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Growth Measurements of Entire and Meromorphic Functions on the Basis of Their Integer Translation

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Abstract: The main purpose of the study of the growths of entire and meromorphic functions have usually been done through their Nevanlinna's characteristic function in association with those of exponential function. But if one is fascinated to compare the growths of any entire and meromorphic function relating to another, the notions of relative growth indicators will come. The field of study in this area may be more considerable through the intensive applications of the theories of slowly increasing functions which in fact means that $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a, i.e., $\lim_{r\to\infty} \frac{L(ar)}{L(r)} = 1$ where $L \equiv L(r)$ is a positive continuous function increasing slowly. Actually in the present paper, using the concepts of relative ${}_pL^*$-order, relative ${}_pL^*$-type and relative ${}_pL^*$-weak type, we establish some results depending on the comparative growth properties of entire and meromorphic functions on the basis of integer translation applied upon them where ${}_pL^*$ is nothing but a weaker assumption of L.

Keywords: Entire function; Meromorphic function; Relative ${}_pL^*$-order; Relative ${}_pL^*$-type; Relative ${}_pL^*$-weak type; Slowly increasing function; Integer translation

1 Introduction

Let $f(z)$ be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to $f(z)$ is defined on $|z| = r$ as $M_f(r) = \max|z| = r|f(z)|$. When $f(z)$ is meromorphic, one may define a different function $T_f(r)$ termed as Nevanlinna's characteristic function of $f(z)$, playing same role as maximum modulus function in the following manner:

$$T_f(r) = N_f(r) + m_f(r),$$

where the function $N_f(r,a)$ $\left(\overline{N}_f(r,a)\right)$ be known as counting function of a-points (distinct a-points) of meromorphic f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a)\log r,$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t} dt + \overline{n}_f(0,a)\log r\right).$$

Moreover, we denote by $n_f(r,a)$ $(\overline{n}_f(r,a))$, the number of a-points (distinct a-points) of f in $|z| \le r$ and an ∞-point is a pole of $f(z)$. In many occasions, $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

Further, the function $m_f(r,\infty)$ alternatively denoted by $m_f(r)$ known as the proximity function of $f(z)$ is defined as follows:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta, \quad \log^+ x = \max(\log x, 0)\, x \geqslant 0.$$

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Also we may denote $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r,a)$.

If $f(z)$ is an entire function, then the Nevanlinna's characteristic function $T_f(r)$ of $f(z)$ is defined as

$$T_f(r) = m_f(r).$$

However, throughout the present paper $\mathbb{N}$ always denote the set of all positive integers. Now considering this, let $f(z)$ be a meromorphic function and $n \in \mathbb{N}$, then the translation of $f(z)$ be denoted by $f(z+n)$. For each $n \in \mathbb{N}$, one may obtain a function with some properties. Let us consider this family by $f_n(z)$, where

$$f_n(z) = \{f(z+n) : n \in \mathbb{N}\}.$$

We should recall that if α is a regular point of an analytic function $f(z)$ and if $f(\alpha) = 0$ then α is called a zero of $f(z)$. The point $z = \alpha$ is called a zero of $f(z)$ of order or multiplicity m (m being a positive integer) if in some neighborhood of α, $f(z)$ can be expanded in a Taylor's series of the form $f(z) = \sum\limits_{n=m}^{\infty} a_n (z-\alpha)^n$, where $a_m \neq 0$.

It is clear that the number of zeros of $f(z)$ may be changed in a finite region after translation but it remains unaltered in the open complex plane $\mathbb{C}$ i.e.,

$$N_{f(z+n)}(r) = N_f(r) + e_n, \tag{1.1}$$

where e_n is a residue term such that $e_n \to 0$ as $r \to \infty$.

Also

$$m_{f(z+n)}(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}+n\right)\right| d\theta, i.e., \ m_{f(z+n)}(r) = m_f(r) + e_n^{'},$$

where $e_n^{'}$ (may be distinct from e_n) be such that $e_n^{'} \to 0$ as $r \to \infty$.

