ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.28(2019) No.1,pp.10-19

The Global Existence of 2D Smooth Compressible Fluid in An Infinitely Expanding Circle

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(Received 19 February 2019, accepted 24 May 2019)

Abstract: We concern with the global existence and large time behavior of compressible fluids in an infinitely expanding circle. Due to the conservation of mass, the fluid in the expanding circle becomes rarefied and eventually tends to a vacuum state. In this paper, we will prove this physical phenomenon for the compressible inviscid and irrotational gases.

Keywords: Compressible Euler equations; Expanding circle; Global existence; Degenerate; Weighted energy estimates; Large time behavior

1 Introduction

In this paper, we consider the behavior of a compressible inviscid fluid in a 2D expanding circle given by $\Omega_0 = \{(t,x) : t \geq 0, |x| = \sqrt{x_1^2 + x_2^2} \leq R_0(t)\}$, where $R_0(t) \in C^6[0,\infty)$ satisfies $R_0(0) = 1, R_0'(0) = 0, R_0''(0) = 0$, and $R_0(t) = 1+Lt$ for $t \geq 1$ with some positive constant L. From the expression of Ω_0, we know that the circle $\mathcal{S}_t^0 = \{x : |x| \leq R_0(t)\}$ at the time t is artificially set by pulling out the initial unit circle $\mathcal{S}^0 = \{x : |x| \leq 1\}$ with a smooth speed and acceleration (see Fig. 1).

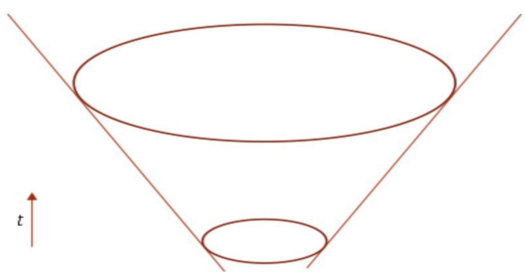

Figure 1: Inviscid gases lie in a 2D expanding circle which expands in a various speed.

Suppose that the movement of fluid in the circle is governed by the 2D compressible isentropic Euler system:

$$\begin{cases} \partial_t \rho + \sum_{i=1}^{2} \partial_i(\rho u_i) = 0, \\ \partial_t(\rho u_i) + \sum_{j=1}^{2} \partial_j(\rho u_i u_j) + \partial_i P = 0, \quad i = 1, 2, \end{cases} \tag{1}$$

where $x = (x_1, x_2)$, $\rho, u = (u_1, u_2)$ and P represent the density, velocity and pressure, respectively. Moreover, assume the state equation $P = A\rho^\gamma$ holds with $A > 0$ and γ $(1 < \gamma < \frac{3}{2})$ being constants. Without loss of generality, take $A = 1$

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IJNS.2019.08.15/1046

and impose the following initial-boundary conditions on

$$\begin{cases} \rho(0,x)=\rho_0(x), \ \ u(0,x)=u_0(x), & \text{for } x\in\mathcal{S}^0, \\ R'(t)=\displaystyle\sum_{i=1}^{2}\frac{x_iu_i}{|x|}, & \text{for } (t,x)\in\partial\Omega_0=\{(t,x):t\geq 0, |x|=R_0(t)\}, \end{cases} \tag{2}$$

where $\rho_0(x)\in H^4(\mathcal{S}^0)$, $u_0(x)\in H_0^4(\mathcal{S}^0)$, and $\rho_0(x)>0$ for $x\in\mathcal{S}^0$. Here, the boundary condition on $\partial\Omega_0$ represents the solid wall condition. In order to solve Eqs.(1) and (2), we first solve an unsteady potential flow equation in the domain $\Omega=\{(t,x):t\geq 0, |x|\leq R(t)\}$ with the initial-boundary conditions (2) . Let $\Phi(t,x)$ be the potential of velocity $u=(u_1,u_2)$, i.e., $u_i=\partial_i\Phi$ $(1\leq i\leq 2)$. Then it follows from the Bernoulli law that

$$\partial_t\Phi+\frac{1}{2}|\nabla_x\Phi|^2+h(\rho)=B_0, \tag{3}$$

where $h(\rho)=\dfrac{c^2(\rho)}{\gamma-1}$ is the specific enthalpy, $c(\rho)=\sqrt{P'(\rho)}$ is the local sound speed, $\nabla_x=(\partial_1,\partial_2)$, $B_0=\dfrac{c^2(\rho_0)}{\gamma-1}$ is the Bernoulli constant of a static state with the constant density ρ_0.

By (3) and the implicit function theorem with $h'(\rho)=\dfrac{c^2(\rho)}{\rho}>0$ for $\rho>0$, the density function $\rho(t,x)$ can be expressed as

$$\rho=h^{-1}\big(B_0-\partial_t\Phi-\frac{1}{2}|\nabla_x\Phi|^2\big)\equiv H(\nabla\Phi), \tag{4}$$

where h^{-1} stands for the inverse function of $h(\rho)$, and $\nabla=(\partial_t,\nabla_x)$. Substituting (4) into the first equation in (1) yields

$$\partial_t(H(\nabla\Phi))+\sum_{i=1}^{2}\partial_i(H(\nabla\Phi)\partial_i\Phi)=0. \tag{5}$$

In fact, for any C^2 solution Φ, (5) can be rewritten into the following second order quasilinear equation

$$\partial_t^2\Phi+2\sum_{k=1}^{2}\partial_k\Phi\partial_{tk}^2\Phi+\sum_{i,j=1}^{2}\partial_i\Phi\partial_j\Phi\partial_{ij}^2\Phi-c^2(\rho)\Delta\Phi=0. \tag{6}$$

