Horseshoe Dynamics in Fractionally Damped Duffing-Vander Pol Oscillator Driven by Nonsinusoidal Forces

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Abstract: The effect of nonsinusoidal forces on the onset of horseshoe chaos is studied both analytically and numerically in the fractionally damped Duffing-vander Pol (DVP) oscillator. The nonsinusoidal periodic forces considered are square-wave, symmetric saw-tooth wave, and asymmetric saw-tooth wave. An analytical threshold condition for the onset of horseshoe is obtained in the DVP oscillator driven by various nonsinusoidal periodic forces using Melnikov method. Melnikov threshold curves are drawn in a parameter space. For all the nonsinusoidal periodic forces, onset of cross-well asymptotic chaos is observed just above the Melnikov threshold curve. Analytical predictions are demonstrated through direct numerical simulation. Period doubling route to chaos, intermittency route to chaos, periodic windows, chaos, reverse period doubling, period-bubbling are found to occur due to the effect of nonsinusoidal periodic force and fractional nonlinear damping. Numerical investigations including computation of stable and unstable manifolds of saddle and threshold curves are used to detect horseshoe chaos and asymptotic chaos.

Keywords: DVP oscillator; Fractional damping; Nonsinusoidal force; Horseshoe chaos; Melnikov method; Chaos

1 Introduction

In recent years, much interest has been focussed in the study of effect of different sinusoidal and nonsinusoidal forces in certain nonlinear systems with linear damping term [1-10]. For a nonlinear system having external excitation, the presence of nonlinear fractional damping may lead to distinct behaviour that cannot be observed in the same nonlinear system without fractional damping. In the past, there are reports on the effect of nonlinear damping in some nonlinear oscillators. For example, Baltanas et al. analyzed the behaviour of a nonlinearly damped Duffing oscillator [11] with a dissipative term proportional to the power of the velocity. Trueba et al. [12] studied the effect of the nonlinear dissipation on the dynamics of certain nonlinear oscillators. In this paper, the nonlinear damping is proportional to the power of velocity in the form of \( \alpha p \dot{x} |\dot{x}|^{p-1} \) where \( p \geq 1 \) is the damping exponent. A similar nonlinear damping term was used previously by Ravindra and Mallik [13] in their work to study the effect of nonlinear damping in some soft Duffing oscillators and in the nonlinearly damped escape oscillator by Sanjuan [14]. Elliot et al. [15] discussed the variety of mechanisms that give rise to the nonlinear damping and their effects. Ge et al. [16] numerically studied the chaotic behaviours in a fractional order modified Duffing system.

Horseshoe is the occurrence of transverse intersections of stable and unstable manifolds of a saddle fixed point in the Poincaré map and is a global phenomenon. Its appearance can be predicted analytically by employing the Melnikov technique [1]. This technique essentially gives a criterion for a transverse intersection of the stable and unstable manifolds of homoclinic/heteroclinic orbits which imply horseshoe chaos. The essence of Melnikov method is to calculate the so-called Melnikov integral, which can be used to predict the regions in the parameter space where Smale-horseshoe chaos occurs [1,17-19]. It is well known that the existence of horseshoe does not imply that trajectories will be asymptotically

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chaotic. The asymptotically chaotic motion is characterized by positive Lyapunov exponent. However the orbits created by horseshoe mechanism display an extreme sensitive dependence on initial conditions and possibly exhibit either a chaotic transient before settling to stable orbits or a strange attractor [1]. In many dynamical systems [18, 19] onset of chaos has been found to occur near the Melnikov threshold curve. Consequently the Melnikov threshold curve is considered as a lower threshold for the onset of chaos.

