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# The Peakons and Periodic Cusp Waves Solutions of the Modified DP Equation

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**Abstract:** In this paper, by the qualitative analysis methods of planar dynamical systems we investigate soliton wave solutions for the Degasperis–Procesi equation with the dispersion term. With the phase portrait bifurcation of traveling wave system, periodic cusp wave solutions and peakon wave solutions are gotten and the graph of the numerical simulation solutions is showed.

Keywords: DP equation with dispersion term; Periodic cusp wave solutions; Peakon wave solutions; Planar dynamical systems

## 1 Introduction

Degasperis–Procesi [1] equation is written in dispersionless form as follows:

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.1}$$

It plays an important role in understanding the physical phenomena of nonlinear systems. Various powerful methods, such as Hirota method, homogeneous balance method, Painleve method, have been presented in finding the explicit exact solution, in particular, solitary wave solutions, of nonlinear evolution equations. It's integrable by constructing its Lax pair and a bi-Hamiltonian structure [2-3]. Multi-peakon solutions is gotten in [4]. Zhou [5] discussed the phenomenon of blow-up and then proved the global existence of the solution. Shock wave solutions were obtained in [6]. Lenells [7] classified all its weak traveling wave solutions. Yu [8-9] discussed the dynamical bifurcation behavior and got loop solutions, breaking kink wave solutions, solitary wave solutions and periodic wave solutions. Vakhnenko and Parkes [10] derived periodic and solitary wave solutions. Qiao [11] obtained the peakons, dehisced solitons, cuspons and new 1-peak solitons. Liang [12] derived travelling wave solutions to the modified DP equation. Rodriguez [13] discussed the general Degasperis-Procesi equation and got solitary wave solutions. Helal [14] investigated stability analysis solutions of the nonlinear the modified Degasperis–Procesi equation. Also, much work has been devoted to the study of the DP equation in [15–18].

We'll consider the following equation with the dispersion term:

$$u_t - u_{xxt} + 4uu_x + \gamma(u - u_{xx})_x = 3u_x u_{xx} + uu_{xxx}, \tag{1.2}$$

which presents a quite rich structure and is effected by the dispersion term.

The remainder of the paper is organized as follows. In Section 2, using the traveling wave transformation, we transform Eq.(1.2) into a planar dynamical system and then discuss bifurcations of phase portraits of this system. In Section 3, we obtain the expressions for the the peakons and periodic cusp waves solutions of Eq.(1.2). A short conclusion is given in Section 4.

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## 2 The bifurcation of phase portraits of DP equation with the dispersion term

Let $\xi = x - ctu = \varphi(x-ct) = \varphi(\xi)$, substituting it into (1.2), we get

$$-c\varphi' + c\varphi''' + 4\varphi\varphi' + \gamma(\varphi' - \varphi''') = 3\varphi'\varphi'' + \varphi\varphi'''. \tag{2.1}$$

Integrating the above equation, we get

$$(\varphi - c + \gamma)\varphi'' = g - (c-\gamma)\varphi + 2\varphi^2 - (\varphi')^2.$$

Let $\frac{d\varphi}{d\xi} = y$, we get the following equations:

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{g-(c-\gamma)\varphi+2\varphi^2-y^2}{\varphi-c+\gamma}. \end{cases} \tag{2.2}$$

In the system (2.2), there is the singular line $\varphi = c - \gamma$. Under the transformation $d\xi = (\varphi - c + \gamma)d\tau$, system (2.2) becomes

$$\begin{cases} \frac{d\varphi}{d\xi} = (\varphi - c + \gamma)y, \\ \frac{dy}{d\xi} = g - (c-\gamma)\varphi + 2\varphi^2 - y^2. \end{cases} \tag{2.3}$$

Let $H(\varphi, y) = (\varphi - c+\gamma)^2(y^2 - \varphi^2 - g)$, we get the same first integral $H(\varphi, y) = h$ to the system (2.2) and (2.3), so system (2.3) have the same topological phase portraits as system (2.3) except the straight line $\varphi = c - \gamma$. We'll consider the phase portraits of (2.3).

