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Third Kind of Elliptic Equation Expansion Method and Exact Traveling Wave Solutions of the Generalized Microstructure Wave Equation and the Sawada–Kotera Equation

Sirendaoerji *
Mathematical Science College, Inner Mongolia Normal University, Huhhot 010022, People's Republic of China
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Abstract: Twelve types of explicit exact solutions expressed by hyperbolic functions, trigonometric functions and Jacobi elliptic functions of the third kind of elliptic equation are presented. These obtained solutions are used to propose the third kind of elliptic equation expansion method for seeking new exact traveling wave solutions of nonlinear equations. Exact traveling wave solutions of the generalized microstructure wave equation and the Sawada–Kotera equation are obtained by using the proposed method.

Keywords: Elliptic equation; Explicit exact solution; Traveling wave solution; Elliptic function solution

1 Introduction

In recent years, the problem of finding exact solutions of nonlinear equations in mathematical physics has become a hot research topic of physics, mathematics and engineering. Based on this background, some direct algebraic methods such as the extended $\tanh$–function method [1–3], F–expansion method [4, 5], G'/G–expansion method [6, 7], Fan sub–equation method [8, 9], the auxiliary equation method [10–13] and the Riccati equation expansion method [14–16] for seeking exact solutions of nonlinear equations have been developed. The effectiveness of these methods depends on the chosen auxiliary ordinary differential equation (AODE) and its solutions. For example, a sub–equation called the third kind of elliptic equation

$$F'^2(\xi) = c_0 + c_1 F(\xi) + c_2 F^2(\xi) + c_3 F^3(\xi), \tag{1}$$

which used in the Fan sub–equation method possesses only an explicit Weierstrass elliptic function solution

$$F(\xi) = \wp(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3), g_2 = -\frac{4c_1}{c_3}, g_3 = -\frac{4c_0}{c_3}, c_3 > 0,$$

where c_i $(i = 0, 1, 2, 3)$ are constants. Therefore, by taking Eq.(1) as an auxiliary ODE one can not find more explicit exact solutions for given nonlinear equations except the Weierstrass elliptic function solution. It remains an open problem as whether the Eq.(1) has some other explicit solutions and how to construct these solutions if they exist.

Recently Liu has studied this problem [17, 18] and obtained some exact solutions of Eq.(1) by direct integral method and complete discrimination system of polynomials. But Liu's solutions are implicit solution which can not lead the explicit exact solutions of nonlinear equations.

Because the exact solutions of Eq.(1) is difficult to find, so the work on this area is very little and the problem of finding explicit exact solutions of Eq.(1) is has not been solved yet.

The goal of this paper is to give explicit exact solutions of Eq.(1) itself rather than its sub equations and establish a direct algebraic method for solving nonlinear equations from the solutions of the third kind of elliptic Eq.(1).

To serve the above goal, this paper is organized as follows. In Sec. 2, we shall give twelve types of new explicit exact solutions expressed by hyperbolic functions, trigonometric functions and Jacobi elliptic functions of Eq.(1) and propose a

* *E-mail address*: siren@imnu.edu.cn

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simple approach for seeking explicit exact solutions of nonlinear equations in mathematical physics. In Sec. 3, the explicit exact solutions of the generalized microstructure wave equation and the Sawada–Kotera equation are obtained by using the proposed method. Finally, Sec. 4 offer a discussion.

2 Solutions of the third kind of elliptic equation and description of the method

In this section, we shall give some explicit exact solutions of Eq.(1) and propose the third kind of elliptic equation expansion method for seeking exact traveling wave solutions of nonlinear equations in mathematical physics.

By using a direct detection technique, we find twelve types of explicit exact solutions of Eq.(1) as follows.