The Melnikov technique was first applied by Holmes [20] to study the chaotic behaviour of a periodically driven Duffing oscillator with linear damping. During the past decade, this method has been applied to certain nonlinear systems with linear damping term [21-25]. Recently, the analytical estimates of the effect of nonlinear damping including fractional damping has been attracted by researchers. Numerous papers have been published to study the effect of nonlinear damping in certain nonlinear systems using Melnikov analytical method. In particular, Litak et al. [26] applied the Melnikov criterion to examine a global homoclinic bifurcation and transition to chaos in a case of double-well dynamical system with a nonlinear fractional damping term and external excitation. Awerjewicz et al. [27] applied Melnikov’s method in the presence of dry friction for a stick-slip oscillator. Trueba et al. [12] and Borowice et al. [28] have analyzed the problem of nonlinear damping including fractional damping cases by both the analytical Melnikov method as well as numerical simulations in double-well Duffing oscillator.

Motivated by the above considerations in the present work we wish to analyze the effect of nonlinear damping including fractional damping and nonsinusoidal forces on horseshoe chaos in Duffing-vander Pol oscillator. In the present paper, we consider the perturbed Duffing-vander Pol oscillator with nonlinear damping term,

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= \alpha^2 x - \beta x^3 + \epsilon \left[-\gamma y(1-x^2) \right]|y|^{p-1} + F(t),
\end{align*}
\]

where \( \alpha \) is the natural frequency, \( \beta \) is the nonlinear parameter, \( \gamma \) is the damping parameter, \( \epsilon \) is a small parameter, \( p \) is the damping exponent and \( F(t) \) is an external periodic nonsinusoidal forces. The periodic nonsinusoidal forces of our interest are square-wave, symmetric saw-tooth wave, and asymmetric saw-tooth wave. Fig. 1 depicts the various nonsinusoidal forces considered in our present work.

The paper is organized as follows. In section II we obtain the Melnikov threshold condition for the transverse intersection of homoclinic orbits for the system (1) separately for each of the nonsinusoidal periodic forces. In section III we plot the Melnikov threshold curve in the \((f, \omega)\) parameter space for all the forces where \( f \) and \( \omega \) are the amplitude and frequency of the external force. We verify the analytical prediction with the numerically calculated critical values of \( f \) at which transverse intersections of stable and unstable manifolds of the saddle occur. Finally, we end up with conclusion in section IV.

Figure 1: The shape of the nonsinusoidal forces (a) square-wave (b) symmetric saw-tooth wave and (c) asymmetric saw-tooth wave.

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2 Melnikov analysis

The unperturbed part of system (1) with \( \epsilon = 0 \) has one saddle point \((x^*, y^*) = (0, 0)\) and two center type fixed points \((x^*, y^*) = (\pm \alpha / \sqrt{\beta}, 0)\). The two homoclinic orbits connecting the saddle to itself are given by

\[
W^\pm(x_h(\tau), y_h(\tau)) = (\pm \alpha \sqrt{\frac{2}{\beta}} \text{sech}\alpha \tau, \mp \alpha^2 \sqrt{\frac{2}{\beta}} \text{sech}\alpha \tau \tanh\alpha \tau),
\]

where \( \tau = t - t_0 \). Stable manifolds \((W^s)\) and unstable manifolds \((W^u)\) of homoclinic orbits are indicated in Fig. 2.

The Melnikov function \(M(t_0)\) measures the distance between the stable manifold \((W^s)\) and unstable manifold \((W^u)\) of a saddle. When the two orbits are always separated then the sign of \(M(t_0)\) always remains the same. \(M(t_0)\) oscillates when the orbits \((W^u)\) and \((W^s)\) intersect transversely (horseshoe dynamics). A zero of \(M(t_0)\) corresponds to a tangential intersection. The occurrence of transverse intersections implies that the Poincaré map of the system has the so-called horseshoe chaos [1-3].

For the DVP system (Eq.1), the Melnikov function is

\[
M^\pm(t_0) = \int_{-\infty}^{+\infty} y_h \left[ -\gamma y_h(1-x_h^2) |y_h|^{p-1} + F(\tau + t_0) \right] d\tau.
\]

In the following, we calculate the Melnikov function for the DVP system (Eq.1) with different nonsinusoidal periodic forces.