Therefore from (1.1) and (1), one may obtain that

$$N_{f(z+n)}(r) + m_{f(z+n)}(r) = N_f(r) + e_n + m_f(r) + e_n^{'}, i.e., \ T_{f(z+n)}(r) = T_f(r) + e_n + e_n^{'}.$$

Now if n varies then the Nevanlinna's characteristic function for the family $f_n(z)$ is

$$T_{f_n}(r) = nT_f(r) + n_{(e_n+e_n^{'})}. \tag{1.2}$$

However for any two meromorphic functions $f(z)$ and $g(z)$ the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of $f(z)$ with respect to $g(z)$ in terms of their Nevanlinna's Characteristic functions. The prime motive of the study of the growth analysis of entire and meromorphic functions have usually been done through their Nevanlinna's characteristic function in association with those of exponential function. But if one is fascinated to compare the growths of any entire and meromorphic function relating to another, the notions of relative growth indicators will come. The field of study in this area may be more considerable through the intensive applications of the theories of slowly increasing functions which in fact means that $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a, i.e., $\lim\limits_{r\to\infty} \frac{L(ar)}{L(r)} = 1$ where $L \equiv L(r)$ is a positive continuous function increasing slowly. Actually in this paper using the concepts of relative ${}_pL^*$-order, relative ${}_pL^*$-type and relative ${}_pL^*$-weak type, we aim at investigating some growth properties of entire and meromorphic functions on the basis of integer translation applied upon them where ${}_pL^*$ is nothing but a weaker assumption of L.

2 Definitions and Notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions, which are available in [4, 5, 9, 10]. We also use the standard notations and definitions of the theory of entire functions which are available in [11]. Therefore, we do not explain those in details. However, the notion of the growth indicators such as order and lower order of entire and meromorphic functions, which are generally used in computational purpose are defined in terms of their growth with respect to the exponential function as the following:

Definition 1 *The order $\rho(f)$ and the lower order $\lambda(f)$ of a meromorphic function $f(z)$ are defined as*

$$\begin{matrix}\rho(f) \\ \lambda(f)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log T_f(r)}{\log(r) + O(1)}.$$

The function $f(z)$ is said to be of regular growth when order and lower order of $f(z)$ are the same. Functions which are not of regular growth are said to be of irregular growth.

Somasundaram and Thamizharasi [7] introduced the notions of L-order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly, i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant "a". The more generalized concept of L-order and L-type of meromorphic functions are L^*-order and L^*-type respectively, which are as follows:

Definition 2 *[7] The L^*-order $\rho^{L^*}(f)$ and the L^*-lower order $\lambda^{L^*}(f)$ of a meromorphic function $f(z)$ are defined by*

$$\begin{matrix} \rho^{L^*}(f) \\ \lambda^{L^*}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]}.$$

Definition 3 *[7] The L^*-type $\sigma^{L^*}(f)$ and L^*-lower type $\overline{\sigma}^{L^*}(f)$ a meromorphic function $f(z)$ having non-zero finite L^*-order $\rho^{L^*}(f)$ are defined as*

$$\begin{matrix} \sigma^{L^*}(f) \\ \overline{\sigma}^{L^*}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\rho^{L^*}(f)}}.$$

Analogously in order to determine the relative growth of two meromorphic functions having same non zero finite L^*-lower order one may introduce the definition of L^*-weak type of meromorphic functions having finite positive L^*-lower order in the following way:

Definition 4 *The L^*-weak type denoted by $\tau^{L^*}(f)$ and the growth indicator $\overline{\tau}^{L^*}(f)$ of a meromorphic function $f(z)$ having non-zero finite L^*-lower order $\lambda^{L^*}(f)$ are defined as*

$$\begin{matrix} \overline{\tau}^{L^*}(f) \\ \tau^{L^*}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\lambda^{L^*}(f)}}.$$

Extending the notion of Somasundaram and Thamizharasi [7], one may introduce concept of ${}_pL^*$-order, ${}_pL^*$-type and ${}_pL^*$-weak type of a meromorphic function $f(z)$ which are as follows:

Definition 5 *The ${}_pL^*$-order $\rho_p^{L^*}(f)$ and the ${}_pL^*$-lower order $\lambda_p^{L^*}(f)$ of a meromorphic function $f(z)$ are defined by*

$$\begin{matrix} \rho_p^{L^*}(f) \\ \lambda_p^{L^*}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log T_f(r)}{\log\left[r\exp^{[p]} L(r)\right]},$$

where p is any positive integer.

