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Mathematical Analysis of a Fractional Order SIS Epidemic Model with Double Diseases, Beddington-DeAngelis Functional Response and Time Delay

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Abstract: In this paper, we consider a fractional order SIS epidemic model with double diseases, Beddington-DeAngelis functional response and time delay, where the fractional derivative is defined in the Caputo sense. The positivity and boundedness of solutions in this system has been proved. Using the theory of stability of delayed fractional order differential equations, sufficient conditions, which ensure the stability of the system, are derived. By choosing the time delay as a bifurcation parameter, the Hopf bifurcation is studied and the critical value of the time delay for the occurrence of the Hopf bifurcation is determined.

Keywords: Fractional order; SIS epidemic model; Time delay; Hopf bifurcation; Stability.

1 Introduction

Epidemiology is the study of the spread of an infectious disease in communities, regions, and countries. Mathematical models of infectious disease are important tools and valuable approach to understand disease transmission and choose the best strategies to control epidemics (see, e.g., [1–7]).

In general epidemic models, there exists only one epidemic disease caused by one virus. In fact, there might be two epidemic diseases caused by two different viruses. Recently, many authors studied the epidemic models with double epidemic hypothesis see, e.g., [8–15]. In [10] Miao et al. proposed the following deterministic SIS epidemic model with double epidemic hypothesis and Beddington-DeAngelis type functional

$$\begin{cases} \dot{S}(t) = & \Lambda - \mu S(t) - \frac{\beta_1 S(t) I_1(t)}{1+\alpha_1 S(t)+\gamma_1 I_1(t)} - \frac{\beta_2 S(t) I_2(t)}{1+\alpha_2 S(t)+\gamma_2 I_2(t)} + r_1 I_1(t) + r_2 I_2(t), \quad t \geq 0, \\ \dot{I}_1(t) = & \frac{\beta_1 S(t) I_1(t)}{1+\alpha_1 S(t)+\gamma_1 I_1(t)} - (\mu + a_1 + r_1) I_1(t), \\ \dot{I}_2(t) = & \frac{\beta_2 S(t) I_2(t)}{1+\alpha_2 S(t)+\gamma_2 I_2(t)} - (\mu + a_2 + r_2) I_2(t), \end{cases}$$

where $S(t)$ is the proportion of susceptible individuals at time t, $I_1(t)$ and $I_2(t)$ are the proportion of infected individuals with virus V_1 and V_2 at time t, respectively. Λ is the recruitment rate of susceptible individuals, μ is the natural death rate of the population, β_i is the transmission coefficient between S and I_i, r_i is the treatment cure rate, a_i is the disease related death rate, α_i and γ_i, $i = 1, 2$, are saturation factors measuring the psychological or inhibitory effect. These parameters are all positive constants.

In the last years, Fractional differential equations are increasingly used to modeling many phenomena in different fields such as biological models [16], viscoelastic material models [17], hydrologic models [18], economic models [19], and so on. The importance of modeling real phenomena using the fractional differential equations is due to these fractional differential systems naturally include both memory and nonlocality effects. These effects are quite relevant to epidemic spread. Therefore, large numbers of researchers have started to study the epidemic models using the fractional differential equations, see, e.g., [20–24].

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In this work, we are concerned with the effect of the fractional order and time delay on the dynamics of an SIS epidemic model with double diseases. To this end, we study the following delay fractional order differential equations

$$\begin{cases} D^\alpha S(t) = & \Lambda - \mu S(t) - \frac{\beta_1 S(t) I_1(t-\tau_1)}{1+\alpha_1 S(t)+\gamma_1 I_1(t-\tau_1)} - \frac{\beta_2 S(t) I_2(t-\tau_2)}{1+\alpha_2 S(t)+\gamma_2 I_2(t-\tau_2)} + r_1 I_1(t) + r_2 I_2(t), \\ D^\alpha I_1(t) = & \frac{\beta_1 S(t) I_1(t-\tau_1)}{1+\alpha_1 S(t)+\gamma_1 I_1(t-\tau_1)} - (\mu + r_1) I_1(t), \\ D^\alpha I_2(t) = & \frac{\beta_2 S(t) I_2(t-\tau_2)}{1+\alpha_2 S(t)+\gamma_2 I_2(t-\tau_2)} - (\mu + r_2) I_2(t), \end{cases} \tag{1}$$

where $\alpha \in (0, 1]$ is the order of the fractional derivative, τ_i is the time delay in which the infectious agents of the virus V_i $(i = 1, 2)$ develop in the vector, and it is only after that time that the infected vector can infect a susceptible individual.