Denote the lateral boundary of Ω by $\partial\Omega=\{(t,x):t\geq 0, |x|=R(t)\}$. Then on $\partial\Omega$,

$$\sum_{i=1}^{2}\partial_i\Phi\cdot\frac{x_i}{|x|}=L. \tag{7}$$

Due to the geometric property of Ω, it is convenient to work in the spherical coordinates (r,θ):

$$(x_1,x_2)=(r\cos\theta, r\sin\theta), \tag{8}$$

where $r=\sqrt{x_1^2+x_2^2}$, $0\leq\theta\leq 2\pi$. Under the coordinate transformation (8), (6) becomes

$$\begin{aligned}&\partial_t^2\Phi+2\partial_r\Phi\partial_{tr}^2\Phi+\frac{2}{r^2}\partial_\theta\Phi\partial_{t\theta}^2\Phi+\big((\partial_r\Phi)^2+c^2(\rho)\big)\partial_r^2\Phi+\frac{1}{r^2}\left(\frac{(\partial_\theta\Phi)^2}{r^2}-c^2(\rho)\right)\partial_\theta^2\Phi\\ &+\frac{2}{r^2}\partial_r\Phi\partial_\theta\Phi\partial_{r\theta}^2\Phi-\frac{1}{r^3}\left((\partial_\theta\Phi)^2-\frac{c^2(\rho)}{r}\partial_r\Phi\right)\partial_r\Phi=0.\end{aligned} \tag{9}$$

Meanwhile, the boundary condition (7) becomes

$$\partial_r\Phi=L, \quad \text{on } \partial\Omega. \tag{10}$$

In addition, we impose the following initial perturbation:

$$\Phi(0,x)=\frac{1}{2}L|x|^2+\varepsilon\Phi_0(x), \quad \partial_t\Phi(0,x)=-\frac{1}{2}L^2|x|^2+\varepsilon\Phi_1(x), \tag{11}$$

where $\varepsilon > 0$ is a small parameter, $(\Phi_0(x), \Phi_1(x)) \in (H^5(\mathcal{S}^0), H^4(\mathcal{S}^0))$, and the initial-boundary value conditions (10)-(11) are compatible on $\mathcal{S}^0$. Note that the initial data (11) can be replaced by $(\Phi(0,x), \partial_t\Phi(0,x)) = (\varepsilon\Phi_0(x), \varepsilon\Phi_1(x))$ when $L > 0$ is small. On the other hand, due to $u_i = \partial_i\Phi$ and (4), the initial conditions (11) can be realized by a small perturbation of the initial density and velocity of an irrotational flow.

Theorem 1 *Under the above assumptions on the initial and boundary data, if $\gamma \in (1, \frac{3}{2})$, then there exists a constant $\varepsilon_0 > 0$ depending on L, B_0 and γ such that problem* (6) *with* (10)-(11) *has a global solution $\Phi(t,x) \in C([0,\infty), H^5(\mathcal{S}_t)) \cap C^1([0,\infty), H^4(\mathcal{S}_t))$ for $\varepsilon < \varepsilon_0$, where $\mathcal{S}_t = \{x : r \le R(t)\}$. Moreover, $\rho(t,x) > 0$ and $\lim_{t\to\infty} \rho(t,x) = 0$ hold.*

Remark 1 *The authors in [1] consider the 3-D case with the assumption of $\gamma \in (1, \frac{4}{3})$. Now, we can improve γ to $(1, \frac{3}{2})$ for 2-D case, which can include the important gas - air (for air $\gamma \approx 1.4$).*

Remark 2 *The linearized operator of the quasilinear wave Eq.(6) around the special expanding solution has the approximate form of*

$$\partial_t^2 - \frac{\gamma}{(1+Lt)^{2(\gamma-1)}}(\partial_1^2 + \partial_2^2) + \frac{2L(\gamma-1)}{1+Lt}\partial_t.$$

On the other hand, if one considers the Cauchy problem of (1) for initial data as a small perturbation of a uniform constant density ρ_0 and velocity $(0,0,q_0)$, that is,

$$\begin{cases} \partial_t^2\Phi + 2\sum_{k=1}^{2} \partial_k\Phi\partial_{tk}^2\Phi + \sum_{i,j=1}^{2} \partial_i\Phi\partial_j\Phi\partial_{ij}^2\Phi - c^2(\rho)\Delta\Phi = 0, \\ \Phi(t,x)|_{t=0} = \varepsilon\Phi_0(x), \quad \partial_t\Phi(t,x)|_{t=0} = q_0 + \varepsilon\Phi_1(x), \qquad x \in \mathbb{R}^3, \end{cases} \tag{12}$$

where $\Phi_i(x) \in C_0^\infty(\mathbb{R}^3)$ $(i = 0,1)$, then (12) *does not fulfill the "null-condition" introduced in [2] and [3]. Therefore, according to the results obtained in [4–8], classical solution to* (12) *blows up and then shock forms in finite time. Compared this blowup result with Theorem 1, the global existence of smooth solution to (5) together with a fixed wall condition comes from the rarefaction property of fluid.*

The rest of the paper is organized as follows. In the next section, we will give some basic properties of the background solution. Next, we will reformulate problem (6) together with (10)-(11) by decomposing its solution as a sum of the background solution and a small perturbation so that its linearization can be studied clearly. In Section 3, we will establish a uniform weighted energy estimate for the corresponding linear problem, where an appropriate multiplier is constructed. In Section 4, we complete the proof of Theorem 1 by applying the Sobolev embedding theorem and the continuation argument.