Definition 6 *The ${}_pL^*$-type $\sigma_p^{L^*}(f)$ and ${}_pL^*$-lower type $\overline{\sigma}_p^{L^*}(f)$ of a meromorphic function $f(z)$ having non-zero finite ${}_pL^*$-order $\rho_p^{L^*}(f)$ are defined as*

$$\begin{matrix} \sigma_p^{L^*}(f) \\ \overline{\sigma}_p^{L^*}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{T_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f)}},$$

where p is any positive integer.

Definition 7 *The ${}_pL^*$-weak type $\tau_p^{L^*}(f)$ and the growth indicator $\overline{\tau}_p^{L^*}(f)$ of a meromorphic function $f(z)$ having non-zero finite ${}_pL^*$-lower order $\lambda_p^{L^*}(f)$ are defined as*

$$\begin{matrix} \overline{\tau}_p^{L^*}(f) \\ \tau_p^{L^*}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{T_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\lambda_p^{L^*}(f)}},$$

where p is any positive integer.

Lahiri and Banerjee [6] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 8 *[6] Let $f(z)$ be meromorphic and $g(z)$ be entire. The relative order of $f(z)$ with respect to $g(z)$ denoted by $\rho_g(f)$ is defined as*

$$\begin{aligned}\rho(f,g) &= \inf\{\mu>0 : T_f(r)<T_g(r^{\mu}) \text{ for all sufficiently large } r\}\\ &= \limsup_{r\to\infty}\frac{\log T_g^{-1}(T_f(r))}{\log r}.\end{aligned}$$

The definition coincides with the classical one [6] if $g(z)=\exp z$. Similarly one can define the relative lower order of a meromorphic function $f(z)$ with respect to an entire $g(z)$ denoted by $\lambda_g(f)$ in the following manner :

$$\lambda(f,g)=\liminf_{r\to\infty}\frac{\log T_g^{-1}(T_f(r))}{\log r}.$$

In order to make some progress in the study of relative order, now we introduce relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of a meromorphic function $f(z)$ with respect to an entire $g(z)$ which are as follows:

Definition 9 *[2] The relative ${}_pL^*$-order $\rho_p^{L^*}(f,g)$ and relative ${}_pL^*$-lower order $\lambda_p^{L^*}(f,g)$ of a meromorphic function $f(z)$ with respect to an entire $g(z)$ are defined as*

$$\begin{matrix}\rho_p^{L^*}(f,g)\\ \lambda_p^{L^*}(f,g)\end{matrix} = \lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log T_g^{-1}(T_f(r))}{\log\left[r\exp^{[p]}L(r)\right]},$$

where p is any positive integers.

Further to compare the relative growth of two meromorphic functions having same non zero finite relative ${}_pL^*$-order with respect to another entire function, one may introduce the definitions of relative ${}_pL^*$-type and relative ${}_pL^*$-lower type in the following manner:

Definition 10 *[2, 3] The relative ${}_pL^*$-type $\sigma_p^{L^*}(f,g)$ and relative ${}_pL^*$-lower type $\overline{\sigma}_p^{L^*}(f,g)$ of a meromorphic function $f(z)$ with respect to an entire function $g(z)$ having non-zero finite relative ${}_pL^*$-order $\rho_p^{L^*}(f,g)$ are defined as*

$$\begin{matrix}\sigma_p^{L^*}(f,g)\\ \overline{\sigma}_p^{L^*}(f,g)\end{matrix} = \lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{T_g^{-1}(T_f(r))}{\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,g)}},$$

where p is any positive integers.

Analogously to determine the relative growth of two meromorphic functions having same non zero finite relative ${}_pL^*$-lower order with respect to an entire function one may introduce the definition of relative ${}_pL^*$-weak type in the following way:

Definition 11 *[2, 3] The relative ${}_pL^*$-weak type $\tau_p^{L^*}(f,g)$ and the growth indicator $\overline{\tau}_p^{L^*}(f,g)$ of a meromorphic function $f(z)$ having non-zero finite relative ${}_pL^*$-lower order $\lambda_p^{L^*}(f,g)$ are defined as*

$$\begin{matrix}\overline{\tau}_p^{L^*}(f,g)\\ \tau_p^{L^*}(f,g)\end{matrix} = \lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{T_g^{-1}(T_f(r))}{\left[r\exp^{[p]}L(r)\right]^{\lambda_p^{L^*}(f,g)}},$$

where p is any positive integer.