The initial condition for system (1) is given as

$$S(0) > 0,\ I_i(\theta) = \phi_i(\theta) \geq 0,\ \theta \in [-\tau, 0],\ \phi_i(0) > 0,\ i = 1, 2,\ \ \tau = \max\{\tau_1, \tau_2\}, \tag{2}$$

where $\phi_i \in \mathcal{C}([-\tau, 0], \mathbb{R}_+)$, the *Banach* space of continuous functions defined from $[-\tau, 0]$ into $\mathbb{R}_+$.

Hopf bifurcation phenomena is a very important dynamical behavior of delayed differential equations (integer order and fractional order), which is occurs when the stability of the equilibrium point changes from stable to unstable as a bifurcation parameter crosses a critical value. Recently, The research on the Hopf bifurcation of fractional order differential equations have received growing attention by many authors, see, e.g., [25–28]. In this study, the conditions that guarantee the local stability and existence of Hopf bifurcation of system (1) are obtained. So the rest of this paper is organized as follows. In the next section, some preliminaries are presented. In Section 3, we investigate the local stability and the existence of the Hopf bifurcation occurring at the unique positive equilibrium. A brief conclusion of our paper is in Section 4.

2 Preliminaries results

There are three several forms of definitions of fractional derivative, that is, the Riemann-Liouville fractional derivative, Grünwald-Letnikov fractional derivative, Caputo fractional derivative. The fractional derivative used in model (1) is in the sense of Caputo definition. The main advantage of the Caputo derivative is that the initial values for fractional differential equations with Caputo derivatives take the same form as that for integer order differential equations [29], which is more applicable for mathematical modelling of real-world problems. Also, another advantage of this definition is that the Caputo derivative of a constant is zero.

Definition 1 *[29]. The Caputo fractional derivative of order $\alpha \in (0, 1)$ for a function $f \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$ is defined as*

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds,$$

where Γ is the Gamma function. When $\alpha = 1$, $D^\alpha f(t) = f'(t)$.

The beauty of the Gamma function can be found in its property, that is, $\Gamma(x + 1) = x\Gamma(x)$ for all $x > 0$, and for nonnegative integers k, we have $\Gamma(k + 1) = k!$.

Definition 2 *[30]. Let $\alpha, \beta > 0$. The function E_α defined by*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

and its general form

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

are called Mittag-Leffler functions.

Remark 1 *For $\beta = 1$ we obtain $E_{\alpha,1}(z) = E_\alpha(z)$. Also we notice that*

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Theorem 2 *There exists a unique solution for the fractional order model (1) with the initial condition (2). Moreover, If $(S(t), I_1(t), I_2(t))$ be the solution of system (1) with the initial condition (2), then*

$$S(t) > 0,\ \ I_1(t) > 0 \ \text{ and } \ I_2(t) > 0, \ \ \text{for all } t \geq 0.$$

In addition, for $N(t) = S(t) + I_1(t) + I_2(t)$, we have

$$\lim_{t\to+\infty} N(t) = \frac{\Lambda}{\mu}.$$

Proof. According to the results shown in Theorem 3.1 and Corollary 3.2 of [31], the solution of model (1) with the initial condition (2) exists and is unique on $[0, +\infty)$.

Let $(S(t), I(t), R(t))$ be the solution of system (1) with the initial condition (2). First, we prove that $S(t) > 0$ for all $t > 0$. Assume the contrary, then there exists a positive t^* such that $S(t^*) = 0$ and $S(t) > 0$ for all $t \in [0, t^*)$. Hence, there must have $I_1(t) > 0$ and $I_2(t) > 0$ for any $t \in [0, t^*)$. If this statement is not true, then there exists a $t_1 \in (0, t^*)$ such that $I_1(t_1) = 0$ and $I_1(t) > 0$ for all $t \in [0, t_1)$; and there exists a $t_2 \in (0, t^*)$ such that $I_2(t_2) = 0$ and $I_2(t) > 0$ for all $t \in [0, t_2)$. From the second and third equations of (1), and by Theorem 7.2 (and Remark 7.1) of [30], we have

