environmental pollution has become increasingly serious [17]. A Resource-Economy-Pollution dynamical system (REP) is used to study close relationship between resource, pollution and economic growth [18]. The purpose of this paper is to investigate the REP system with time delayed feedback analytically and

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numerically. Our results show that the stability changes as the delays and feedback strength vary. When the REP system are asymptotically stable, the chaotic attractor is converted into a stable steady state, an unstable periodic orbit or another chaotic attractor again when the delay passes through some values.

There are possibly three contributions of this study. First, this paper using the time delayed feedback control to control the REP system. We regarding the time delay τ and feedback strength k as parameters, the effect of time delay on the dynamics of this system is explored. Second, through set the scenario, we study the local stability and the existence of Hopf bifurcation of the REP system. Third, we demonstrate that chaos vanishes as the time delayed reaches a certain value and we try to extend these results to control the REP system.

The organization of this paper is as follows. In Section 2, we have a brief description of REP system. In Section 3, we investigate the problem of controlling chaos of the REP system. We consider the stability of one of the equilibrium points and determine the range of delay τ at which the equilibrium point of the chaotic REP system can be controlled to be stable. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Finally, a brief conclusion is given in Section 5.

2 Description of Resource-Economy-Pollution system

The governing equations of the system are:

$$\begin{cases} \dot{x} = a_1x + a_2y - a_3yz, \\ \dot{y} = b_1x(1 - x/M) - b_2y - b_3z, \\ \dot{z} = c_1xy - c_2z, \end{cases} \tag{1}$$

where $x(t)$, $y(t)$, $z(t)$ represents the total consumed in a region during a given period, economy scale, and the pollution variable, respectively. a_i, b_i, c_i, M are positive constant parameters. M is the maximum value of resource consumption [18].

3 Bifurcation analysis of energy prices system with delayed feedback force

In this section, we investigate the problem of controlling chaos of system (1). We add the control function $k(y(t) - y(t - \tau_1))$, $k(z(t) - z(t - \tau_2))$ to the second equation and the third equation of system (1), where $\tau_1, \tau_2 > 0$ is a constant time delayed, k is the feedback strength. Then the delayed feedback control system can be written as:

$$\begin{cases} \dot{x} = a_1x + a_2y - a_3yz, \\ \dot{y} = b_1x(1 - x/M) - b_2y - b_3z + k(y(t) - y(t - \tau_1)), \\ \dot{z} = c_1xy - c_2z + k(z(t) - z(t - \tau_2)), \end{cases} \tag{2}$$

where τ_1, τ_2 is a positive constant number and $k \epsilon R$.

By the linear transform, system (2) as following:

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y + a_{13}z, \\ \dot{y} = a_{21}x + a_{22}y + a_{23}z + a_{24}y(t - \tau_1), \\ \dot{z} = a_{31}x + a_{32}y + a_{33}z + a_{34}z(t - \tau_2), \end{cases} \tag{3}$$

where

$$a_{11} = a_1, a_{12} = a_2 - a_3z^*, a_{13} = -a_3y^*, a_{21} = b_1 - 2b_1x^*/M, a_{22} = -b_2 + k,$$

$$a_{23} = -b_3, a_{24} = -k, a_{31} = c_1y^*, a_{32} = c_1x^*, a_{33} = -c_2 + k, a_{34} = -k.$$

Therefore, the corresponding characteristic equation of system (2) is given by

$$\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 + e^{-\lambda\tau_1}(B_2\lambda^2 + B_1\lambda + B_0) + e^{-\lambda\tau_2}(C_2\lambda^2 + C_1\lambda + C_0) + e^{-\lambda(\tau_1+\tau_2)}(D_1\lambda + D_0) = 0, \tag{4}$$

where

$$A_2 = -a_{33} - a_{22} - a_{11}, A_1 = a_{22}a_{33} + a_{11}a_{33} + a_{11}a_{22} - a_{13}a_{31} - a_{12}a_{21} - a_{23}a_{32}, B_1 = a_{24}a_{33} + a_{11}a_{24},$$
$$A_0 = a_{13}a_{31}a_{22} + a_{12}a_{21}a_{33} + a_{23}a_{32}a_{11} - a_{21}a_{32}a_{13} - a_{31}a_{12}a_{23} - a_{11}a_{22}a_{33}, B_2 = -a_{24}, B_0 = a_{13}a_{31}a_{24} - a_{11}a_{24}a_{33},$$
$$C_2 = -a_{34}, C_1 = a_{11}a_{34} + a_{22}a_{34}, C_0 = a_{12}a_{21}a_{34} - a_{11}a_{22}a_{34}, D_1 = a_{24}a_{34}, D_0 = -a_{11}a_{24}a_{34}.$$

In order to study the distribution of the roots of the exponential polynomial (4), we state a result due to [5], which is the following theorem.