Let $A_\pm(c-\gamma, \pm\sqrt{(c-\gamma)^2+g})$, $B_\pm(\frac{c-\gamma\pm\sqrt{(c-\gamma)^2-8g}}{4}, 0)$, then $H(A_\pm) = 0$,

$$H(B_+) = \frac{c-\gamma+\sqrt{(c-\gamma)^2-8g}}{4}\left[\frac{\sqrt{(c-\gamma)^2-8g}-3(c-\gamma)}{4}\right]^3,$$

$$H(B_-) = -\frac{c-\gamma-\sqrt{(c-\gamma)^2-8g}}{4}\left[\frac{\sqrt{(c-\gamma)^2-8g}+3(c-\gamma)}{4}\right]^3.$$

When $g > \frac{(c-\gamma)^2}{8}$, system (2.3) has two singular points $A_\pm$.
When $g = \frac{(c-\gamma)^2}{8}$, system (2.3) has three singular points $(c-\gamma, \pm\frac{3(c-\gamma)}{2\sqrt{2}})$, $(\frac{c-\gamma}{4}, 0)$.
When $-(c-\gamma)^2 < g < \frac{(c-\gamma)^2}{8}$, system (2.3) has four singular points $A_\pm B_\pm$.
When $g = -(c-\gamma)^2$, system (2.3) has two singular points $(c-\gamma, 0)$, $(-\frac{c-\gamma}{2}, 0)$.
When $g < -(c-\gamma)^2$, system (2.3) has two singular points $B_\pm$.

According to the qualitative theory of differential equations, we get the following facts:
$(i) A_\pm$ are saddle points.
$(ii)(c-\gamma, \pm\frac{3(c-\gamma)}{2\sqrt{2}})$ is a saddle point, $(\frac{c-\gamma}{4}, 0)$ is a cusp point.
$(iii)$ When $\gamma > c$, $B_+$is a saddle point, $B_-$is a center point; when $\gamma < c$, $B_-$is a saddle point, $B_+$is a center point.
$(iv)(-\frac{c-\gamma}{2}, 0)$ is a saddle point.
$(v) B_\pm$ are saddle points.

According to the qualitative theory of dynamical systems and the results in proposition, we draw the bifurcation of phase portraits of system (2.3) as Fig.1 and Fig.2.

## 3 The peakons

### 3.1 Peakons from the limit of solitary waves

Since the solitary wave solution of Eq.(1.2) corresponds to a homoclinic orbit of system (2.2), when the parameter expression of the homoclinic orbit is $\varphi = \varphi(\xi)$ and $y = y(\xi)$, then $u = \varphi(\xi)$ with $\xi = x - ct$ is a solitary wave solution

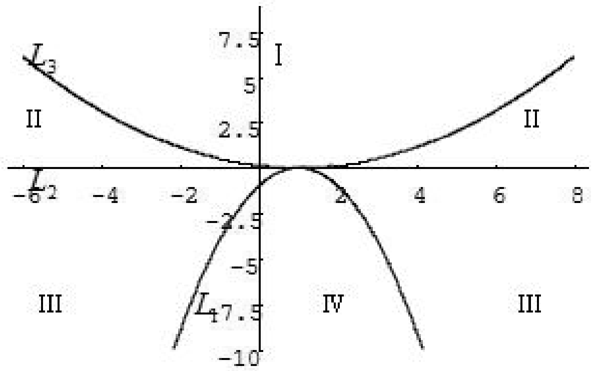
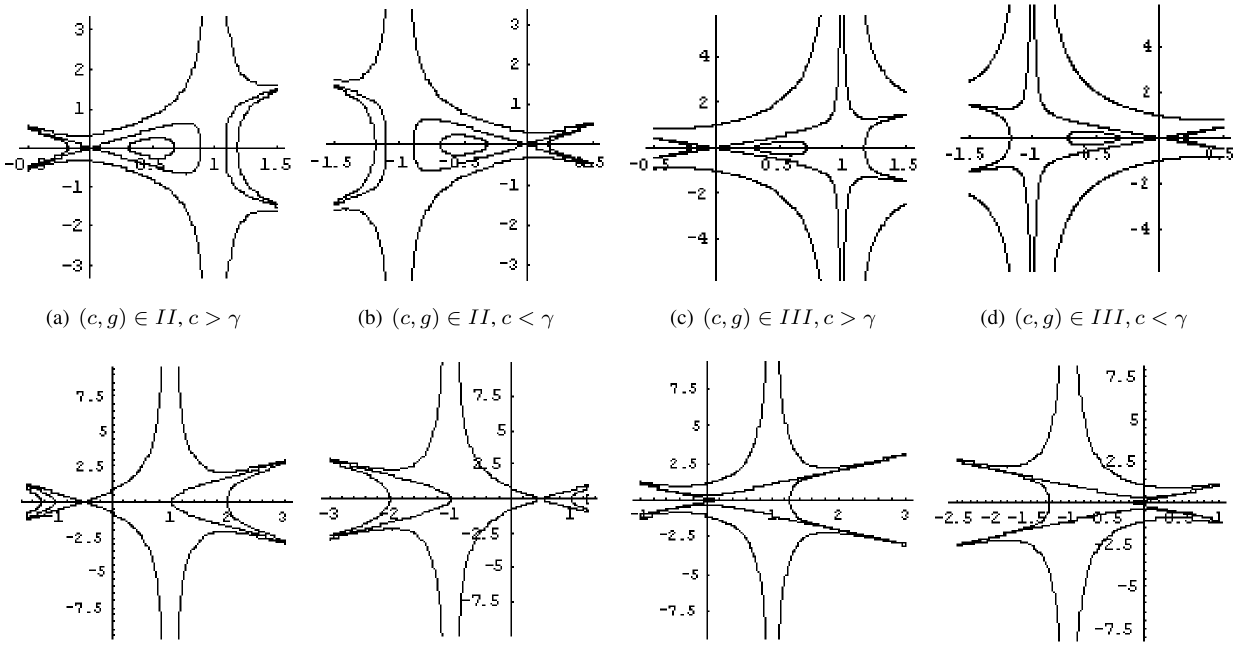