2.1 Square-wave

For the system (1) driven by square wave,

\[
F(t) = F(t + 2\pi/\omega) = f \text{sgn}(\sin \omega t),
\]

where \text{sgn}(y) is sign of \(y\). Its Fourier series is

\[
F(t) = \frac{4f}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\omega t}{(2n-1)}.
\]

Using Eq.(5), we can calculate the Melnikov function as follows

\[
M^\pm(t_0) = -\gamma \int_{-\infty}^{+\infty} |y_h|^{p+1}(1-x_h^2) d\tau + \int_{-\infty}^{+\infty} y_h \frac{4f}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\omega(\tau + t_0)}{(2n-1)} d\tau
\]

\[
= I_1 + I_2.
\]
By the application of some algebraic techniques the integrals $I_1$ and $I_2$ are calculated as follows

\[
I_1 = -\gamma \int_{-\infty}^{+\infty} |y_h|^p+1(1-x_h^2) d\tau = -\gamma \int_{-\infty}^{+\infty} |y_h|^p+1 d\tau + \gamma \int_{-\infty}^{+\infty} |y_h|^p+1 x_h^2 d\tau
\]

(7)

\[
I_1 = I_{11} + I_{12}.
\]

The evaluation of the integrals $I_1$ and $I_2$ gives the following results

\[
I_{11} = -\gamma (\alpha^2)^{p+\frac{1}{2}} \left[ \frac{2}{\beta} \right]^\frac{p+1}{2} B \left[ \frac{p+2}{2}, \frac{p+1}{2} \right],
\]

(8a)

\[
I_{12} = \gamma (\alpha^2)^{p+\frac{1}{2}} \left[ \frac{2}{\beta} \right]^\frac{p+1}{2} B \left[ \frac{p+2}{2}, \frac{p+3}{2} \right].
\]

(8b)

Adding Eqs. 8(a) and 8(b), we get

\[
I_1 = -\gamma (\alpha^2)^{p+\frac{1}{2}} \left[ \frac{2}{\beta} \right]^\frac{p+1}{2} B \left[ \frac{p+2}{2}, \frac{p+1}{2} \right] + \gamma (\alpha^2)^{p+\frac{1}{2}} \left[ \frac{2}{\beta} \right]^\frac{p+1}{2} B \left[ \frac{p+2}{2}, \frac{p+3}{2} \right],
\]

(9)

where $B(m, n)$ is the Euler beta function which is defined as

\[
B(m, n) = \left( \frac{m}{n} \right) \left( \frac{n}{m+n} \right),
\]

(10)

where $\Gamma(n)$ denotes the Euler Gamma function.

\[
\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0.
\]

(11)

Similarly the integral $I_2$ is worked out to be

\[
I_2 = \int_{-\infty}^{+\infty} y_h \frac{4f}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\omega(\tau + t_0)}{(2n-1)} d\tau
\]

\[
= \frac{4f}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \int_{-\infty}^{+\infty} y_h \sin(2n-1)\omega\tau \cos(2n-1)\omega t_0 d\tau
\]

\[
+ \frac{4f}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \int_{-\infty}^{+\infty} y_h \cos(2n-1)\omega\tau \sin(2n-1)\omega t_0 d\tau
\]

\[
= I_{21} + I_{22}.
\]

The evaluation of the integrals $I_{21}$ and $I_{22}$ gives the following results.