Theorem 1 *Consider the exponential polynomial:*

$$P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) = \lambda^n + p_1^0\lambda^{n-1} + \cdots + p_{n-1}^0\lambda + p_n^0 + (p_1^1\lambda^{n-1} + \cdots + p_{n-1}^1\lambda + p_n^1)e^{-\lambda\tau_1} + \cdots + (p_1^m\lambda^{n-1} + \cdots + p_{n-1}^m\lambda + p_n^m)e^{-\lambda\tau_m}, \quad (5)$$

where $\tau_j > 0 (j = 1, 2, \cdots, m)$ and $p_k^j (j = 1, 2, \cdots, m; k = 1, 2, \cdots, n)$ are constants. As $(\tau_1, \tau_2, \cdots, \tau_m)$ vary, the sum of the order of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ on the open right half plane can change only if a zero appears on or crosses the imaginary axis.

By setting $\tau_1 = \tau_2 = \tau \neq 0$ in Eq. (4), the corresponding characteristic equation reduces to:

$$\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 + e^{-\lambda\tau}[(B_2 + C_2)\lambda^2 + (B_1 + C_1)\lambda + (B_0 + C_0)] + e^{-2\lambda\tau}(D_1\lambda + D_0) = 0, \quad (6)$$

According to the Hopf bifurcation theory [19-20] , we let $\lambda = \pm i * \omega$ is a root of the Eq. (6), then we can obtain

$$\begin{cases} (\omega^3 + D_1\omega - A_1\omega)sin\omega_1\tau_1 + (A_0 + D_0 - A_2\omega^2)cos\omega_1\tau_1 = B_2\omega^2 - B_0, \\ (-\omega^3 + A_1\omega + D_1\omega)cos\omega_1\tau_1 + (A_0 - A_2\omega^2 - D_0)sin\omega_1\tau_1 = -B_1\omega. \end{cases} \quad (7)$$

From (7), we can get :

$$sin\omega\tau = \frac{F_5\omega^5 + F_3\omega^3 + F_1\omega}{\omega^6 + G_4\omega^4 + G_2\omega^2 + G_0}, cos\omega\tau = \frac{F_4\omega^5 + F_2\omega^3 + F_0\omega}{\omega^6 + G_4\omega^4 + G_2\omega^2 + G_0},$$

where

$$F_5 = B_2, F_4 = B_1 - A_2B_2, F_3 = A_2B_1 - B_0 - (A_1 + C_1)A_2, F_2 = (A_0 - C_2)B_2 + (C_1 - A_1)B_1 + A_2B_0,$$
$$F_1 = (A_1 + C_1)B_0 - (A_0 + C_2)B_1, F_0 = (C_2 - A_0)B_0, G_4 = A_2^2 - 2A_1, G_2 = A_1^2 - C_1^2 - 2A_0A_2, G_0 = A_0^2 - C_2^2.$$

By taking square of each equation in Eq. (7) and then adding them up, we obtains:

$$\omega^{12} + E_5\omega^{10} + E_4\omega^8 + E_3\omega^6 + E_2\omega^4 + E_1\omega + E_0 = 0, \quad (8)$$

where

$$E_5 = 2G_4 - F_5^2, E_4 = G_4^2 + 2G_2 - F_4^2 - 2F_3F_5, E_3 = 2G_0 + 2G_2G_4 - F_3^2 - 2F_1F_5 - 2F_2F_4,$$
$$E_2 = G_2^2 + 2G_0G_4 - F_2^2 - 2F_0F_4 - 2F_1F_3, E_1 = 2G_0G_2 - F_1^2 - 2F_0F_2, E_0 = G_0^2 - F_0^2.$$

Let $u_3 = \omega^2$, then Eq. (8) becomes

$$u_3^6 + E_{35}u_3^5 + E_{34}u_3^4 + E_{33}u_3^3 + E_{32}u_3^2 + E_{31}u_3 + E_{30} = 0. \quad (9)$$

From Eq. (9), we have

$$h_3(u_2) = u_3^6 + E_{35}u_3^5 + E_{34}u_3^4 + E_{33}u_3^3 + E_{32}u_3^2 + E_{31}u_3 + E_{30}. \quad (10)$$