Case 1 *If* $c_0 = \frac{(c_2+8)(c_2-4)^2}{27c_3^2}, c_1 = \frac{c_2^2-16}{3c_3}$, *then*

$$F_1(\xi) = -\frac{c_2+8}{3c_3} + \frac{4}{c_3}\tanh^2\xi,$$
$$F_2(\xi) = -\frac{c_2+8}{3c_3} + \frac{4}{c_3}\coth^2\xi.$$

Case 2 *If* $c_0 = \frac{(c_2-8)(c_2+4)^2}{27c_3^2}, c_1 = \frac{c_2^2-16}{3c_3}$, *then*

$$F_3(\xi) = -\frac{c_2-8}{3c_3} + \frac{4}{c_3}\tan^2\xi,$$
$$F_4(\xi) = -\frac{c_2-8}{3c_3} + \frac{4}{c_3}\cot^2\xi.$$

Case 3 *If* $c_0 = \frac{(4m^2+c_2-2)\left[32m^2(1-m^2)-4c_2m^2+(c_2+1)^2\right]}{27c_3^2}$, $c_1 = \frac{16m^2(1-m^2)+c_2^2-1}{3c_3}$, *then*

$$F_5(\xi) = -\frac{4m^2+c_2+1}{3c_3} + \frac{2}{c_3\,(1+\mathrm{cn}(\xi,m))},$$
$$F_6(\xi) = -\frac{4m^2+c_2-5}{3c_3} - \frac{2}{c_3\,(1+\mathrm{nc}(\xi,m))}.$$

Case 4 *If* $c_0 = \frac{(-2m^2+c_2+4)(m^4+(2c_2+32)m^2+c_2^2-4c_2-32)}{27c_3^2}$, $c_1 = \frac{m^2(16-m^2)+c_2^2-16}{3c_3}$, *then*

$$F_7(\xi) = -\frac{m^2+c_2+4}{3c_3} + \frac{2m^2}{c_3\,(1+\mathrm{dn}(\xi,m))},$$
$$F_8(\xi) = \frac{5m^2-c_2-4}{3c_3} - \frac{2m^2}{c_3\,(1+\mathrm{nd}(\xi,m))}.$$

Case 5 *If* $c_0 = \frac{(-2m^2+c_2-2)(m^2+6m+c_2+1)(m^2-6m+c_2+1)}{27c_3^2}$, $c_1 = -\frac{m^4+14m^2-c_2^2+1}{3c_3}$, *then*

$$F_9(\xi) = \frac{5m^2-c_2-1}{3c_3} - \frac{2(m^2-1)}{c_3\,(1+\mathrm{sn}(\xi,m))},$$
$$F_{10}(\xi) = -\frac{m^2+c_2-5}{3c_3} + \frac{2(m^2-1)}{c_3\,(1+\mathrm{ns}(\xi,m))},$$
$$F_{11}(\xi) = \frac{5m^2-c_2-1}{3c_3} - \frac{2(m^2-1)}{c_3\,(1+\mathrm{cd}(\xi,m))},$$
$$F_{12}(\xi) = -\frac{m^2+c_2-5}{3c_3} + \frac{2(m^2-1)}{c_3\,(1+\mathrm{dc}(\xi,m))},$$

where m $(0 < m < 1)$ is the modulus of the Jacobi elliptic functions.

Clearly, these solutions are determined on the two restricted relations among the coefficients of Eq.(1) and requires to satisfy the condition $c_3 \neq 0$.

Now let us briefly describe our simple method which is called the third kind of elliptic equation expansion method for solving nonlinear equations in mathematical physics.

The main steps of the third kind of elliptic equation expansion method introduced here can be divided into the following five steps.

(1) Suppose that the wave transformation

$$u(x,t) = u(\xi), \xi = x - \omega t, \tag{2}$$

transforms the given nonlinear equation

$$P(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt},\cdots) = 0, \tag{3}$$

into the following ODE

$$Q(u,u',u'',\cdots) = 0, \tag{4}$$

where ω is an arbitrary constant.

(2) We set the exact solution of Eq.(3) in the form

$$u(\xi) = \sum_{j=0}^{n} a_j F^j(\xi), \tag{5}$$

where a_j $(j = 1,2,\cdots,n)$ are constants to be determined, $F(\xi)$ is solution of Eq.(1), n is a positive integer which is determined by the homogeneous balance principal. In general, if $F(\xi)$ satisfies the following auxiliary equation

$$F'^2(\xi) = \sum_{i=0}^{m} c_i F^i(\xi), \tag{6}$$

then it is not difficult to calculate from (5) and (6) that the highest degree of $\frac{d^p u}{d\xi^p}$ is

$$\begin{cases} O\left(\dfrac{d^p u}{d\xi^p}\right) = n + \left(\dfrac{m}{2} - 1\right)p, p = 1,2,3,\cdots, \\ O\left(u^q \dfrac{d^p u}{d\xi^p}\right) = (q+1)n + \left(\dfrac{m}{2} - 1\right)p, p = 1,2,3,\cdots; q = 0,1,2,\cdots, \end{cases} \tag{7}$$

where m is a positive integer.