The integral value of $I_{22}$ is zero because of the odd parity with the function $y_h$. Then the integral $I_{21}$ is worked out to be

\[
I_{21} = -4f \sqrt{\frac{2}{\beta}} \omega \sum_{n=1}^{\infty} \cos(2n-1)\omega t_0 \sech \left[ \frac{(2n-1)\pi\omega}{2\alpha} \right].
\]

(12)

Therefore the integral value of $I_2$ is

\[
I_2 = -4f \sqrt{\frac{2}{\beta}} \omega \sum_{n=1}^{\infty} \cos(2n-1)\omega t_0 \sech \left[ \frac{(2n-1)\pi\omega}{2\alpha} \right].
\]

(13)
Then the Melnikov function for the system (1) driven by square-wave force is

\[
M^\pm(t_0) = -\gamma (\alpha^2)^{p+\frac{3}{2}} \left[ \frac{2\pi}{\beta} B \left[ \frac{p+2}{2} \frac{p+1}{2} \right] + \gamma (\alpha^2)^{p+\frac{3}{2}} \left[ \frac{2\pi}{\beta} B \left[ \frac{p+2}{2} \frac{p+3}{2} \right] \right) \right] -4f \sqrt{\frac{2}{\beta}} \delta \sum_{n=1}^{\infty} \cos(2n-1)\omega t_0 \sech \left[ \frac{(2n-1)\pi \omega}{2\alpha} \right],
\]

where

\[
M^\pm(t_0) = A + B \pm C \cos(2n-1)\omega t_0,
\]

where

\[
A = -\gamma (\alpha^2)^{p+\frac{3}{2}} \left[ \frac{2\pi}{\beta} B \left[ \frac{p+2}{2} \frac{p+1}{2} \right] \right],
\]

\[
B = \gamma (\alpha^2)^{p+\frac{3}{2}} \left[ \frac{2\pi}{\beta} B \left[ \frac{p+2}{2} \frac{p+3}{2} \right] \right],
\]

\[
C = -4f \sqrt{\frac{2}{\beta}} \delta \sum_{n=1}^{\infty} \sech \left[ \frac{(2n-1)\pi \omega}{2\alpha} \right].
\]

### 2.2 Symmetric saw-tooth wave

The mathematical expression for the symmetric saw-tooth wave is

\[
F(t) = F(t + \frac{2\pi}{\omega}) = \begin{cases} 
4ft/T & 0 \leq t \leq \frac{T}{4}
-4ft/T + 2f & \frac{T}{4} < t \leq \frac{T}{2}
-4ft/T - 4f & \frac{T}{2} < t \leq \frac{3T}{4}
\end{cases}
\]

and its Fourier series is

\[
F(t) = \frac{8f}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(2n-1)\omega t}{(2n-1)^2}.
\]

For the symmetric saw-tooth wave, the Eq.(3) becomes.

\[
M^\pm(t_0) = -\gamma \int_{-\infty}^{+\infty} |y_n|^{p+1}(1-x_n^2) \, dt + \int_{-\infty}^{+\infty} \frac{8f}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(2n-1)\omega(t + t_0)}{(2n-1)^2} \, dt
= -I_1 + I_3.
\]

The integral value of \(I_1\) is given in Eq.(9). The integral value of \(I_3\) is

\[
I_3 = \frac{8f}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \int_{-\infty}^{+\infty} y_n \sin(2n-1)\omega (\tau + t_0) \, d\tau
= \frac{8f}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \int_{-\infty}^{+\infty} y_n \sin(2n-1)\omega \tau \cos(2n-1)\omega t_0 \, d\tau
+ \frac{8f}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \int_{-\infty}^{+\infty} y_n \cos(2n-1)\omega \tau \sin(2n-1)\omega t_0 \, d\tau
= I_{31} + I_{32}.
\]

The integral of \(I_{32}\) is zero and the value of \(I_{31}\) is worked out to be

\[
I_{31} \quad= \quad \frac{8f}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \int_{-\infty}^{+\infty} y_n \sin(2n-1)\omega \tau \cos(2n-1)\omega t_0 \, d\tau
\]

\[
= \quad \frac{8f}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \cos(2n-1)\omega t_0 \int_{-\infty}^{+\infty} y_n \sin(2n-1)\omega \tau d\tau
\]

\[
= \quad -\frac{8f}{\pi} \sqrt{\frac{2}{\beta}} \delta \sum_{n=1}^{\infty} \sech \left[ \frac{(2n-1)\pi \omega}{2\alpha} \right] \frac{(-1)^{n+1}}{(2n-1)^2} \cos(2n-1)\omega t_0.
\]
Therefore, the value of $I_3$ is

$$I_3 = -\frac{8f}{\pi} \sqrt{\frac{2}{\beta}} \omega (-1)^{n+1} \sum_{n=1}^{\infty} \frac{\text{sech} \left[ \frac{(2n-1)\pi \omega}{2\alpha} \right]}{2n-1} \cos(2n-1)\omega t_0. \tag{19}$$