$$\begin{aligned} I_1(t_1) &= I_1(0)\, E_\alpha(-(\mu + r_1)t_1^\alpha) \\ &\quad + \int_0^{t_1} \frac{\beta_1 S(t_1 - s) I_1(t_1 - s - \tau_1)}{1 + \alpha_1 S(t_1 - s) + \gamma_1 I_1(t_1 - s - \tau_1)} s^{\alpha-1} E_{\alpha,\alpha}(-(\mu + r_1)s^\alpha) ds, \\ I_2(t_2) &= I_2(0)\, E_\alpha(-(\mu + r_2)t_2^\alpha) \\ &\quad + \int_0^{t_2} \frac{\beta_2 S(t_2 - s) I_2(t_2 - s - \tau_2)}{1 + \alpha_2 S(t_2 - s) + \gamma_2 I_2(t_2 - s - \tau_2)} s^{\alpha-1} E_{\alpha,\alpha}(-(\mu + r_2)s^\alpha) ds, \end{aligned}$$

which is contradiction with $I_1(t_1) = 0$ and $I_2(t_2) = 0$. Hence $I_1(t) > 0$ and $I_2(t) > 0$ for all $t \in [0, t^*)$. Therefore, by the first equation of (1), we have that

$$D^\alpha S(t^*) = \Lambda + r_1 I_1(t^*) + r_2 I_2(t^*) > 0,$$

which is, by Lemma 2 in [32], a contradiction to the assumption that $S(t) > 0 = S(t^*)$ for any $t \in [0, t^*)$. It follows that $S(t)$ is always positive for $t > 0$. By the previous stepping, it is obviously that $I_1(t) > 0$ and $I_2(t) > 0$ for all $t > 0$.

The fractional derivative of the total population $N(t)$ is given by

$$D^\alpha N(t) = \Lambda - \mu N(t).$$

Again by Theorem 7.2 (and Remark 7.1) of [30], we have

$$N(t) = N(0)\, E_\alpha(-\mu t^\alpha) + \Lambda \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\mu s^\alpha) ds.$$

Then

$$\begin{aligned} N(t) &= N(0)\, E_\alpha(-\mu t^\alpha) + \Lambda \int_0^t s^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-\mu t^\alpha)^k}{\Gamma(\alpha k + \alpha)} ds \\ &= N(0)\, E_\alpha(-\mu t^\alpha) + \Lambda \sum_{k=0}^{\infty} \frac{(-\mu)^k t^{\alpha k + \alpha}}{(\alpha k + \alpha)\Gamma(\alpha k + \alpha)} \\ &= N(0)\, E_\alpha(-\mu t^\alpha) + \Lambda \sum_{k=0}^{\infty} \frac{(-\mu)^k t^{\alpha(k+1)}}{\Gamma\left[\alpha(k+1) + 1\right]} \\ &= N(0)\, E_\alpha(-\mu t^\alpha) - \frac{\Lambda}{\mu} \sum_{k=1}^{\infty} \frac{(-\mu t^\alpha)^k}{\Gamma(\alpha k + 1)} \\ &= N(0)\, E_\alpha(-\mu t^\alpha) - \frac{\Lambda}{\mu} \left(E_\alpha(-\mu t^\alpha) - 1\right). \end{aligned}$$

Hence $\lim_{t\to+\infty} N(t) = \frac{\Lambda}{\mu}$ since $\lim_{t\to+\infty} E_\alpha(-\mu t^\alpha) = 0$. This complete the proof. ■

Consider the following n-dimensional linear fractional differential system with multiple time delays

$$D^\alpha x(t) = Ax(t) + Bx(t-\tau),\ \ t \geq 0, \tag{3}$$

where $\alpha \in (0,1]$, $A = (a_{ij})_{i,j=1,2,\ldots,n} \in \mathbb{R}^{n\times n}$, $B = (b_{ij})_{i,j=1,2,\ldots,n} \in \mathbb{R}^{n\times n}$, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \mathbb{R}^n$, $x(t-\tau) = (x_1(t-\tau_1), x_2(t-\tau_2), \ldots, x_n(t-\tau_n)) \in \mathbb{R}^n$ and $(\tau_i)_{i=1,2,\ldots,n} \in \mathbb{R}_+$.