2 Background solution and reformulation of problem (6) with (10) and (11)

In this section, we analyze the background solution to (6) with (10)-(11) when the initial data (11) are

$$\hat\Phi(0,x) = \frac{1}{2}L|x|^2, \quad \partial_t\hat\Phi(0,x) = -\frac{1}{2}L^2|x|^2. \tag{13}$$

In this case, the density $\rho(x)$ and velocity $u(t,x) = \nabla_x\Phi(t,x)$ in Ω take the form: $\rho(t,x) = \hat\rho(t,r)$, $u(t,x) = \frac{x}{r}\hat U(t,r)$. Consequently, problem (6) with (10) and (13) is equivalent to

$$\begin{cases} r\partial_t\hat\rho + \partial_r(r\hat\rho\hat U) = 0, \\ \partial_t(r\hat\rho\hat U) + \partial_r(r\hat\rho\hat U) + r\partial_r P = 0, \\ \hat\rho(0,r) = 1, \quad \hat U(0,r) = Lr. \end{cases} \tag{14}$$

One can easily check that (14) has a solution

$$\hat\rho(t,r) = \frac{1}{R^2(t)}, \quad \hat U(t,r) = \frac{Lr}{R(t)}. \tag{15}$$

Then for $1 < \gamma < \dfrac{3}{2}$, it follows from $u_i = \partial_i \Phi$, (3) and (15) that (6) with (10) and (13) has a solution

$$\hat{\Phi}(t,r) = \frac{\gamma}{(\gamma-1)(3-2\gamma)L} + B_0 t + \frac{Lr^2}{2(1+Lt)} - \frac{\gamma}{(\gamma-1)(3-2\gamma)L}(1+Lt)^{3-2\gamma}, \tag{16}$$

where $B_0 = \dfrac{\gamma}{\gamma - 1}$.

Next, we reformulate (6) with (10)-(11). Let $\dot{\Phi} = \Phi - \hat{\Phi}$. Then (6) can be reduced to

$$\mathcal{L}\dot{\Phi} = \dot{f} \quad \text{in } \Omega, \tag{17}$$

where

$$\begin{cases} \mathcal{L}\dot{\Phi} = \partial_t^2 \dot{\Phi} + 2\sum\limits_{i=1}^{2} \partial_i \hat{\Phi} \partial_{ti}^2 \dot{\Phi} + \sum\limits_{i,j=1}^{2} \partial_i \hat{\Phi} \partial_j \hat{\Phi} \partial_{ij}^2 \dot{\Phi} - \hat{c}^2 \Delta \dot{\Phi} + \dfrac{2L(\gamma-1)}{R(t)} (\partial_t \dot{\Phi} + \sum\limits_{i=1}^{2} \partial_i \hat{\Phi} \partial_i \dot{\Phi}), \\ \dot{f} = \sum\limits_{i=1}^{2} f_{0i} \partial_{ti}^2 \dot{\Phi} + \sum\limits_{1 \le i \ne j \le 2} f_{ij} \partial_{ij}^2 \dot{\Phi} + \sum\limits_{i=1}^{2} f_{ii} \partial_i^2 \dot{\Phi} + f_0, \end{cases} \tag{18}$$

with

$$\begin{cases} f_{0i} = -2\partial_i \dot{\Phi}, \\ f_{ij} = -\partial_i \dot{\Phi} \partial_j \dot{\Phi} - 2\partial_i \hat{\Phi} \partial_j \dot{\Phi}, \\ f_{ii} = -(\partial_i \dot{\Phi})^2 - (\gamma-1)(\partial_t \dot{\Phi} + \sum\limits_{j=1}^{2} \partial_j \hat{\Phi} \partial_j \dot{\Phi} + \dfrac{1}{2} \sum\limits_{j=1}^{2} (\partial_j \dot{\Phi})^2) - 2\partial_i \hat{\Phi} \partial_i \dot{\Phi}, \\ f_0 = -\dfrac{\gamma L}{R(t)} \sum\limits_{i=1}^{2} (\partial_i \dot{\Phi})^2. \end{cases}$$

For later analysis, we rewrite $\mathcal{L}\dot{\Phi}$ as follows:

$$\begin{aligned} \mathcal{L}\dot{\Phi} \quad = \quad & \partial_t^2 \dot{\Phi} + 2\partial_r \hat{\Phi} \partial_{tr}^2 \dot{\Phi} + ((\partial_r \hat{\Phi})^2 - \hat{c}^2) \partial_r^2 \dot{\Phi} - \frac{\hat{c}^2}{r^2} \partial_\theta^2 \dot{\Phi} + 2(\gamma-1) \partial_r^2 \hat{\Phi} \partial_t \dot{\Phi} \\ & + \big(2\partial_{tr}^2 \hat{\Phi} - \frac{\hat{c}^2}{r} + 2(\gamma-1)(\partial_r \hat{\Phi})^2\big) \partial_r \dot{\Phi}. \end{aligned} \tag{19}$$

On the lateral boundary $\partial\Omega$ of Ω, $\dot{\Phi}$ satisfies

$$\partial_r \dot{\Phi} = 0. \tag{20}$$

In addition, we have the following initial data of $\dot{\Phi}$ from (11)

$$\dot{\Phi}(0,x) = \varepsilon \Phi_0(x), \quad \partial_t \dot{\Phi}(0,x) = \varepsilon \Phi_1(x). \tag{21}$$

3 The weighted energy estimate and reformulation of (19)-(21)

In this section, we derive the weighted energy estimate of $\nabla_{t,x}\dot{\Phi}$ for the linear part (19) together with (20)-(21). Set $\Omega_T = \Omega \cap \{0 < t < T\}$, $\mathcal{S}_T = \Omega \cap \{t = T\}$ and $\mathcal{B}_T = \partial\Omega \cap \{0 < t < T\}$. Then, we have the following theorem.