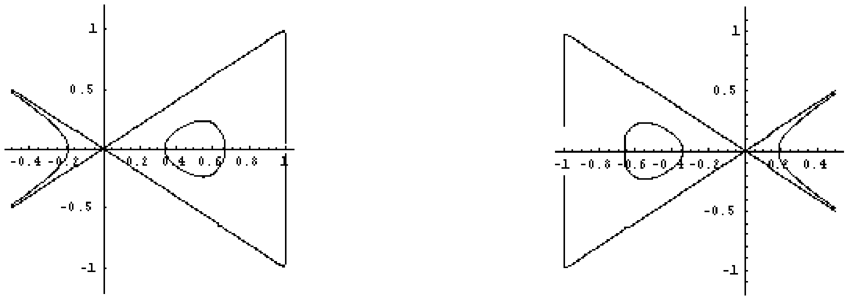
Figure 1: $L_1 : g = -(c-\gamma)^2$, $L_2 : g = 0$, $L_3 : g = \frac{(c-\gamma)^2}{8}$ in the plane $c - g$.



(a) $(c, g) \in II, c > \gamma$ (b) $(c, g) \in II, c < \gamma$ (c) $(c, g) \in III, c > \gamma$ (d) $(c, g) \in III, c < \gamma$

(e) $(c, g) \in L_1, c > \gamma$ (f) $(c, g) \in L_1, c < \gamma$ (g) $(c, g) \in L_3, c > \gamma$ (h) $(c, g) \in L_3, c < \gamma$

Figure 2: The phase portrait bifurcation of system (2.3).



(a) $c > \gamma, g = 0$ (b) $c < \gamma, g = 0$

Figure 3: The homoclinic orbits of system (2.3).

of Eq.(1.2). From the Fig.2 $(a)$ and $(b)$, when $0 < g < \frac{(c-\gamma)^2}{8}$, there exists a homoclinic orbit and when $g \to 0$, the homoclinic orbit changes to a triangle.

Integrating $\frac{d\varphi}{d\xi} = y$ along the homoclinic orbit, we get

$$\int_{\varphi}^{c-\gamma} \frac{(s-c+\gamma)ds}{(s-\frac{c-\gamma-\sqrt{(c-\gamma)^2-8g}}{4})\sqrt{s^2-(\frac{3c-3\gamma+\sqrt{(c-\gamma)^2-8g}}{2})s+\frac{3(c-\gamma)^2-4g+5(c-\gamma)\sqrt{(c-\gamma)^2-8g}}{8}}} = -|\xi|, c > \gamma. \quad (3.1)$$

$$\int_{c-\gamma}^{\varphi} \frac{(s-c+\gamma)ds}{(s-\frac{c-\gamma+\sqrt{(c-\gamma)^2-8g}}{4})\sqrt{s^2-(\frac{3c-3\gamma-\sqrt{(c-\gamma)^2-8g}}{2})s+\frac{3(c-\gamma)^2-4g-5(c-\gamma)\sqrt{(c-\gamma)^2-8g}}{8}}} = -|\xi|, c < \gamma. \quad (3.2)$$