Suppose that Eq. (10) has six positive roots, which are denoted as $u_1, u_2, u_3, u_4, u_5, u_6$. Then Eq. (8) has six positive roots $\omega_k = \sqrt{u_k}, k = 1, 2, 3$. The corresponding critical value of time delay τ_k^j is

$$\tau_k^j = \frac{1}{\omega_k} * arccos(\frac{F_4\omega_k^4 + F_2\omega_k^2 + F_0}{\omega_k^6 + G_4\omega_k^4 + G_2\omega_k^2 + G_0 + 2\pi j}), \quad (11)$$

where $k = 1, 2, 3, 4, 5, 6$ and $j = 0, 1, 2 \cdots$.

In the following, we differentiate the two side of Eq. (4) with respect to τ to verify the transversally condition. Taking the derivative of λ with respect to τ in Eq. (4) and substituting $\lambda = i\omega_0$, we get

$$Re(\frac{d\lambda}{d\tau})^{-1}_{\lambda=i\omega_0} = Re(\frac{H_1 + H_2 i}{H_3 + H_4 i}) = \frac{H_1H_3 + H_2H_4}{H_3^2 + H_4^2},$$

where

$$H_1 = (A_1 - 3\omega_0^2)cos\omega_0\tau_0 - 2A_2\omega_0 sin\omega_0\tau_0 + D_1 cos\omega_0\tau_0 + (B_1 + C_1),$$
$$H_2 = (A_1 - 3\omega_0^2)sin\omega_0\tau_0 + 2A_2\omega_0 cos\omega_0\tau_0 - D_1 sin\omega_0\tau_0 + 2(B_2 + C_2)\omega_0,$$
$$H_3 = (A_1 - D_1 - \omega_0^2)\omega_0^2 cos\omega_0\tau_0 + (D_0 + A_0 - A_2\omega_0^2)\omega_0 sin\omega_0\tau_0,$$
$$H_4 = (A_1 + D_1 - \omega_0^2)\omega_0^2 sin\omega_0\tau_0 + (D_0 - A_0 + A_2\omega_0^2)\omega_0 cos\omega_0\tau_0.$$

Obviously, if the following condition holds:

$$H_1H_3 + H_2H_4 \neq 0.$$

Then we have $\frac{d(Re\lambda)}{d\tau_0}_{\lambda=i\omega_0} \neq 0$. By the above analysis, we have the following results:

Theorem 2 *For system (2), with $\tau_1 = \tau_2 = \tau$.*
(1) If $\tau\epsilon[0, \tau_1^0) \bigcup (\tau > \tau_2^0)$, the equilibrium S is unstable.
(2) If $\tau\epsilon(\tau_1^0, \tau_2^0)$, the equilibrium S is asymptotically stable.
Furthermore, the system (2) undergoes a Hopf bifurcation at the equilibrium S when $\tau = \tau_1^0$ or τ_2^0.

4 Numerical example

In this section, to verify and demonstrate the effectiveness and the feasibility of the presented control method, the simulation results have been performed. Let $a_1 = 0.056, a_2 = 0.19, a_3 = 0.98, b_1 = 0.53, b_2 = 0.2154, b_3 = 0.21, c_1 = 0.022, c_2 = 0.094, M = 5.7$. Then we consider the following system:

$$\begin{cases} \dot{x} = 0.056x + 0.19y - 0.98yz, \\ \dot{y} = 0.53x(1 - x/5.7) - 0.2154y - 0.21z, \\ \dot{z} = 0.022xy - 0.094z. \end{cases} \quad (12)$$