(3) Taking (5) with Eq.(1) into (4) and setting the coefficients of all powers of $F^i F'^j$ $(i = 0,1,2,\cdots; j = 0,1)$ to zero, yields a set of algebraic equations for unknown a_i, c_j $(i = 0,1,\cdots,n; j = 0,1,2,3)$ and ω.

(4) To take the expressions of c_0, c_1 given by *Case 1* to *Case 5* into the obtained algebraic equations in step three in separately, yields a new algebraic equations for a_i, c_j $(i = 0,1,\cdots,n; j = 2,3)$ and ω. And then solve this obtained new algebraic equations.

(5) Inserting the solutions of algebraic equations obtained in step four and the corresponding solutions $F_j(\xi)$ into (5), then we obtain the exact solutions of Eq.(3).

3 Illustrative examples

To illustrate our method, in this section we shall present the explicit exact traveling wave solutions of the generalized microstructure wave equation and the Sawada–Kotera equation by using the proposed third kind of elliptic equation expansion method.

3.1 The generalized microstructure wave equation

Let us consider the generalized microstructure wave equation

$$u_{tt} - bu_{xx} - \frac{\mu}{2}\left(u^2\right)_{xx} - \delta\left(\beta u_{tt} - \gamma u_{xx}\right)_{xx} = 0, \tag{8}$$

where $b, \mu, \delta, \beta, \gamma$ are constants, $u(x,t)$ is the microstrain wave function. This equation describes wave propagation in microstructured solids and first introduced by Leto and Choudhury [19]. The exact solitary wave solutions of Eq.(8) were obtained by Gao [20] using the G'/G–expansion method. Also the stability of the solitary wave solutions for Eq.(8) was studied by Zhao and Cui [21]. More recently, Wang and Zheng [22] obtained the traveling wave solutions including bright and dark soliton solutions of Eq.(8) by using the approach of dynamical system.

Under transformation (2), Eq.(8) can be changed into the following ODE

$$\left(\omega^2 - b\right) u'' - \frac{\mu}{2}\left(u^2\right)'' - \delta\left(\beta\omega^2 - \gamma\right) u^{(4)} = 0. \tag{9}$$

Setting $m = 3$ and using (7), we obtain that $O\left(u^{(4)}\right) = n + 2$ and $O\left(\left(u^2\right)''\right) = 2n + 1$. This shows that the balance between $u^{(4)}$ with $\left(u^2\right)''$ in (9) yields $n = 1$. Therefore, the solution of Eq.(9) can be taken as

$$u(\xi) = a_0 + a_1 F(\xi), \tag{10}$$

where a_0, a_1 are constants and $F(\xi)$ expresses the solution of Eq.(1).

Substituting (10) with Eq.(1) into Eq.(9) and setting the coefficients of F^j $(j = 0, 1, 2, 3)$ to zero, we obtain a set of algebraic equations

$$\begin{cases} \frac{15}{2}\delta\gamma a_1 c_3^2 - \frac{15}{2}\delta\omega^2\beta a_1 c_3^2 - \frac{5}{2}\mu a_1^2 c_3 = 0, \\ \frac{15}{2}\delta\gamma a_1 c_2 c_3 - \frac{15}{2}\delta\omega^2\beta a_1 c_2 c_3 - \frac{3}{2}\mu a_1 a_0 c_3 + \frac{3}{2}\omega^2 a_1 c_3 - \frac{3}{2} b a_1 c_3 - 2\mu a_1^2 c_2 = 0, \\ \frac{9}{2}\delta\gamma a_1 c_3 c_1 - \delta\omega^2\beta a_1 c_2^2 - \frac{9}{2}\delta\omega^2\beta a_1 c_3 c_1 - \mu a_1 a_0 c_2 + \delta\gamma a_1 c_2^2 + \omega^2 a_1 c_2 - b a_1 c_2 - \frac{3}{2}\mu a_1^2 c_1 = 0, \\ -\frac{1}{2}\delta\omega^2\beta a_1 c_2 c_1 - 3\delta\omega^2\beta a_1 c_3 c_0 + \frac{1}{2}\delta\gamma a_1 c_2 c_1 - \mu a_1^2 c_0 + 3\delta\gamma a_1 c_3 c_0 - \frac{1}{2}\mu a_1 a_0 c_1 - \frac{1}{2} b a_1 c_1 + \frac{1}{2}\omega^2 a_1 c_1 = 0. \end{cases}$$