The characteristic equation of system (3) is

$$\Delta(s) = \det\begin{pmatrix} s^\alpha - a_{11} - b_{11}e^{-s\tau_1} & -a_{12} - b_{12}e^{-s\tau_2} & \cdots & -a_{1n} - b_{1n}e^{-s\tau_n} \\ -a_{21} - b_{21}e^{-s\tau_1} & s^\alpha - a_{22} - b_{22}e^{-s\tau_2} & \cdots & -a_{2n} - b_{2n}e^{-s\tau_n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} - b_{n1}e^{-s\tau_1} & -a_{n2} - b_{n2}e^{-s\tau_2} & \cdots & s^\alpha - a_{nn} - b_{nn}e^{-s\tau_n} \end{pmatrix} = 0.$$

If $\tau_i = 0$, system (3) can be expressed as

$$D^\theta x(t) = Mx(t),$$

where the coefficient matrix $M = A + B$.

We have the following two stability results.

Lemma 3 *[33]. If all the roots of the characteristic equation $\Delta(s) = 0$ have negative real parts, then the zero solution of system (3) is Lyapunov globally asymptotically stable.*

Lemma 4 *[34]. If all the eigenvalues λ of M satisfy $|\arg(\lambda)| > \frac{\pi}{2}$ and the characteristic equation $\Delta(s) = 0$ has no purely imaginary roots for any $\tau_i > 0$, $i = 1, 2, \ldots, n$, then the zero solution of system (3) is Lyapunov globally asymptotically stable.*

3 Stability and Hopf bifurcation

It is obvious that system (1) has a unique positive equilibrium $E^* = (S^*, I_1^*, I_2^*)$ defined by

$$\begin{aligned} S^* &= \frac{\frac{\Lambda}{\mu} + \frac{1}{\gamma_1} + \frac{1}{\gamma_2}}{1 + \frac{\beta_1 - \alpha_1(\mu+r_1)}{\gamma_1(\mu+r_1)} + \frac{\beta_2-\alpha_2(\mu+r_2)}{\gamma_2(\mu+r_2)}}, \\ I_1^* &= \frac{[\beta_1 - \alpha_1(\mu+r_1)]\,S^* - (\mu+r_1)}{\gamma_1(\mu+r_1)}, \\ I_2^* &= \frac{[\beta_2 - \alpha_2(\mu+r_2)]\,S^* - (\mu+r_2)}{\gamma_2(\mu+r_2)}, \end{aligned}$$

provided

$(H1)$ $\beta_i > \alpha_i(\mu + r_i)$ for $i = 1, 2$, and $S^* > \max_{i=1,2}\left\{\frac{\mu+r_i}{\beta_i - \alpha_i(\mu+r_i)}\right\}$.

Assume that $(H1)$ holds. Let $x(t) = S(t) - S^*$, $y_1(t) = I_1(t) - I_1^*$ and $y_2(t) = I_2(t) - I_2^*$. Then by linearizing system (1) around E^*, we get the following system

$$\begin{cases} D^\alpha x(t) = & -(\mu + m_1 + m_2)x(t) - m_3 y_1(t-\tau_1) - m_4 y_2(t-\tau_2) + r_1 y_1(t) + r_2 y_2(t), \\ D^\alpha y_1(t) = & m_1 x(t) + m_3 y_2(t-\tau_1) - m_5 y_1(t), \\ D^\alpha y_2(t) = & m_2 x(t) + m_4 y_2(t-\tau_2) - m_6 y_2(t), \end{cases} \tag{4}$$

where

$$\begin{aligned} m_1 &= \frac{\beta_1 I_1^*(1+\gamma_1 I_1^*)}{(1+\alpha_1 S^*+\gamma_1 I_1^*)^2} > 0, \\ m_2 &= \frac{\beta_2 I_2^*(1+\gamma_2 I_2^*)}{(1+\alpha_2 S^*+\gamma_2 I_2^*)^2} > 0, \\ m_3 &= \frac{\beta_1 S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^*+\gamma_1 I_1^*)^2} > 0, \\ m_4 &= \frac{\beta_2 S^*(1+\alpha_2 S^*)}{(1+\alpha_2 S^*+\gamma_2 I_2^*)^2} > 0, \\ m_5 &= \mu + r_1 > 0, \\ m_6 &= \mu + r_2 > 0. \end{aligned}$$