Then the Melnikov function for the system (1) driven by symmetric saw-tooth wave is

$$M^\pm(t_0) = I_1 - \frac{8f}{\pi} \sqrt{\frac{2}{\beta}} \omega (-1)^{n+1} \sum_{n=1}^{\infty} \frac{\text{sech} \left[ \frac{(2n-1)\pi \omega}{2\alpha} \right]}{2n-1} \cos(2n-1)\omega t_0, \tag{20a}$$

$$M^\pm(t_0) = A + B \pm D f \cos(2n-1)\omega t_0, \tag{20b}$$

where

$$D = -\frac{8}{\pi} \sqrt{\frac{2}{\beta}} \omega \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \text{sech} \left[ \frac{(2n-1)\pi \omega}{2\alpha} \right]. \tag{20c}$$

$A$ and $B$ are given in Eqs.(14(b)) and (14(c)).

### 2.3 Asymmetric saw-tooth wave

The mathematical representation of the asymmetric saw-tooth wave is

$$F(t) = F(t + \frac{2\pi}{\omega}) = \begin{cases} 2ft/T & 0 \leq t < \frac{\pi}{\omega} \\ 2ft/T - 2f & \frac{\pi}{\omega} \leq t \leq \frac{2\pi}{\omega} \end{cases} \tag{21}$$

and its Fourier series is

$$F(t) = \frac{2f}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nw t}{n}. \tag{22}$$

For the asymmetric saw-tooth wave, the Eq.(3) becomes

$$M^\pm(t_0) = -\gamma \int_{-\infty}^{+\infty} |y_h|^p \left(1 - x_h^2\right) d\tau + \int_{-\infty}^{+\infty} y_h \frac{2f}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nw(\tau + t_0)}{n} d\tau \tag{23}$$

$$= I_1 + I_4. \tag{23}$$

The integral value of $I_1$ is given in Eq.(9) and the value of $I_4$ is calculated as follows

$$I_4 = \int_{-\infty}^{+\infty} y_h \frac{2f}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nw(\tau + t_0)}{n} d\tau$$

$$= \frac{2f}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_{-\infty}^{+\infty} y_h \sin nw\tau \cos nw t_0 + \cos nw\tau \sin nw t_0 d\tau,$$

$$I_4 = \frac{2f}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_{-\infty}^{+\infty} y_h \sin nw\tau \cos nw t_0 d\tau$$

$$+ \frac{2f}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_{-\infty}^{+\infty} y_h \cos nw\tau \sin nw t_0 d\tau$$

$$= I_{41} + I_{42}. \tag{23}$$

The integral value of $I_{42}$ is zero. The integral value of $I_{41}$ is worked out to be

$$I_{41} = \frac{2f}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cos nw t_0 \int_{-\infty}^{+\infty} \frac{\sqrt{2}}{\beta} \text{sech} \alpha \tau \tanh \alpha \tau \sin nw\tau d\tau$$
\[
= -\frac{2f}{\pi} \alpha^2 \sqrt{\frac{2}{\beta}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cos n \omega t_0 \int_{-\infty}^{+\infty} \text{sech} \tau \ tanh \alpha \tau \ sin n \omega d\tau.
\]

After simplifying the above integral, the integral value of \( I_4 \) is

\[
I_4 = -2f \sqrt{\frac{2}{\beta}} \omega \sum_{n=1}^{\infty} (-1)^{n+1} \text{sech} \left( \frac{\pi n \omega}{2\alpha} \right) \cos n \omega t_0.
\]