If we take the expressions of c_0, c_1 in *Case 1* to *Case 5* into the above algebraic equations, then they will lead a same solution as below

$$\begin{cases} a_0 = \dfrac{c_2\delta\left(\gamma - \beta\omega^2\right) + \omega^2 - b}{\mu}, \\ a_1 = \dfrac{3c_3\delta\left(\gamma - \beta\omega^2\right)}{\mu}. \end{cases} \tag{11}$$

Now the explicit exact solutions of Eq.(8) can be obtained by inserting (11) with $F_i(i = 1, \cdots, 12)$ into (10) and they are

$$\begin{aligned} u_1(x,t) &= \frac{12\delta(\gamma - \beta\omega^2)}{\mu}\tanh^2(x - \omega t) + \frac{8\delta(\beta\omega^2 - \gamma) + \omega^2 - b}{\mu}, \\ u_2(x,t) &= \frac{12\delta(\gamma - \beta\omega^2)}{\mu}\coth^2(x - \omega t) + \frac{8\delta(\beta\omega^2 - \gamma) + \omega^2 - b}{\mu}, \\ u_3(x,t) &= \frac{12\delta(\gamma - \beta\omega^2)}{\mu}\tan^2(x - \omega t) + \frac{8\delta(\gamma - \beta\omega^2) + \omega^2 - b}{\mu}, \\ u_4(x,t) &= \frac{12\delta(\gamma - \beta\omega^2)}{\mu}\cot^2(x - \omega t) + \frac{8\delta(\gamma - \beta\omega^2) + \omega^2 - b}{\mu}, \\ u_5(x,t) &= \frac{\delta(\beta\omega^2 - \gamma)(4m^2 + 1) + \omega^2 - b}{\mu} + \frac{6\delta(\gamma - \beta\omega^2)}{\mu\left[1 + \mathrm{cn}(x - \omega t, m)\right]}, \end{aligned}$$

$$
\begin{aligned}
u_6(x,t) &= \frac{\delta(\beta\omega^2-\gamma)(4m^2-5)+\omega^2-b}{\mu} - \frac{6\delta(\gamma-\beta\omega^2)}{\mu\left[1+\mathrm{nc}(x-\omega t,m)\right]},\\
u_7(x,t) &= \frac{\delta(\beta\omega^2-\gamma)(m^2+4)+\omega^2-b}{\mu} + \frac{6m^2\delta(\gamma-\beta\omega^2)}{\mu\left[1+\mathrm{dn}(x-\omega t,m)\right]},\\
u_8(x,t) &= \frac{\delta(\gamma-\beta\omega^2)(5m^2-4)+\omega^2-b}{\mu} - \frac{6m^2\delta(\gamma-\beta\omega^2)}{\mu\left[1+\mathrm{nd}(x-\omega t,m)\right]},\\
u_9(x,t) &= \frac{\delta(\gamma-\beta\omega^2)(5m^2-1)+\omega^2-b}{\mu} - \frac{6\delta(m^2-1)(\gamma-\beta\omega^2)}{\mu\left[1+\mathrm{sn}(x-\omega t,m)\right]},\\
u_{10}(x,t) &= \frac{\delta(\beta\omega^2-\gamma)(m^2-5)+\omega^2-b}{\mu} + \frac{6\delta(m^2-1)(\gamma-\beta\omega^2)}{\mu\left[1+\mathrm{ns}(x-\omega t,m)\right]},\\
u_{11}(x,t) &= \frac{\delta(\gamma-\beta\omega^2)(5m^2-1)+\omega^2-b}{\mu} - \frac{6\delta(m^2-1)(\gamma-\beta\omega^2)}{\mu\left[1+\mathrm{cd}(x-\omega t,m)\right]},\\
u_{12}(x,t) &= \frac{\delta(\beta\omega^2-\gamma)(m^2-5)+\omega^2-b}{\mu} + \frac{6\delta(m^2-1)(\gamma-\beta\omega^2)}{\mu\left[1+\mathrm{dc}(x-\omega t,m)\right]},
\end{aligned}
$$

where ω is an arbitrary constants.