3 Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 1 *[1] Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with regular growth and non zero finite order. If $f_n(z)=f(z+n)$ and $g_m(z)=g(z+m)$ for $m,n\in\mathbb{N}$, then*

$$\lim_{r\to\infty}\frac{\log T_{g_m}^{-1}(T_{f_n}(r))}{\log T_g^{-1}(T_f(r))}=1.$$

Lemma 2 *[1] Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function of regular growth and non zero finite type. If $f_n(z) = f(z+n)$ and $g_m(z) = g(z+m)$ for $m, n \in \mathbb{N}$, then*

$$\lim_{r\to\infty} \frac{T_{g_m}^{-1}(T_{f_n}(r))}{T_g^{-1}(T_f(r))} = \left(\frac{n}{m}\right)^{\frac{1}{\rho_g}}.$$

Lemma 3 *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with regular growth. If $f_n(z) = f(z+n)$ and $g_m(z) = g(z+m)$ for $m, n \in \mathbb{N}$, then for any positive integer p, the relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of $f_n(z)$ with respect to $g_m(z)$ are same as those of $f(z)$ with respect to $g(z)$.*

Proof. By Lemma 1, we obtain that

$$\begin{aligned}
\rho_p^{L^*}(f_n, g_m) &= \limsup_{r\to\infty} \frac{\log T_{g_m}^{-1}(T_{f_n}(r))}{\log\left[r\exp^{[p]} L(r)\right]} \\
&= \limsup_{r\to\infty} \left\{ \frac{\log T_g^{-1}(T_f(r))}{\log\left[r\exp^{[p]} L(r)\right]} \cdot \frac{\log T_{g_m}^{-1}(T_{f_n}(r))}{\log T_g^{-1} T_f(r)} \right\} \\
&= \limsup_{r\to\infty} \frac{\log T_g^{-1}(T_f(r))}{\log\left[r\exp^{[p]} L(r)\right]} \cdot \lim_{r\to\infty} \frac{\log T_{g_m}^{-1}(T_{f_n}(r))}{\log T_g^{-1} T_f(r)} \\
&= \rho_p^{L^*}(f, g) \cdot 1 \\
&= \rho_p^{L^*}(f, g).
\end{aligned}$$

In a similar manner, $\lambda_p^{L^*}(f_n, g_m) = \lambda_p^{L^*}(f, g)$.

This proves the lemma. ■

Lemma 4 *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with regular growth and non zero finite type. If $f_n(z) = f(z+n)$ and $g_m(z) = g(z+m)$ for $m, n \in \mathbb{N}$, then for any positive integer p, the relative ${}_pL^*$-type and relative ${}_pL^*$-lower type of $f_n(z)$ with respect to $g_m(z)$ are $\left(\frac{n}{m}\right)^{\frac{1}{\rho_g}}$ times that of $f(z)$ with respect to $g(z)$ if $\rho_p^{L^*}(f, g)$ is positive finite.*

Proof. By Lemma 2, Lemma 3, and above we get that

$$\begin{aligned}
\sigma_p^{L^*}(f_n, g_m) &= \limsup_{r\to\infty} \frac{T_{g_m}^{-1}(T_{f_n}(r))}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f_n, g_m)}} \\
&= \lim_{r\to\infty} \frac{T_{g_m}^{-1}(T_{f_n}(r))}{T_g^{-1} T_f(r)} \cdot \limsup_{r\to\infty} \frac{T_g^{-1}(T_f(r))}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f, g)}} \\
&= \left(\frac{n}{m}\right)^{\frac{1}{\rho_g}} \cdot \sigma_p^{L^*}(f, g).
\end{aligned}$$

Similarly $\overline{\sigma}_p^{L^*}(f_n, g_m) = \left(\frac{n}{m}\right)^{\frac{1}{\rho_g}} \cdot \overline{\sigma}_p^{L^*}(f, g)$.

This proves the lemma. ■

In the line of Lemma 4, we may state the following lemma without its proof.