Then the Melnikov function for the system (1) driven by asymmetric saw-tooth wave is

\[
M^\pm(t_0) = I_1 = 2f \sqrt{\frac{2}{\beta}} \omega \sum_{n=1}^{\infty} (-1)^{n+1} \text{sech} \left( \frac{\pi n \omega}{2\alpha} \right) \cos n \omega t_0 \tag{24a}
\]

\[
M^\pm(t_0) = A + B \pm E \cos n \omega t_0, \tag{24b}
\]

where \( A \) and \( B \) are given in Eqs.(14(b)) and (14(c))

\[
E = -2 \sqrt{\frac{2}{\beta}} \omega \sum_{n=1}^{\infty} (-1)^{n+1} \text{sech} \left( \frac{\pi n \omega}{2\alpha} \right). \tag{24c}
\]

### 3 Horseshoe chaos and strange attractors

In this section, we compute the Melnikov threshold value for horseshoe chaos and compare it with numerical prediction. From the Eqs.(14), (20), and (24), we obtain the threshold curve for getting horseshoe chaos in the parameters space \((\alpha, \omega, \gamma, \beta, f, p)\). For example, fixing \(\alpha, \beta, \gamma,\) and \(p\) we have the following threshold curve in the \((\omega, f)\) plane.

**i) Square-wave**

\[
f_{sq} \geq f_M = \gamma \frac{2^{p+2}}{\beta^2} \frac{(\alpha^2)^{p+\frac{1}{2}}}{\omega} \left\{ \frac{2}{\beta} \alpha^2 \ B \left[ \frac{p+2}{2}, \frac{p+3}{2} \right] - B \left[ \frac{p+2}{2}, \frac{p+1}{2} \right] \right\} \sum_{n=1}^{\infty} \cosh \left( \frac{(2n-1)\pi \omega}{2\alpha} \right). \tag{25}
\]

**ii) Symmetric saw-tooth wave**

\[
f_{sst} \geq f_M = \gamma \frac{2^{p+2}}{\beta^2} \frac{(\alpha^2)^{p+\frac{1}{2}}}{\omega} \left\{ \frac{2}{\beta} \alpha^2 \ B \left[ \frac{p+2}{2}, \frac{p+3}{2} \right] - B \left[ \frac{p+2}{2}, \frac{p+1}{2} \right] \right\} \sum_{n=1}^{\infty} (-1)^{n+1} \cosh \left( \frac{(2n-1)\pi \omega}{2\alpha} \right). \tag{26}
\]

**iii) Asymmetric saw-tooth wave**

\[
f_{ast} \geq f_M = \gamma \frac{2^{p+2}}{\beta^2} \frac{(\alpha^2)^{p+\frac{1}{2}}}{\omega} \left\{ \frac{2}{\beta} \alpha^2 \ B \left[ \frac{p+2}{2}, \frac{p+3}{2} \right] - B \left[ \frac{p+2}{2}, \frac{p+1}{2} \right] \right\} \sum_{n=1}^{\infty} \cosh \left( \frac{\pi n \omega}{2\alpha} \right) \tag{27}
\]

Eqs.(25)-(27) are the necessary conditions for the occurrence of horseshoe. The sufficient condition requires the existence of simple zeros of \(M(t_0)\). When Eqs.(25)-(27) become an equality, the zero of \(M\) is nontransverse and this corresponds to tangential intersection where \(dM/dt = 0\) at \(t = t_0\). We consider sufficiently large number of terms, say, 100 terms in the summation of equation for \(M(t_0)\). Fig. 3 shows the plot of \(f_M\) versus \(n\), the number of terms in the summation (Eq.(14)) for the square-wave force. \(f_M\) converges to a constant value with increase in \(n\). For \(n > 10\), the variation of \(f_M\) is negligible. Similar result is found for symmetric saw-tooth wave and asymmetric saw-tooth wave also. Hence in our numerical calculation, we fix \(n = 50\). For our further analysis we fix the other parameters values in Eq.(1) as
Figure 3: $f_M$ versus $n$, the number of terms in the summation in Eq.(14) when the system (1) is driven by square-wave force. The variation in $f_M$ converges to a constant value with increase in $n$. The values of the other parameters in Eq.(1) are $\alpha = 1.0$, $\beta = 5.0$, $\gamma = 0.4$, $p = 1.0$ and $\omega = 1.0$.