When $m \to 1$, the above Jacobi elliptic function solutions will degenerate to the solitary wave solutions of Eq.(8) as following

$$
\begin{aligned}
u(x,t) &= \frac{5\delta(\beta\omega^2-\gamma)+\omega^2-b}{\mu} + \frac{6\delta(\gamma-\beta\omega^2)}{\mu\left[1+\mathrm{sech}(x-\omega t)\right]},\\
u(x,t) &= \frac{\delta(\gamma-\beta\omega^2)+\omega^2-b}{\mu} - \frac{6\delta(\gamma-\beta\omega^2)}{\mu\left[1+\cosh(x-\omega t)\right]}.
\end{aligned}
$$

When $m \to 0$, the above Jacobi elliptic function solutions are reduced to the following periodic traveling wave solutions of Eq. (8)

$$
\begin{aligned}
u(x,t) &= \frac{\delta(\beta\omega^2-\gamma)+\omega^2-b}{\mu} + \frac{6\delta(\gamma-\beta\omega^2)}{\mu\left[1+\cos(x-\omega t)\right]},\\
u(x,t) &= \frac{5\delta(\gamma-\beta\omega^2)+\omega^2-b}{\mu} - \frac{6\delta(\gamma-\beta\omega^2)}{\mu\left[1+\sec(x-\omega t)\right]},\\
u(x,t) &= \frac{\delta(\beta\omega^2-\gamma)+\omega^2-b}{\mu} + \frac{6\delta(\gamma-\beta\omega^2)}{\mu\left[1+\sin(x-\omega t)\right]},\\
u(x,t) &= \frac{5\delta(\gamma-\beta\omega^2)+\omega^2-b}{\mu} - \frac{6\delta(\gamma-\beta\omega^2)}{\mu\left[1+\csc(x-\omega t)\right]}.
\end{aligned}
$$

3.2 The Sawada–Kotera equation

The Sawada–Kotera equation

$$u_t + u_{xxxxx} + 15uu_{xxx} + 15u_xu_{xx} + 45u^2u_x = 0, \tag{12}$$

is a special case of the fifth order KdV equation [23] which applied in quantum mechanics, nonlinear optics, motions of long waves in shallow water under gravity and in a one–dimensional nonlinear lattice. The exact solutions of Eq.(12) were obtained by using the exp–function method [24], the Hirota bilinear method [25] and the Riccati equation mapping method [26], etc.

Making the wave transformation (2), Eq.(12) is converted into the following ODE

$$-\omega u' + u^{(5)} + 15uu''' + 15u'u'' + 45u^2u' = 0. \tag{13}$$

Setting $m = 3$ and using (7) we obtain that $O\left(u^{(5)}\right) = n + 5/2$ and $O\left(u^2u'\right) = 3n + 1/2$. This shows that the balance between $u^{(5)}$ with u^2u' gives $n = 1$. Thus we can set the solution of Eq.(13) in the form

$$u(\xi) = a_0 + a_1F(\xi), \tag{14}$$

where a_0, a_1 are undetermined constants and $F(\xi)$ is the solution of Eq.(1).

Substituting (14) along with (1) into (13) and setting the coefficients of $F^j F'\ (j = 0, 1, 2)$ to zero, we get a set of algebraic equations

$$\begin{cases} \frac{45}{2}a_1c_3^2 + \frac{135}{2}a_1^2c_3 + 45a_1^3 = 0, \\ -\omega a_1 + a_1c_2^2 + \frac{9}{2}a_1c_1c_3 + 15a_0a_1c_2 + \frac{15}{2}a_1^2c_1 + 45a_0^2a_1 = 0, \\ 90a_0a_1^2 + 45a_0a_1c_3 + 30a_1^2c_2 + 15a_1c_2c_3 = 0. \end{cases} \tag{15}$$