When $0 < g < \frac{(c-\gamma)^2}{8}$ and $g \to 0$, we get $\varphi(\xi) \to (c-\gamma)e^{-|\xi|}$ from the integral equalities above. When $g = 0$, we get the peakon solution of Eq.(1.2).

$$u_1(x,t) = (c-\gamma)e^{-|x-ct|}. \quad (3.3)$$

The peakons expressed by $u_1(x,t)$ are shown in Fig.4 under some parameter conditions. $(c = 2, \gamma = 1)$

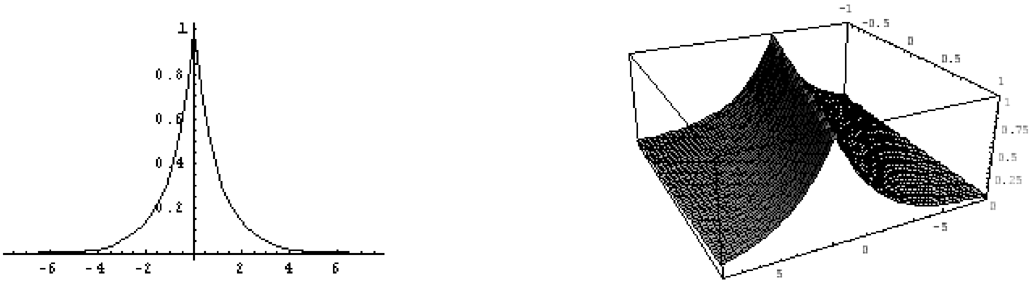
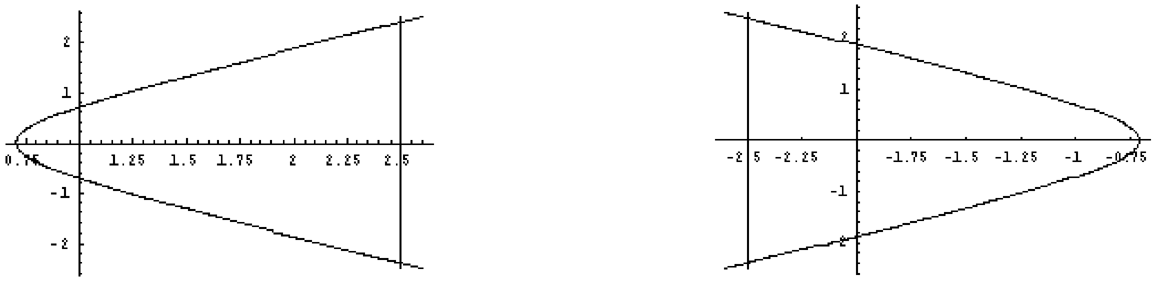


Figure 4: The peaked solitary wave solution for Eq. (1.2): $c = 2, \gamma = 1$.

## 3.2 Peakons from the limit of periodic cusp waves

From Fig.2 $(c)$ and $(d)$, when $-(c-\gamma)^2 < g < 0$, system (2.3) has a periodic orbit which consist of an arc $y^2 = \varphi^2 + g$ and a line segment $\varphi = c - \gamma$ (Fig. 5).



(a) $-(c-\gamma)^2 < g < 0, c > \gamma$

(b) $-(c-\gamma)^2 < g < 0, c < \gamma$

Figure 5: The periodic orbit of system (2.3) when $-(c-\gamma)^2 < g < 0$.

Note that when $-(c-\gamma)^2 < g < 0$ and $g \to 0$, the periodic orbits lose their smoothness and become non-smooth periodic orbits, and when $g = 0$, the periodic orbits become periodic cusp orbits. Integrating $\frac{d\varphi}{d\xi} = y$ along the periodic orbit, we get

$$u_2(\xi, c, \gamma, g) = \begin{cases} v_1(\xi + 2nT, c, \gamma), & c > \gamma, \\ v_2(\xi + 2nT, c, \gamma), & c < \gamma, \end{cases} \quad (3.4)$$