Theorem 2 *Let $\dot{\Phi} \in C^2(\bar{\Omega}_T)$ satisfy the boundary condition (20) and initial data condition (21). Then for $1 < \gamma < \frac{3}{2}$, there exists a multiplier $\mathcal{M}\dot{\Phi} = R(t)^\mu a(t) D_t \dot{\Phi}$ such that for fixed constant $\mu = 4\gamma - 6$ we have*

$$\begin{aligned} & R(T)^\mu \int_{\mathcal{S}_T} (D_t \dot{\Phi})^2 dx + R(T)^{\mu - 2(\gamma-1)} \int_{\mathcal{S}_T} (\nabla_x \dot{\Phi})^2 dx \\ & + C \int_{\Omega_T} \big(R(t)^{\mu-1-\delta} (D_t \dot{\Phi})^2 + R(t)^{\mu-1-2(\gamma-1)} (\nabla_x \dot{\Phi})^2\big) dt dx \\ & \le \int_{\Omega_T} \mathcal{L}\dot{\Phi} \cdot \mathcal{M}\dot{\Phi} dt dx + C\varepsilon^2, \end{aligned} \tag{22}$$

where $D_t = \partial_t + \sum_{i=1}^{2} \partial_i \hat{\Phi} \partial_i = \partial_t + \frac{Lr}{R(t)} \partial_r$ *is the material derivative,* $C > 0$ *is a generic positive constant depending only on the initial data, and* $\delta > 0$ *is a small fixed constant.*

Proof. Choosing $\mathcal{M}\dot{\Phi} = R(t)^\mu (a(t,r)\partial_t \dot{\Phi} + b(t,r)\partial_r \dot{\Phi})$, where the non-negative functions $a(t,r)$ and $b(t,r)$ will be determined later, then

$$\int_{\Omega_T} \mathcal{L}\dot{\Phi} \cdot \mathcal{M}\dot{\Phi} dt dx = I + II + III, \tag{23}$$

where

$$\begin{aligned}
I &= \int_{\mathcal{B}_T} \frac{R(t)^\mu}{2\sqrt{1+L^2}} (La(t,r) - b(t,r)) \bigg((\partial_t \dot{\Phi})^2 - \frac{\hat{c}^2}{r^2} (\partial_\theta \dot{\Phi})^2 \\
&\quad + 2((L^2 - \hat{c}^2)a(t,r) - Lb(t,r))\partial_t \dot{\Phi} \partial_r \dot{\Phi} + (L(L^2 - \hat{c}^2)a(t,r) - (L^2 + \hat{c}^2)b(t,r))(\partial_r \dot{\Phi})^2 \bigg) dS, \\
II &\equiv II_1 - II_2, \\
III &= \int_{\Omega_T} \bigg(A(t,r)(\partial_t \dot{\Phi})^2 + B(t,r)\partial_t \dot{\Phi} \partial_r \dot{\Phi} + C(t,r)(\partial_r \dot{\Phi})^2 + \frac{1}{r^2} D(t,r)(\partial_\theta \dot{\Phi})^2 \bigg) dt dx,
\end{aligned}$$

with

$$\begin{aligned}
II_1 &= \int_{\mathcal{S}_T} R(t)^\mu \bigg(\frac{1}{2} a(t,r)(\partial_t \dot{\Phi})^2 + b(t,r)\partial_t \dot{\Phi} \partial_r \dot{\Phi} + (b(t,r)\partial_r \hat{\Phi} - \frac{1}{2}((\partial_r \hat{\Phi})^2 - \hat{c}^2)a(t,r))(\partial_r \dot{\Phi})^2 \\
&\quad + \frac{\hat{c}^2}{2r^2} a(t,r)(\partial_\theta \dot{\Phi})^2 \bigg) dx, \\
II_2 &= \int_{\mathcal{S}_0} \bigg(\frac{1}{2} a(0,r)(\partial_t \dot{\Phi})^2 + b(0,r)\partial_t \dot{\Phi} \partial_r \dot{\Phi} + (b(0,r)\partial_r \hat{\Phi} - \frac{1}{2}((\partial_r \hat{\Phi})^2 - \hat{c}^2)a(0,r))(\partial_r \dot{\Phi})^2 \\
&\quad + \frac{\hat{c}^2}{2r^2} a(0,r)(\partial_\theta \dot{\Phi})^2 \bigg) dx, \\
A(t,r) &= -\frac{1}{2}\partial_t \Big(R(t)^\mu a(t,r) \Big) - R(t)^\mu \partial_r \Big(a(t,r)\partial_r \hat{\Phi} \Big) - R(t)^\mu r^{-1} a(t,r)\partial_r \hat{\Phi} \\
&\quad + r^{-2}\partial_r \Big(\frac{1}{2} r^2 R(t)^\mu b(t,r) \Big) + 2(\gamma - 1)\partial_r^2 \hat{\Phi} R(t)^\mu a(t,r), \\
B(t,r) &= -R(t)^\mu \partial_r \Big(((\partial_r \hat{\Phi})^2 - \hat{c}^2)a(t,r) \Big) - \frac{1}{r} R(t)^\mu a(t,r) \Big((\partial_r \hat{\Phi})^2 - \hat{c}^2 \Big) - \partial_t \Big(R(t)^\mu b(t,r) \Big) \\
&\quad + R(t)^\mu a(t,r) \Big(2\partial_{tr}^2 \hat{\Phi} - \frac{\hat{c}^2}{r} + 2(\gamma - 1)(\partial_r \hat{\Phi})^2 + 2(\gamma - 1)\partial_r^2 \hat{\Phi} \partial_r \hat{\Phi} \Big), \\
C(t,r) &= \frac{1}{2}\partial_t \Big(((\partial_r \hat{\Phi})^2 - \hat{c}^2) R(t)^\mu a(t,r) \Big) - r^{-2}\partial_r \Big(\frac{1}{2} R(t)^\mu b(t,r) r^2 ((\partial_r \hat{\Phi})^2 - \hat{c}^2) \Big), \\
&\quad - \partial_t \Big(R(t)^\mu b(t,r)\partial_r \hat{\Phi} \Big) + R(t)^\mu b(t,r) \Big(2\partial_{tr}^2 \hat{\Phi} - \frac{\hat{c}^2}{r} + 2(\gamma - 1)\partial_r \hat{\Phi} \partial_r^2 \hat{\Phi} \Big), \\
D(t,r) &= -\frac{1}{2}\partial_t \Big(\hat{c}^2 R(t)^\mu a(t,r) \Big) - \frac{1}{2}\partial_r \Big(\frac{\hat{c}^2}{r^2} R(t)^\mu b(t,r) \Big).
\end{aligned}$$