\[
\alpha = 1.0, \beta = 5.0, \gamma = 0.4, \omega = 1, p = 0.1, 0.5, 1.0, \text{ and } 2.0.
\]

Fig. 4 shows the plot of the threshold curves for horseshoe chaos in the $(f - \omega)$ parameter plane and for various $p$ values. In the parameter regions above the threshold curve transverse intersection of the stable and unstable manifolds of the saddle occurs and below the threshold curve no transverse intersection of orbits of the saddle occurs. The Melnikov threshold values ($f_M$) and the corresponding numerical value ($f_N$) of the amplitude of the square-wave, symmetric, and asymmetric saw-tooth waves for $p = 0.5$ and $p = 1.0$ are given in Table 1. This numerical threshold $f_N$ for all the forces was found by plotting stable and unstable manifolds of the saddle in the Poincaré map for various values of $p$. From Fig. 4, it is evident that for a fixed value of $p$ as $\omega$ increases the $f_M$ value decreases and approaches a limiting value. Now we compare the analytical predictions of Melnikov threshold ($f_M$) values given by the Eqs.(25)-(27) with the numerical prediction ($f_N$) where the stable and unstable orbits are tangential to each other in the system (1) for various values of $p$. In Figs. 5 and 6, we plotted the orbits of the saddle for two values of $f$: one for $f < f_M$ and another for $f > f_M$ for each of the forces. For clarity only part of the manifolds are shown. To find both the stable and unstable manifolds of the perturbed system we have used totally 400
Table 1: Critical values of $f_M$ and $f_N$ for fractionally damped DVP oscillator driven by nonsinusoidal forces Eq.(1) for $p = 0.5$ and $p = 1.0$ with $\alpha = 1.0, \gamma = 0.4, \beta = 5.0$ and $\omega = 1.0$.

<table>
<thead>
<tr>
<th>$p$ values</th>
<th>Square-wave</th>
<th>Symmetric saw-tooth wave</th>
<th>Asymmetric saw-tooth wave</th>
</tr>
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<tbody>
<tr>
<td>$p = 0.5$</td>
<td>$0.1884$</td>
<td>$0.2959$</td>
<td>$0.3768$</td>
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<tr>
<td>$p = 1.0$</td>
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</tbody>
</table>

Initial conditions along the eigenvector associated with the stable and unstable manifolds of the saddle point. The unstable manifold is computed by integrating the system (1) in the forward time direction. The stable manifold is computed by integrating the system (1) in the backward time direction. In the left side subplots (Figs. 5 and 6), for $f < f_M$ the stable and unstable orbits are well separated. In the right side subplots (Figs. 5 and 6, for $f > f_M$ we can clearly notice transverse intersections of orbits at certain places. As an example when the system (1) driven by square-wave force, for $p = 1.0$ (Fig. 5(a) and (b)) and $p = 2.0$ (Fig. 6(a) and (b)), show part of such orbits for two chosen values of $f$. For $f < 0.0889$ and $f < 0.02210$, the stable and unstable orbits which are joined smoothly in the absence of perturbations (see for example fig.2) are now separated for which the Melnikov distance is always positive. This is shown in Figs. 5(a) and 6(a) for $f = 0.01$.