Lemma 5 *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with regular growth and non zero finite type. If $f_n(z) = f(z+n)$ and $g_m(z) = g(z+m)$ for $m, n \in \mathbb{N}$, then for $0 < \lambda_p^{L^*}(f, g) < \infty$, $\tau_p^{L^*}(f_n, g_m)$ and $\overline{\tau}_p^{L^*}(f_n, g_m)$ are $\left(\frac{n}{m}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g, i.e.,*

$$\tau_p^{L^*}(f_n, g_m) = \left(\frac{n}{m}\right)^{\frac{1}{\rho_g}} \cdot \tau_p^{L^*}(f, g),$$

and

$$\overline{\tau}_p^{L^*}(f_n, g_m) = \left(\frac{n}{m}\right)^{\frac{1}{\rho_g}} \cdot \overline{\tau}_p^{L^*}(f, g).$$

4 Main Results

In this section, we present the main results of the paper.

Theorem 6 *Suppose $f(z), g(z)$ be any two meromorphic functions and $h(z), k(z)$ be any two entire functions such that $0 < \overline{\sigma}_p^{L^*}(f,h) \leq \sigma_p^{L^*}(f,h) < \infty$, $0 < \overline{\sigma}_p^{L^*}(g,k) \leq \sigma_p^{L^*}(g,k) < \infty$ and $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g,k)$ where p is any positive integer. Also let $h(z)$ and $k(z)$ be of regular growth having non zero finite type. If $f_n(z) = f(z+n)$, $g_m(z) = g(z+m)$, $h_l(z) = h(z+l)$ and $k_q(z) = k(z+q)$ for $m, n, l, q \in \mathbb{N}$, then*

$$\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \sigma_p^{L^*}(g,k)} \leq \liminf_{r\to\infty} \frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))} \leq$$

$$\min\left\{\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \overline{\sigma}_p^{L^*}(g,k)}, \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \sigma_p^{L^*}(g,k)}\right\} \leq$$

$$\max\left\{\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \overline{\sigma}_p^{L^*}(g,k)}, \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \sigma_p^{L^*}(g,k)}\right\} \leq$$

$$\limsup_{r\to\infty} \frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))} \leq \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \overline{\sigma}_p^{L^*}(g,k)}.$$

Proof. From the definition of $\overline{\sigma}_p^{L^*}(f_n, h_l)$ and in view of Lemma 3 and Lemma 4, we have for arbitrary positive ε and for all sufficiently large values of r that

$$T_{h_l}^{-1}(T_{f_n}(r)) \geq \left(\overline{\sigma}_p^{L^*}(f_n, h_l) - \varepsilon\right)\left(r \exp^{[p]} L(r)\right)^{\rho_p^{L^*}(f_n,h_l)},$$

$$i.e.,\ T_{h_l}^{-1}(T_{f_n}(r)) \geq \left(\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h) - \varepsilon\right)\left(r \exp^{[p]} L(r)\right)^{\rho_p^{L^*}(f,h)}. \tag{4.1}$$

Also we obtain from the definition of $\sigma_p^{L^*}(g_m, k_q)$ for all sufficiently large values of r and for arbitrary positive ε that

$$T_{k_q}^{-1}(T_{g_m}(r)) \leq \left(\sigma_p^{L^*}(g_m, k_q) + \varepsilon\right)\left(r \exp^{[p]} L(r)\right)^{\rho_p^{L^*}(g_m,k_q)},$$

$$i.e.,\ T_{k_q}^{-1}(T_{g_m}(r)) \leq \left(\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \sigma_p^{L^*}(g,k) + \varepsilon\right)\left(r \exp^{[p]} L(r)\right)^{\rho_p^{L^*}(g,k)}. \tag{4.2}$$

Now from (4.1), (4.2) and the condition $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g,k)$, it follows all sufficiently large values of r that

$$\frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))} \geqslant \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h) - \varepsilon}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \sigma_p^{L^*}(g,k) + \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to\infty} \frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))} \geqslant \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \sigma_p^{L^*}(g,k)}. \tag{4.3}$$