In view of the boundary condition (20), we have

$$I = \int_{\mathcal{B}_T} \frac{1}{2\sqrt{1+L^2}} (La(t,R(t)) - b(t,R(t))) \big((\partial_t \dot{\Phi})^2 - \frac{\hat{c}^2}{r^2} (\partial_\theta \dot{\Phi})^2 \big) dS.$$

To guarantee $I \geq 0$, it requires that on the boundary $r = R(t)$,

$$b(t,R(t)) = La(t,R(t)). \tag{24}$$

In this case,

$$I = 0. \tag{25}$$

Next, we consider II. To fulfill $II_1 > 0$, it requires that on $t = T$

$$\begin{cases} a(t,r) > 0, \\ b(t,r)^2 - a(t,r)\Big(2b(t,r)\partial_r\hat{\Phi} - ((\partial_r\hat{\Phi})^2 - \hat{c}^2)a(t,r)\Big) \le 0. \end{cases} \tag{26}$$

This means that on $\{t = T\}$,

$$\begin{cases} a(t,r) > 0, \\ \partial_r\hat{\Phi} - \hat{c} \le \dfrac{b(t,r)}{a(t,r)} \le \partial_r\hat{\Phi} + \hat{c}. \end{cases} \tag{27}$$

Thus, combining (24) and (27) yields

$$b(t,r) = \partial_r\hat{\Phi} \cdot a(t,r). \tag{28}$$

On the other hand, by $\hat{c}^2 = \gamma R(t)^{-2(\gamma-1)}$, we have

$$\begin{aligned} II_1 &= \int_{\mathcal{S}_T} \frac{1}{2}R(t)^{\mu}a(T,r)\Big((\partial_t\dot{\Phi} + \partial_r\hat{\Phi}\partial_r\dot{\Phi})^2 + \hat{c}^2\big((\partial_r\dot{\Phi})^2 + \frac{1}{r^2}(\partial_\theta\dot{\Phi})^2\big)\Big)dS \\ &\ge C\int_{\mathcal{S}_T} a(T,r)\Big(R(T)^{\mu}(D_t\dot{\Phi})^2 + R(T)^{\mu-2(\gamma-1)}\big((\partial_r\dot{\Phi})^2 + \frac{1}{r^2}(\partial_\theta\dot{\Phi})^2\big)\Big)dS. \end{aligned} \tag{29}$$

It follows from initial data (21) that

$$|II_2| \le C\varepsilon^2. \tag{30}$$

Finally, we deal with III. In fact, we only need to choose $a(t,r) \equiv a(t)$. In this case, direct computation yields

$$\begin{cases} A(t,r) = \dfrac{1}{2}R(t)^{\mu-1}\Big(L(4\gamma-6-\mu)a(t) - R(t)a'(t)\Big), \\ B(t,r) = LrR(t)^{\mu-2}\Big(L(4\gamma-6-\mu)a(t) - R(t)a'(t)\Big), \\ C(t,r) = \dfrac{1}{2}R(t)^{\mu-3}\Big(\big(L(4\gamma-6-\mu)a(t) - R(t)a'(t)\Big) + \dfrac{\hat{c}^2}{R}(t)^{\mu-1}\Big(\big(L(2\gamma-2-\mu)a(t) - R(t)a'(t)\Big), \\ D(t,r) = \dfrac{\hat{c}^2}{2r^2}R(t)^{\mu-1}\Big(L(2\gamma-2-\mu)a(t) - R(t)a'(t)\Big). \end{cases}$$