By inserting c_0, c_1 in *Case 1* and *Case 2* into Eq.(15), then it gives a same solution

$$a_0 = -\frac{c_2}{6} \pm \frac{\sqrt{5\omega + 20}}{15}, a_1 = -\frac{c_3}{2}. \tag{16}$$

Now the explicit exact traveling wave solution of Eq.(12) can be obtained by taking (16) with $F_j\ (j = 1, 2, 3, 4)$ into (14) and they are

$$\begin{aligned} u_1(x,t) &= \frac{4}{3} \pm \frac{\sqrt{5\omega + 20}}{15} - 2\tanh^2(x - \omega t), \\ u_2(x,t) &= \frac{4}{3} \pm \frac{\sqrt{5\omega + 20}}{15} - 2\coth^2(x - \omega t), \\ u_3(x,t) &= -\frac{4}{3} \pm \frac{\sqrt{5\omega + 20}}{15} - 2\tan^2(x - \omega t), \\ u_4(x,t) &= -\frac{4}{3} \pm \frac{\sqrt{5\omega + 20}}{15} - 2\cot^2(x - \omega t), \end{aligned}$$

where $\omega > -4$ is an arbitrary constant.

When substitution of c_0, c_1 in *Case 3* into (15), it can be solved that

$$a_0 = -\frac{c_2}{6} \pm \frac{\sqrt{80m^4 - 80m^2 + 20\omega + 5}}{30}, a_1 = -\frac{c_3}{2}, \tag{17}$$

$$a_0 = \pm\frac{\sqrt{80m^4 - 80m^2 + 20\omega + 5}}{15}, a_1 = -\frac{c_3}{2}, c_2 = \mp\frac{\sqrt{80m^4 - 80m^2 + 20\omega + 5}}{5}. \tag{18}$$

Taking (17) or (18) with $F_j\ (j = 5, 6)$ into (14), we obtain the explicit exact traveling wave solutions of Eq.(12) as follows

$$\begin{aligned} u_5(x,t) &= \frac{2m^2}{3} + \frac{1}{6} \pm \frac{\sqrt{80m^4 - 80m^2 + 20\omega + 5}}{30} - \frac{1}{1 + cn\,(x - \omega t, m)}, \\ u_6(x,t) &= \frac{2m^2}{3} - \frac{5}{6} \pm \frac{\sqrt{80m^4 - 80m^2 + 20\omega + 5}}{30} + \frac{1}{1 + nc\,(x - \omega t, m)}, \end{aligned}$$

where $\omega > -4m^4 + 4m^2 - \frac{1}{4}$.

If we take c_0, c_1 in *Case 4* into (15), then it solves that

$$a_0 = -\frac{c_2}{6} \pm \frac{\sqrt{5m^4 - 80m^2 + 20\omega + 80}}{30}, a_1 = -\frac{c_3}{2}, \tag{19}$$

$$a_0 = \pm\frac{\sqrt{5m^4 - 80m^2 + 20\omega + 80}}{15}, a_1 = -\frac{c_3}{2}, c_2 = \mp\frac{\sqrt{5m^4 - 80m^2 + 20\omega + 80}}{5}. \tag{20}$$

Taking (19) or (20) with $F_j\ (j = 7, 8)$ into (14) yields the explicit exact traveling wave solutions of Eq.(12) as following

$$\begin{aligned} u_7(x,t) &= \frac{m^2}{6} + \frac{2}{3} \pm \frac{\sqrt{5m^4 - 80m^2 + 20\omega + 80}}{30} - \frac{m^2}{1 + dn\,(x - \omega t, m)}, \\ u_8(x,t) &= -\frac{5m^2}{6} + \frac{2}{3} \pm \frac{\sqrt{5m^4 - 80m^2 + 20\omega + 80}}{30} + \frac{m^2}{1 + nd\,(x - \omega t, m)}, \end{aligned}$$

where $\omega > -\frac{1}{4}m^4 + 4m^2 - 4$.