![Figure 5: Numerically computed stable and unstable manifolds of the saddle fixed point of the system (1) driven by (a-b) square-wave (c-d) symmetric saw-tooth wave and (e-f) asymmetric saw-tooth wave forces for $p = 1.0$. The values of the other parameters in Eq.(1) are $\alpha = 1.0, \beta = 5.0, \gamma = 0.4$, and $\omega = 1.0$.](image)

In this parametric regime one may expect regular behaviour. Further for $f > 0.0889$ and $f > 0.02210$, we find transverse intersections of two orbits where the Melnikov distance oscillates between positive and negative values. For example Figs. 5(b) and 6(b) show two such transverse intersections for $f = 0.2$ and 0.03. Thus for $f > 0.0889$ and $f > 0.02210$ it is possible to have either asymptotic chaos or transient chaos followed by asymptotically periodic motion.

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Similar results were observed for symmetric and asymmetric saw-tooth wave forces also which are presented in Figs. 5(c)-(f) and 6(c)-(f). For all the forces, the analytical prediction is in good agreement with the actual numerical analysis of the system.

In order to know the nature of attractors of the system near the Smale-horseshoe threshold curve, we have further numerically investigated Eq. (1) and the onset of chaos therein. Figs. 7 and 8 show the bifurcation diagrams of the system (1) driven by (a,d) square-wave, (b,e) symmetric saw-tooth wave and (c,f) asymmetric saw-tooth wave for two values of $p$ such as $p < 1$ and $p \geq 1$. For all the forces, as $f$ is increased from zero, a stable period-$T$ (= $2\pi/\omega$) orbit occurs then it loses its stability giving birth to further period-$2T$, $4T$, $8T$ etc. orbits. This cascade of bifurcation accumulate at certain critical values $(f_c)$. At this critical value of $f$ onset of chaotic motion occurs. When the parameter $f$ is further increased from $(f_c)$ one finds that the chaotic orbit persists for a range of $f$ values interspersed by periodic windows, period-doubling windows, reverse period-doubling, intermittency route to chaos, period bubbling, band merging, and sudden widening. For clarity, the chaotic orbit in the $(x - \dot{x})$ plane and the strange attractor in the Poincaré map of the system (1) driven by nonsinusoidal forces are presented in Fig. 9 for $p = 1.0$ and $p = 2.0$. 

**Figure 6:** Numerically computed stable and unstable manifolds of the saddle fixed point of the system (1) driven by (a-b) square-wave (c-d) symmetric saw-tooth wave and (e-f) asymmetric saw-tooth wave forces for $p = 2.0$. The values of the other parameters in Eq.(1) are $\alpha = 1.0$, $\beta = 5.0$, $\gamma = 0.4$, and $\omega = 1.0$. 

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In the present paper we considered the fractionally damped Duffing-vander Pol oscillator driven by nonsinusoidal periodic forces. We studied the effect of nonsinusoidal periodic forces on horseshoe chaos and asymptotic chaos. For each nonsinusoidal periodic force, we obtained the threshold condition for onset of horseshoe chaos that is, transverse intersection of stable and unstable branches of homoclinic orbits using Melnikov analytical method. Threshold curves are drawn in a parameter space for each nonperiodic force. This numerical threshold $f_N$ for all the forces was found by plotting stable and unstable manifolds of the saddle in the Poincaré map for various values of $p$. Near the Melnikov threshold curves onset of asymptotic chaos is observed. For typical parametric values, we have observed that the increase of damping

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Figure 9: Chaotic orbits for the system (1) driven by (a-b) square-wave (c-d) symmetric saw-tooth wave and (e-f) asymmetric saw-tooth wave for $p = 1.0$ and $p = 2.0$. The values of the other parameters in Eq.(1) are $\alpha = 1.0, \beta = 5.0, \gamma = 0.4$ and $\omega = 1.0$. 

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exponent (p) from small value gives the decrease of onset of horseshoe chaos for all the forces. It is important to study the effect of nonlinear fractional damping in DVP oscillator driven by amplitude and frequency modulated forces. This will be investigated in future.

References


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