Again in view of Lemma 3 and Lemma 4 we get for a sequence of values of r tending to infinity that

$$T_{h_l}^{-1}\left(T_{f_n}\left(r\right)\right) \leq \left(\overline{\sigma}_p^{L^*}\left(f_n, h_l\right) + \varepsilon\right)\left(r \exp^{[p]} L\left(r\right)\right)^{\rho_p^{L^*}\left(f_n, h_l\right)},$$

$$i.e.,\ T_{h_l}^{-1}\left(T_{f_n}\left(r\right)\right) \leq \left(\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}\left(f, h\right) + \varepsilon\right)\left(r \exp^{[p]} L\left(r\right)\right)^{\rho_p^{L^*}\left(f, h\right)}, \quad (4.4)$$

and for all sufficiently large values of r that

$$T_{k_q}^{-1}\left(T_{g_m}\left(r\right)\right) \geq \left(\overline{\sigma}_p^{L^*}\left(g_m, k_q\right) - \varepsilon\right)\left(r \exp^{[p]} L\left(r\right)\right)^{\rho_p^{L^*}\left(g_m, k_q\right)},$$

$$i.e.,\ T_{k_q}^{-1}\left(T_{g_m}\left(r\right)\right) \geq \left(\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \overline{\sigma}_p^{L^*}\left(g, k\right) - \varepsilon\right)\left(r \exp^{[p]} L\left(r\right)\right)^{\rho_p^{L^*}\left(g, k\right)}. \quad (4.5)$$

Combining (4.4) and (4.5) and the condition $\rho_p^{L^*}\left(f, h\right) = \rho_p^{L^*}\left(g, k\right)$, we get for a sequence of values of r tending to infinity that

$$\frac{T_{h_l}^{-1}\left(T_{f_n}\left(r\right)\right)}{T_{k_q}^{-1}\left(T_{g_m}\left(r\right)\right)} \leq \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}\left(f, h\right) + \varepsilon}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \overline{\sigma}_p^{L^*}\left(g, k\right) - \varepsilon}.$$

Since $\varepsilon\left(> 0\right)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{T_{h_l}^{-1}\left(T_{f_n}\left(r\right)\right)}{T_{k_q}^{-1}\left(T_{g_m}\left(r\right)\right)} \leq \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}\left(f, h\right)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \overline{\sigma}_p^{L^*}\left(g, k\right)}. \quad (4.6)$$

Further, in view of Lemma 3 and Lemma 4, and for a sequence of values of r tending to infinity it follows that

$$T_{k_q}^{-1}\left(T_{g_m}\left(r\right)\right) \leq \left(\overline{\sigma}_p^{L^*}\left(g_m, k_q\right) + \varepsilon\right)\left(r \exp^{[p]} L\left(r\right)\right)^{\rho_p^{L^*}\left(g_m, k_q\right)},$$

$$i.e.,\ T_{k_q}^{-1}\left(T_{g_m}\left(r\right)\right) \leq \left(\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \overline{\sigma}_p^{L^*}\left(g, k\right) + \varepsilon\right)\left(r \exp^{[p]} L\left(r\right)\right)^{\rho_p^{L^*}\left(g, k\right)}. \quad (4.7)$$

Now from (4.1), (4.7) and the condition $\rho_p^{L^*}\left(f, h\right) = \rho_p^{L^*}\left(g, k\right)$, we obtain for a sequence of values of r tending to infinity that

$$\frac{T_{h_l}^{-1}\left(T_{f_n}\left(r\right)\right)}{T_{k_q}^{-1}\left(T_{g_m}\left(r\right)\right)} \geq \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}\left(f, h\right) - \varepsilon}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \overline{\sigma}_p^{L^*}\left(g, k\right) + \varepsilon}.$$

As $\varepsilon\left(> 0\right)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty} \frac{T_{h_l}^{-1}\left(T_{f_n}\left(r\right)\right)}{T_{k_q}^{-1}\left(T_{g_m}\left(r\right)\right)} \geq \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}\left(f, h\right)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \overline{\sigma}_p^{L^*}\left(g, k\right)}. \quad (4.8)$$

Also in view of Lemma 3 and Lemma 4, it follows for all sufficiently large values of r that

$$T_{h_l}^{-1}\left(T_{f_n}\left(r\right)\right) \leq \left(\sigma_p^{L^*}\left(f_n, h_l\right) + \varepsilon\right)\left(r \exp^{[p]} L\left(r\right)\right)^{\rho_p^{L^*}\left(f_n, h_l\right)},$$

$$i.e.,\ T_{h_l}^{-1}\left(T_{f_n}\left(r\right)\right) \leq \left(\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}\left(f, h\right) + \varepsilon\right)\left(r \exp^{[p]} L\left(r\right)\right)^{\rho_p^{L^*}\left(f, h\right)}. \quad (4.9)$$