where $n = 0, \pm 1, \pm 2, \cdots, \xi \in [(2n-1)T, (2n+1)T]$,
and

$$v_1(\xi, c, \gamma, g) = \frac{1}{2}(c - \gamma - \sqrt{(c-\gamma)^2 + g})e^{|\xi|} + \frac{1}{2}(c - \gamma + \sqrt{(c-\gamma)^2 + g})e^{-|\xi|},$$

$$v_2(\xi, c, \gamma, g) = \frac{1}{2}(c - \gamma + \sqrt{(c-\gamma)^2 + g})e^{|\xi|} + \frac{1}{2}(c - \gamma - \sqrt{(c-\gamma)^2 + g})e^{-|\xi|},$$

$$-T < \xi < T, T = \left|\ln(\sqrt{-g}) - \ln\left|c - \gamma + \sqrt{(c-\gamma)^2 + g}\right|\right|.$$

When $-(c-\gamma)^2 < g < 0, c > \gamma$, let $g \to 0$, then $T \to \infty, \frac{1}{2}(c - \gamma - \sqrt{(c-\gamma)^2 + g}) \to 0, \frac{1}{2}(c - \gamma + \sqrt{(c-\gamma)^2 + g}) \to c - \gamma$.

When $-(c-\gamma)^2 < g < 0, c < \gamma$, let $g \to 0$, then $T \to \infty, \frac{1}{2}(c - \gamma + \sqrt{(c-\gamma)^2 + g}) \to 0, \frac{1}{2}(c - \gamma - \sqrt{(c-\gamma)^2 + g}) \to c - \gamma$.

So we get

$$u_2(\xi, c, \gamma, g) \to u_1(\xi, c, \gamma, g) = (c-\gamma)e^{-|\xi|}. \tag{3.5}$$

The result (3.5) is identical to (3.3). The graph of some periodic wave for Eq.(1.2) is shown under some parameter condition (Fig.6).

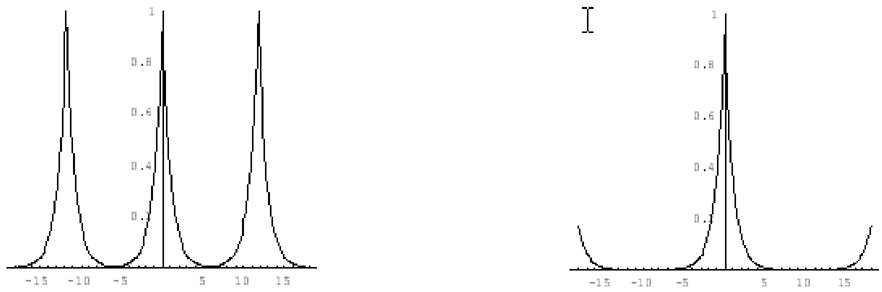


Figure 6: The periodic cusp wave solution for Eq.(1.2): $c = 2, \gamma = 1$.

# 4 Conclusion

In this paper the qualitative analysis methods of dynamical system are used to investigate the peaked wave solutions Eq.(1.2). By the phase portrait bifurcation of the traveling wave system, We obtain the peaked solitary wave solution: $u_1(\xi) = (c-\gamma)e^{-|\xi|}$ and the periodic cusp wave solution: $u_2(\xi, c, \gamma, g) = \begin{cases} v_1(\xi + 2nT, c, \gamma), & c > \gamma, \\ v_2(\xi + 2nT, c, \gamma), & c < \gamma. \end{cases}$ where $n = 0, \pm 1, \pm 2, \cdots\cdots, \xi \in [(2n-1)T, (2n+1)T]$,
and

$$v_1(\xi, c, \gamma, g) = \frac{1}{2}(c - \gamma - \sqrt{(c-\gamma)^2 + g})e^{|\xi|} + \frac{1}{2}(c - \gamma + \sqrt{(c-\gamma)^2 + g})e^{-|\xi|},$$

$$v_2(\xi, c, \gamma, g) = \frac{1}{2}(c - \gamma + \sqrt{(c-\gamma)^2 + g})e^{|\xi|} + \frac{1}{2}(c - \gamma - \sqrt{(c-\gamma)^2 + g})e^{-|\xi|},$$

$$-T < \xi < T, T = \left|\ln(\sqrt{-g}) - \ln\left|c - \gamma + \sqrt{(c-\gamma)^2 + g}\right|\right|.$$

Finally when $-(c-\gamma)^2 < g < 0$ and $g \to 0$, the periodic wave solutions $u_2(\xi)$ tends to the peakons $u_1(\xi)$.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