In order to have $III > 0$, we choose

$$A(t,r) > 0, \quad B(t,r)^2 - 4A(t,r)C(t,r) < 0, \quad D(t,r) > 0. \tag{31}$$

From this, we naturally set

$$\mu = 4\gamma - 6, \quad a(t) > 0 \quad \text{and} \quad a'(t) < 0. \tag{32}$$

If we choose

$$a(t) = 1 + R(t)^{-\delta} \quad \text{with} \quad \delta > 0, \tag{33}$$

then

$$\begin{aligned} III &= \int_{\Omega_T}\Big\{\frac{\delta}{2}R(t)^{\mu-1-\delta}(\partial_t\dot{\Phi} + \partial_r\hat{\Phi}\partial_r\dot{\Phi})^2 \\ &\quad + \frac{\gamma}{2}R(t)^{\mu-1-2(\gamma-1)}\big(L(4-2\gamma)a(t) - R(t)a'(t)\big)\Big((\partial_r\dot{\Phi})^2 + \frac{1}{r^2}(\partial_\theta\dot{\Phi})^2\Big)\Big\}dtdx \\ &\ge C\int_{\Omega_T}\Big\{R(t)^{\mu-1-\delta}(D_t\dot{\Phi})^2 + R(t)^{\mu-1-2(\gamma-1)}\Big((\partial_r\dot{\Phi})^2 + \frac{1}{r^2}(\partial_\theta\dot{\Phi})^2\Big)\Big\}dtdx. \end{aligned} \tag{34}$$

Substituting (25), (29)-(30) and (33) into (23) yields

$$\begin{aligned}
&R(T)^{\mu}\int_{\mathcal{S}_T}(D_t\dot{\Phi})^2dS+R(T)^{\mu-2(\gamma-1)}\int_{\mathcal{S}_T}\big((\partial_r\dot{\Phi})^2+\frac{1}{r^2}(\partial_\theta\dot{\Phi})^2\big)dS\\
&+C\int_{\Omega_T}\Big(R(t)^{\mu-1-\delta}(D_t\dot{\Phi})^2+R(t)^{\mu-1-2(\gamma-1)}\big((\partial_r\dot{\Phi})^2+\frac{1}{r^2}(\partial_\theta\dot{\Phi})^2\big)\Big)dtdx\\
&\leq\int_{\Omega_T}\mathcal{L}\dot{\Phi}\cdot\mathcal{M}\dot{\Phi}dtdx+C\varepsilon^2.
\end{aligned}\tag{35}$$

This together with (22) completes the proof of the theorem. ■

Next, we will derive the higher order energy estimates of $\dot{\Phi}$ to (17) with (20)-(21). For this, we need to take care of the difficulties coming from the Neumann boundary condition (20), and the different decay rates of $D_t\dot{\Phi}$ and $\nabla_x\dot{\Phi}$.

Theorem 3 *Let $\dot{\Phi}\in C^4(\bar{\Omega}_T)$ be the solution to (17) with (20)-(21), and assume with some $\delta>0$ and $1<\gamma<\frac{4}{3}$,*

$$\begin{cases}|\nabla_x\dot{\Phi}|\leq M\varepsilon R(t)^{-2(\gamma-1)+\frac{\delta}{2}}, \quad |R(t)^{l-1}D_t^l\dot{\Phi}|\leq M\varepsilon R(t)^{-2(\gamma-1)}, \quad \textit{for } l=1,2,\\ |R(t)\nabla_xD_t\dot{\Phi}|\leq M\varepsilon R(t)^{-2(\gamma-1)+\frac{\delta}{2}}, \quad |R(t)\nabla_x^2\dot{\Phi}|\leq M\varepsilon R(t)^{-\frac{2(\gamma-1)-\delta}{2}},\end{cases}\tag{36}$$

in addition, for $r>\dfrac{1}{3}R(t)$, further assume

$$|\partial_\theta\dot{\Phi}|\leq M\varepsilon R(t)^{1-2(\gamma-1)}, |\partial_\theta D_t\dot{\Phi}|\leq M\varepsilon R(t)^{-2(\gamma-1)}, |\nabla_x\partial_\theta\dot{\Phi}|\leq M\varepsilon R(t)^{-(\gamma-1)}.\tag{37}$$

Then for sufficiently small $\varepsilon>0$ and $0\leq k\leq 2$, we have

$$\begin{aligned}
&\int_{\mathcal{S}_T}\Big(R(T)^{\mu+2k}(\nabla_{t,x}^kD_t\dot{\Phi})^2+R(T)^{\mu-2(\gamma-1)+2k}(\nabla_{t,x}^k\nabla_x\dot{\Phi})^2\Big)dS\\
&\quad+\int_{\Omega_T}\Big(R(t)^{\mu-1-\delta+2k}(\nabla_{t,x}^kD_t\dot{\Phi})^2+R(t)^{\mu-1-2(\gamma-1)+2k}(\nabla_{t,x}^k\nabla_x\dot{\Phi})^2\Big)dtdx\\
&\leq C\varepsilon^2,
\end{aligned}\tag{38}$$

and

$$\begin{aligned}
&\int_{\mathcal{S}_T}\Big(R(T)^{\mu-\delta+6}(\nabla_{t,x}^3D_t\dot{\Phi})^2+R(T)^{\mu-\delta-2(\gamma-1)+6}(\nabla_{t,x}^3\nabla_x\dot{\Phi})^2\Big)dS\\
&\quad+\int_{\Omega_T}\Big(R(t)^{\mu+5-\delta}(\nabla_{t,x}^3D_t\dot{\Phi})^2+R(t)^{\mu-\delta+5-2(\gamma-1)}(\nabla_{t,x}^3\nabla_x\dot{\Phi})^2\Big)dtdx\\
&\leq C\varepsilon^2,
\end{aligned}\tag{39}$$

where $\mu=4\gamma-6$, $0<\delta\leq\frac{2(\gamma-1)}{5}$, $C>0$ is independent of M, and the domains $\Omega_T,\mathcal{S}_T$ are defined in Section 4.