In view of the condition $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g,k)$, it follows from (4.5) and (4.9) for all sufficiently large values of r that

$$\frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))} \le \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)+\varepsilon}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \overline{\sigma}_p^{L^*}(g,k)-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))} \le \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \overline{\sigma}_p^{L^*}(g,k)}. \quad (4.10)$$

Further, in view of Lemma 3 and Lemma 4, we get from the definition of $\sigma_{k_q}(g_m)$, for a sequence of values of r tending to infinity that

$$T_{k_q}^{-1}(T_{g_m}(r)) \ge \left(\sigma_p^{L^*}(g_m,k_q)-\varepsilon\right)\left(r\exp^{[p]}L(r)\right)^{\rho_p^{L^*}(g_m,k_q)},$$

$$i.e.,\ T_{k_q}^{-1}(T_{g_m}(r)) \ge \left(\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \sigma_p^{L^*}(g,k)-\varepsilon\right)\left(r\exp^{[p]}L(r)\right)^{\rho_p^{L^*}(g,k)}. \quad (4.11)$$

Now from (4.9), (4.11) and the condition $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g,k)$, it follows for a sequence of values of r tending to infinity that

$$\frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))} \le \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)+\varepsilon}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \sigma_p^{L^*}(g,k)-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))} \le \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \sigma_p^{L^*}(g,k)}. \quad (4.12)$$

Again in view of Lemma 3 and Lemma 4, we get for a sequence of values of r tending to infinity that

$$T_{h_l}^{-1}(T_{f_n}(r)) \geqslant \left(\sigma_p^{L^*}(f_n,h_l)-\varepsilon\right)\left(r\exp^{[p]}L(r)\right)^{\rho_p^{L^*}(f_n,h_l)},$$

$$i.e.,\ T_{h_l}^{-1}(T_{f_n}(r)) \geqslant \left(\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)-\varepsilon\right)\left(r\exp^{[p]}L(r)\right)^{\rho_p^{L^*}(f,h)}. \quad (4.13)$$

So combining (4.2) and (4.13), in view of the condition $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g,k)$, we get for a sequence of values of r tending to infinity that

$$\frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))} \geqslant \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)-\varepsilon}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \sigma_p^{L^*}(g,k)+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty}\frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))} \geqslant \frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}} \cdot \sigma_p^{L^*}(g,k)}. \quad (4.14)$$

Thus the theorem follows from (4.3), (4.6), (4.8), (4.10), (4.12), and (4.14). ■

Now in the line of Theorem 6 and with the help of Lemma 5, one can easily prove the following theorem using the notion of relative weak type and therefore its proof is omitted.

Theorem 7 *Suppose $f(z)$, $g(z)$ be any two meromorphic functions and $h(z)$, $k(z)$ be any two entire functions such that $0<\tau_p^{L^*}(f,h)\le\overline{\tau}_p^{L^*}(f,h)<\infty$, $0<\tau_p^{L^*}(g,k)\le\overline{\tau}_p^{L^*}(g,k)<\infty$ and $\lambda_p^{L^*}(f,h)=\lambda_p^{L^*}(g,k)$, where p is any positive integer. Also let $h(z)$ and $k(z)$ be of regular growth having non zero finite type. If $f_n(z)=f(z+n)$, $g_m(z)=g(z+m)$, $h_l(z)=h(z+l)$ and $k_q(z)=k(z+q)$ for $m,n,l,q\in\mathbb{N}$, then*

$$\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\tau_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\overline{\tau}_p^{L^*}(g,k)}\le\liminf_{r\to\infty}\frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))}\le$$

$$\min\left\{\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\tau_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\tau_p^{L^*}(g,k)},\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\overline{\tau}_p^{L^*}(g,k)}\right\}\le$$

$$\max\left\{\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\tau_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\tau_p^{L^*}(g,k)},\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\overline{\tau}_p^{L^*}(g,k)}\right\}\le$$

$$\limsup_{r\to\infty}\frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))}\le\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\tau_p^{L^*}(g,k)}.$$

We may now state the following two theorems without their proofs based on relative type and relative weak type.