Inserting c_0, c_1 in *Case 5* into (15), then it solves that

$$a_0 = -\frac{c_2}{6} \pm \frac{\sqrt{5m^4 + 70m^2 + 20\omega + 5}}{30}, a_1 = -\frac{c_3}{2}, \tag{21}$$

$$a_0 = \pm\frac{\sqrt{5m^4 + 70m^2 + 20\omega + 5}}{15}, a_1 = -\frac{c_3}{2}, c_2 = \mp\frac{\sqrt{5m^4 + 70m^2 + 20\omega + 5}}{5}. \tag{22}$$

Substituting (21) or (22) with F_j $(j = 9, 10, 11, 12)$ into (14) yields the following explicit exact traveling wave solutions of Eq.(12)

$$u_9(x,t) = \frac{1-5m^2}{6} \pm \frac{\sqrt{5m^4 + 70m^2 + 20\omega + 5}}{30} - \frac{1-m^2}{1 + sn\,(x-\omega t, m)},$$

$$u_{10}(x,t) = \frac{m^2-5}{6} \pm \frac{\sqrt{5m^4 + 70m^2 + 20\omega + 5}}{30} + \frac{1-m^2}{1 + ns\,(x-\omega t, m)},$$

$$u_{11}(x,t) = \frac{1-5m^2}{6} \pm \frac{\sqrt{5m^4 + 70m^2 + 20\omega + 5}}{30} - \frac{1-m^2}{1 + cd\,(x-\omega t, m)},$$

$$u_{12}(x,t) = \frac{m^2-5}{6} \pm \frac{\sqrt{5m^4 + 70m^2 + 20\omega + 5}}{30} + \frac{1-m^2}{1 + dc\,(x-\omega t, m)},$$

where $\omega > -\frac{1}{4}m^4 - \frac{7}{2}m^2 - \frac{1}{4}$.

When $m \to 1$ and $m \to 0$, the above obtained Jacobi elliptic function solutions will degenerate respectively to the solitary wave solutions

$$u(x,t) = \frac{5}{6} \pm \frac{\sqrt{20\omega + 5}}{30} - \frac{1}{1 + \operatorname{sech}(x-\omega t)},$$

$$u(x,t) = -\frac{1}{6} \pm \frac{\sqrt{20\omega + 5}}{30} + \frac{1}{1 + \cosh(x-\omega t)},$$

and the periodic solutions

$$u(x,t) = \frac{1}{6} \pm \frac{\sqrt{20\omega + 5}}{30} - \frac{1}{1 + \cos(x-\omega t)},$$

$$u(x,t) = -\frac{5}{6} \pm \frac{\sqrt{20\omega + 5}}{30} + \frac{1}{1 + \sec(x-\omega t)},$$

$$u(x,t) = \frac{1}{6} \pm \frac{\sqrt{20\omega + 5}}{30} - \frac{1}{1 + \sin(x-\omega t)},$$

$$u(x,t) = -\frac{5}{6} \pm \frac{\sqrt{20\omega + 5}}{30} + \frac{1}{1 + \csc(x-\omega t)},$$

for Eq.(12), with an arbitrary constant ω satisfying the condition $\omega > -\frac{1}{4}$.

4 Conclusions

We have showed that the third kind of elliptic Eq.(1) has some explicit exact hyperbolic function solutions, trigonometric function solutions and Jacobi elliptic solutions and they can lead some new explicit exact traveling wave solutions for nonlinear equations. We believe that the new solutions for nonlinear equations found by our method may be used to explain some new features of the governing nonlinear equations and it is worthy to further study.

It is worth noting that our proposed method may cause two special cases which does not appear in other direct methods. First, just like the microstructure wave equation, the same solution of the algebraic equations in *Step 5* may produce different solutions of nonlinear equation. Second, the different solutions of the algebraic equations in *Step 5* may produce same solution of the nonlinear equation as like the Sawada–Kotera equation. Therefore, it is necessary to be more careful when you use it to solve nonlinear equations.

It is also noted that our approach can give more trigonometric periodic solutions for nonlinear equations than those were obtained by the rational sine–cosine method [27]. In Ref. [24], the rational $\cosh-\sinh$-type solutions of Sawada–Kotera equation were obtained by using the exp–function method but they failed to find the rational sech-type solution. But our method provides some different types of solutions, especially the explicit exact Jacobi elliptic function solutions, for nonlinear equation in a unified way and it is independent of other direct methods.

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