Proof. The proof is very similar in [1], for reader's convenience, we only give main steps. We will apply the induction on k in (38)-(39) to establish the following estimates respectively:

(i) $D_tS^k\dot{\Phi}$ and $\nabla_xS^k\dot{\Phi}$ with $S^k=S_0^{l_1}\partial_\theta^{l_2}(S_0=R(t)D_t)$ and $1\leq k=l_1+l_2\leq 3$ (in this case, all the tangent derivatives of $\nabla_x\dot{\Phi}$ up to the third order are estimated, where the tangent derivative means the one of boundary $\mathcal{B}_T$);

(ii) $D_tS_1\dot{\Phi}$ and $\nabla_xS_1\nabla_x\dot{\Phi}$ with $S_1=r\partial_r$ (in this case, together with the case $k=1$ in (i), all the second order derivatives $\nabla_{t,x}^2\dot{\Phi}$ are analyzed);

(iii) $D_tSS_1\dot{\Phi}$, $\nabla_xSS_1\dot{\Phi}$, $D_tS_1^2\dot{\Phi}$ and $\nabla_xS_1^3\dot{\Phi}$ (in this case, together with the case $k=2$ in (i), all the estimates of third order derivatives $\nabla_{t,x}^3\dot{\Phi}$ are given);

(iv) $D_tS^2S_1\dot{\Phi}$, $\nabla_xS^2S_1\dot{\Phi}$, $D_tSS_1^2\dot{\Phi}$, $\nabla_xSS_1^2\dot{\Phi}$, $D_tS_1^3\dot{\Phi}$ and $\nabla_xS_1^3\dot{\Phi}$ (in this case, together with the case $k=3$ in (i), all the fourth order derivatives $\nabla_{t,x}^4\dot{\Phi}$ are estimated). ■

4 Proof of Theorem 1

To complete the proof of Theorem 1, as in [9], the following estimate is needed.

Lemma 1 *For $1 \le t \le T_0$ and $k_0 \ge 4$, we have that for a smooth function $\varphi(t,x) \in H^{k_0}(\Omega_T)$*

$$\sum_{0\le l\le k_0-4} |t^l \nabla^{l+1}\varphi(t,x)|^2 \le C_0 t^{-2} \int_{\mathcal{S}_T} \sum_{0\le l\le k_0-1} |t^l \nabla^{l+1}\varphi(t,x)|^2 dx. \tag{40}$$

We now first prove Theorem 1 as follows.

Proof. It follows from Lemma 6 that, for $0 \le t \le T$,

$$\begin{cases} \sum_{0\le l\le 1} |R(t)^l \nabla_x^{l+1}\dot\Phi|^2 \le CR(t)^{-2} \int_{\mathcal{S}_t} \sum_{0\le l\le 3} |R(t)^l \nabla_x^{l+1}\dot\Phi|^2 dx, \\ |R(t) D_t^2 \dot\Phi|^2 \le CR(t)^{-2} \int_{\mathcal{S}_t} \sum_{0\le l\le 3} |R(t)^{1+l}\nabla_x^l D_t^2\dot\Phi|^2 dx, \\ \sum_{0\le l\le 1} |R(t)^l \nabla_x^l D_t\dot\Phi|^2 \le CR(t)^{-2} \int_{\mathcal{S}_t} \sum_{0\le l\le 3} |R(t)^l \nabla_x^l D_t\dot\Phi|^2 dx. \end{cases} \tag{41}$$

On the other hand, (38) and (39) give

$$\begin{cases} \int_{\mathcal{S}_t} \sum_{0\le l\le 2} |R(t)^l \nabla_x^{l+1}\dot\Phi|^2 dx \le C\varepsilon^2 R(t)^{-\mu+2(\gamma-1)}, \\ \int_{\mathcal{S}_t} \sum_{0\le l\le 2} |R(t)^l \nabla_x^l D_t\dot\Phi|^2 dx \le C\varepsilon^2 R(t)^{-\mu}, \end{cases} \tag{42}$$

and

$$\begin{cases} \int_{\mathcal{S}_t} \sum_{0\le l\le 2} |R(t)^{1+l} \nabla_x^{l+2}\dot\Phi|^2 dx \le C\varepsilon^2 R(t)^{-\mu+2(\gamma-1)+\delta}, \\ \int_{\mathcal{S}_t} \sum_{0\le l\le 2} |R(t)^{1+l} \nabla_x^{1+l} D_t\dot\Phi|^2 dx \le C\varepsilon^2 R(t)^{-\mu+\delta}, \\ \int_{\mathcal{S}_t} \sum_{0\le l\le 3} |R(t)^{1+l} \nabla_x^l D_t^2\dot\Phi|^2 dx \le C\varepsilon^2 R(t)^{-\mu+\delta}. \end{cases} \tag{43}$$

Hence, we obtain

$$\begin{cases} |\nabla_x\dot\Phi| \le C\varepsilon R(t)^{-(\gamma-1)}, \quad |R(t)^{l-1} D_t^l\dot\Phi| \le C\varepsilon R(t)^{-2(\gamma-1)}, (l=1,2) \\ |R(t)\nabla_x D_t\dot\Phi| \le C\varepsilon R(t)^{-2(\gamma-1)+\frac{\delta}{2}}, \quad |R(t)\nabla_x^2\dot\Phi| \le C\varepsilon R(t)^{-\frac{2(\gamma-1)-\delta}{2}}. \end{cases} \tag{44}$$