Theorem 8 *Suppose $f(z)$, $g(z)$ be any two meromorphic functions and $h(z)$, $k(z)$ be any two entire functions such that $0<\overline{\sigma}_p^{L^*}(f,h)\le\sigma_p^{L^*}(f,h)<\infty$, $0<\tau_p^{L^*}(g,k)\le\overline{\tau}_p^{L^*}(g,k)<\infty$ and $\rho_p^{L^*}(f,h)=\lambda_p^{L^*}(g,k)$ where p is any positive integer. Also let $h(z)$ and $k(z)$ be of regular growth having non zero finite type. If $f_n(z)=f(z+n)$, $g_m(z)=g(z+m)$, $h_l(z)=h(z+l)$ and $k_q(z)=k(z+q)$ for $m,n,l,q\in\mathbb{N}$, then*

$$\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\overline{\tau}_p^{L^*}(g,k)}\le\liminf_{r\to\infty}\frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))}\le$$

$$\min\left\{\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\tau_p^{L^*}(g,k)},\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\overline{\tau}_p^{L^*}(g,k)}\right\}\le$$

$$\max\left\{\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\tau_p^{L^*}(g,k)},\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\overline{\tau}_p^{L^*}(g,k)}\right\}\le$$

$$\limsup_{r\to\infty}\frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))}\le\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\tau_p^{L^*}(g,k)}.$$

Theorem 9 *Suppose $f(z)$, $g(z)$ be any two meromorphic functions and $h(z)$, $k(z)$ be any two entire functions such that $0<\tau_p^{L^*}(f,h)\le\overline{\tau}_p^{L^*}(f,h)<\infty$, $0<\overline{\sigma}_p^{L^*}(g,k)\le\sigma_p^{L^*}(g,k)<\infty$ and $\lambda_p^{L^*}(f,h)=\rho_p^{L^*}(g,k)$ where p is any positive integer. Also let $h(z)$ and $k(z)$ be of regular growth having non zero finite type. If $f_n(z)=f(z+n)$, $g_m(z)=g(z+m)$, $h_l(z)=h(z+l)$ and $k_q(z)=k(z+q)$ for $m,n,l,q\in\mathbb{N}$, then*

$$\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\tau_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\sigma_p^{L^*}(g,k)}\le\liminf_{r\to\infty}\frac{T_{h_l}^{-1}(T_{f_n}(r))}{T_{k_q}^{-1}(T_{g_m}(r))}\le$$

$$\min\left\{\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\tau_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\overline{\sigma}_p^{L^*}(g,k)},\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\sigma_p^{L^*}(g,k)}\right\}\leq$$

$$\max\left\{\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\tau_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\overline{\sigma}_p^{L^*}(g,k)},\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\sigma_p^{L^*}(g,k)}\right\}\leq$$

$$\limsup_{r\to\infty}\frac{T_{h_l}^{-1}\left(T_{f_n}(r)\right)}{T_{k_q}^{-1}\left(T_{g_m}(r)\right)}\leq\frac{\left(\frac{n}{l}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_p^{L^*}(f,h)}{\left(\frac{m}{q}\right)^{\frac{1}{\rho_k}}\cdot\overline{\sigma}_p^{L^*}(g,k)}.$$

5 Conclusion

In this paper, some comparative growth properties of entire and meromorphic functions on the basis of integer translation applied upon them are studied using the notions of relative ${}_pL^*$-order, relative ${}_pL^*$-type, and relative ${}_pL^*$-weak type. Actually, we are trying to extend and modify the notion of L-order, L-type, and L-weak type to relative ${}_pL^*$-order, relative ${}_pL^*$-type, and relative ${}_pL^*$-weak type respectively and generalize the growth properties of entire and meromorphic functions under certain different conditions. However, the concepts of (p,q)-φ order, (p,q)-φ lower order of entire, and meromorphic functions were introduced by Shen et al. [8], where $p\geq q\geq 1$ and $\varphi:[0,+\infty)\to(0,+\infty)$ be a non-decreasing unbounded function. For details about (p,q)-φ order and (p,q)-φ lower order, one may see [8]. Revisiting the ideas developed by Shen et al. [8] one can introduce the definitions of relative (p,q)-φ order, relative (p,q)-φ type, and relative (p,q)-φ weak type. Accordingly the results which we have established in this paper may be modified by the treatment of the notions of relative (p,q)-φ order, relative (p,q)-φ type, and relative (p,q)-φ weak type.

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