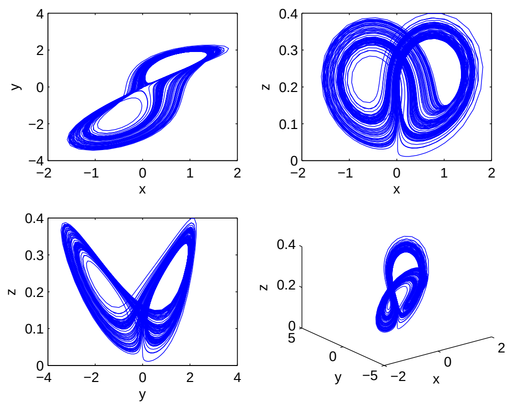
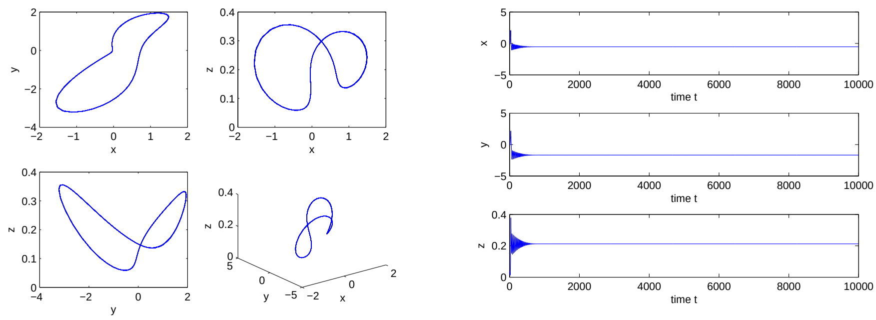
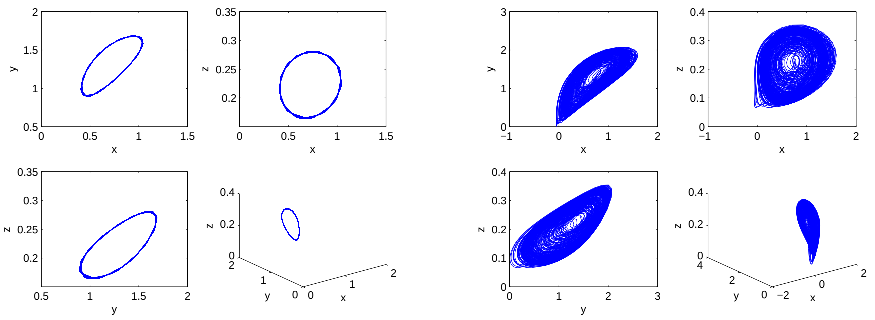


Figure 1: *A chaotic attractor of the REP system for $\tau\epsilon[0, \tau_1^0)$.*



(a) (b)

Figure 2: Phase diagram and time series of the system (12). (a):A period solution of the REP system for $\tau < \tau_1^0$. (b):The equilibrium point S is stable for $\tau \in (\tau_1^0, \tau_2^0)$.



(a) (b)

Figure 3: Phase diagram of the system (12). (a):The period solution of the system. (b):Chaos occurs again of the system for $\tau > \tau_2^0$.

For the purpose of controlling the chaos, we take $k = -0.1$. By using Eqs. (8) and (11), one obtain the following:

$$\omega_1^0 = 0.11180, \tau_1^0 = 2.9374; \omega_2^0 = 0.46456, \tau_2^0 = 13.0420.$$

From Theorem 2, we know that the system still chaotic when $\tau\epsilon[0, \tau_1^0)$ as shown in Fig. 1. When τ approaches τ_1^0 gradually, the chaotic attractor of the system vanishes and the period solution appear in Fig. 2(a). By taking $\tau\epsilon(\tau_1^0, \tau_2^0)$, the solution approaches to the equilibrium point S, namely this system is stable, as seen in Fig. 2(b). Moreover, when we take $\tau > \tau_2^0$, the equilibrium point loses its stability and a periodic solution bifurcates from it as display in Fig. 3(a). From Fig. 3(b), one conclude that when $\tau > \tau_2^0$, the bifurcating periodic solution disappear and chaos occurs again.

5 Conclusions

In this paper, we study the REP system at Hopf bifurcation occurs and the stability of equilibrium. Our theoretical results and numerical simulations show that the chaos phenomena of REP system can be controlled delay. As the delayed increases further, the numerical simulations show that the periodic solution disappears and the chaos attractor appears again. The obtained results can also be applied to the control and anti-control of chaos phenomena of REP system. The two time delay τ_1, τ_2 and feedback strength k are supposed to control REP chaotic system. We designed k and τ_1, τ_2 to eliminate the chaotic behaviors. The local stability of the equilibria S is discussed by analyzing the distribution of the roots of associated characteristic equation for REP system. The numerical simulations show that, if $k = -0.1$ and τ satisfies the corresponding conditions of Theorem 2, the chaotic phenomenon of REP system can be controlled, the existence of local Hopf bifurcations are existed when the delay passes though critical values of τ. It is hoped that the results that are reported and illustrated in this paper increase our knowledge of the chaos control of REP system with time delay feedback, which can be used to control the pollution status of an area and improve economic development.

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