On the other hand, by the stream line equation

$$\frac{dx_i(t)}{dt} = \frac{Lx_i}{R(t)}, \quad x_i(0) = x_i^0, \quad (i=1,2), \tag{45}$$

we have

$$x_i(t) = x_i^0 R(t) \quad (i=1,2), \tag{46}$$

where (x_1^0, x_2^0) is the initial point. Then integrating along the stream line, we have for $1 < \gamma < \frac{3}{2}$ that

$$|R(t)\nabla_x\dot\Phi(t,x(t))| \le |\nabla_x\dot\Phi(0,x(0))| + \int_0^t |D_t(R(t)\nabla_x\dot\Phi)| dt \le C\varepsilon\big(1 + R(t)^{1-2(\gamma-1)+\frac{\delta}{2}}\big),$$

which implies

$$|\nabla_x\dot\Phi| \le C\varepsilon\big(R(t)^{-1} + R(t)^{-2(\gamma-1)+\frac{\delta}{2}}\big) \le C\varepsilon R(t)^{-2(\gamma-1)+\frac{\delta}{2}}. \tag{47}$$

Thus, the first inequality in (36) is proved. On the other hand, it follows Theorem 5 that

$$\begin{aligned}&\sum_{0\le l_1+l_2\le 2}\Big(R(t)^{\mu}\int_{\mathcal{S}_t}(D_tS^{l_1}S_1^{l_2}\dot\Phi)^2dx+R(t)^{\mu-2(\gamma-1)}\int_{\mathcal{S}_t}(\nabla_xS^{l_1}S_1^{l_2}\dot\Phi)^2dx\Big)\\&+\sum_{l_1+l_2=3,l_1\ge1}\Big(R(t)^{\mu}\int_{\mathcal{S}_t}(D_tS^{l_1}S_1^{l_2}\dot\Phi)^2dx+R(t)^{\mu-2(\gamma-1)}\int_{\mathcal{S}_t}(\nabla_xS^{l_1}S_1^{l_2}\dot\Phi)^2dx\Big)\\&\le C\varepsilon^2.\end{aligned}\tag{48}$$

Noticing that for $r>\dfrac{1}{3}R(t)$, $r\sim R(t)$ holds. Then (48) implies

$$\begin{aligned}&\sum_{0\le l\le 2}\Big(R(t)^{\mu+2l}\int_{\mathcal{S}_t\bigcap\{r>\frac13R(t)\}}\big(\nabla_x^l\partial_\theta D_t\dot\Phi\big)^2dx+R(t)^{\mu-2(\gamma-1)+2l}\int_{\mathcal{S}_t\bigcap\{r>\frac13R(t)\}}\big(\nabla_x^{l+1}\partial_\theta\dot\Phi\big)^2dx\Big)\\&\le C\varepsilon^2.\end{aligned}$$

Together with Lemma 6, this yields for $r>\frac13R(t)$,

$$\begin{aligned}&|\partial_\theta D_t\dot\Phi|^2\le CR(t)^{-3}\int_{\mathcal{S}_t\bigcap\{r>\frac13R(t)\}}\sum_{0\le l\le2}|R(t)^l\nabla_x^l\partial_\theta D_t\dot\Phi|^2dx\le C\varepsilon R(t)^{-\mu-3},\\&|\nabla_xD_t\dot\Phi|^2\le CR(t)^{-3}\int_{\mathcal{S}_t\bigcap\{r>\frac13R(t)\}}\sum_{0\le l\le2}|R(t)^l\nabla_x^{l+1}\partial_\theta\dot\Phi|^2dx\le C\varepsilon R(t)^{-\mu+2(\gamma-1)-3}.\end{aligned}$$

Subsequently, one has

$$|\partial_\theta D_t\dot\Phi|\le C\varepsilon R(t)^{-2(\gamma-1)},\quad |\nabla_xD_t\dot\Phi|\le C\varepsilon R(t)^{-(\gamma-1)}.$$

In addition, for $1<\gamma<\dfrac32$ and $r>\dfrac13R(t)$, integrating along the stream yields

$$|\partial_\theta\dot\Phi(t,x(t))|\le|\partial_\theta\dot\Phi(0,x(0))|+\int_0^t|D_t(\partial_\theta\dot\Phi)|dt\le C\varepsilon\big(1+R(t)^{1-2(\gamma-1)}\big)\le C\varepsilon R(t)^{1-2(\gamma-1)}.$$

Note that the generic constant $C>0$ appeared in this section depends only on the initial data. Then we can choose the constant $M=2C$ in (36)-(37) for small $\varepsilon>0$ so that (36) and (37) hold. In this case, by the Bernoulli law (3), we have $c^2(\rho)=c^2(\hat\rho)-(\gamma-1)D_t\dot\Phi-\frac{\gamma-1}{2}|\nabla_x\dot\Phi|^2$, which gives $CR(t)^{2(1-\gamma)}-C\varepsilon R(t)^{2(1-\gamma)}<c^2(\rho)<CR(t)^{2(1-\gamma)}+C\varepsilon R(t)^{2(1-\gamma)}$. Thus, one obtains $c^2(\rho)\sim R(t)^{2(1-\gamma)}>0$ for any $t\ge0$ and small $\varepsilon>0$. Therefore, the proof of Theorem 1 is completed by the local existence result and continuation argument. ■

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