Characteristic equation which is associated with system (4) is given by

$$\Delta(s) = \det \begin{pmatrix} s^\alpha + \mu + m_1 + m_2 & m_3 e^{-s\tau_1} - r_1 & m_4 e^{-s\tau_2} - r_2 \\ -m_1 & s^\alpha - m_3 e^{-s\tau_1} + m_5 & 0 \\ -m_2 & 0 & s^\alpha - m_4 e^{-s\tau_2} + m_6 \end{pmatrix} = 0,$$

which leads to

$$(s^\alpha + \mu)\left[s^{2\alpha} + a_1 s^\alpha + a_0 - (b_1 s^\alpha + b_0)e^{-s\tau_1} - (c_1 s^\alpha + c_0)e^{-s\tau_2} + d_0 e^{-s(\tau_1+\tau_2)}\right] = 0, \tag{5}$$

where

$$\begin{aligned} a_1 &= m_1 + m_2 + m_5 + m_6, \\ a_0 &= m_1 m_6 + m_2 m_5 + m_5 m_6, \\ b_1 &= m_3, \\ b_0 &= m_3(m_2 + m_6), \\ c_1 &= m_4, \\ c_0 &= m_4(m_1 + m_5), \\ d_0 &= m_3 m_4. \end{aligned}$$

The system (1) has two time delays, then we discuss the following various cases.
Case 1 ($\tau_1 = \tau_2 = 0$).
In absence of both delays, the characteristic equation of the coefficient matrix M of system (4) is

$$(\lambda + \mu)\left[\lambda^2 + (a_1 - b_1 - c_1)\lambda + a_0 - b_0 - c_0 + d_0\right] = 0, \tag{6}$$

where

$$\begin{aligned} a_1 - b_1 - c_1 = {} & m_1 + m_2 + (m_5 - m_3) + (m_6 - m_4), \\ a_0 - b_0 - c_0 + d_0 = {} & (m_1 + m_5 - m_3)(m_6 - m_4) + m_2(m_5 - m_3). \end{aligned}$$

Clearly, $\lambda = -\mu < 0$ is always a root of Eq. (6). The other roots of (6) are determined by the following equation

$$\lambda^2 + (a_1 - b_1 - c_1)\lambda + a_0 - b_0 - c_0 + d_0 = 0.$$

Note that

$$m_5 - m_3 = \frac{\gamma_1 \beta_1 S^* I_1^*}{(1+\alpha_1 S^* + \gamma_1 I_1^*)^2} > 0, \tag{7}$$

$$m_6 - m_4 = \frac{\gamma_2 \beta_2 S^* I_2^*}{(1+\alpha_2 S^* + \gamma_2 I_2^*)^2} > 0, \tag{8}$$

then it is easy to show that $a_1 - b_1 - c_1 > 0$ and $a_0 - b_0 - c_0 + d_0 > 0$. Thus all roots λ_i $(i = 1, 2, 3)$ of (6) have negative real part, so that all the eigenvalues of M of system (4) satisfy $|\arg(\lambda_i)| > \frac{\pi}{2} \geq \frac{\alpha\pi}{2}$ for any $\alpha \in (0, 1]$ if $(H1)$ holds. Therefore the equilibrium E^* is locally asymptotically stable for $\tau_1 = \tau_2 = 0$ if $(H1)$ holds.

Case 2 (**$\tau_1 > 0$, $\tau_2 = 0$**).
In this case, equation (5) becomes

$$(s^\alpha + \mu)\left[s^{2\alpha} + p_1 s^\alpha + p_0 - (q_1 s^\alpha + q_0)e^{-s\tau_1}\right] = 0, \tag{9}$$

where

$$\begin{aligned} p_1 &= a_1 - c_1 = m_1 + m_2 + m_5 + (m_6 - m_4) > 0, \\ p_0 &= a_0 - c_0 = (m_1 + m_5)(m_6 - m_4) + m_2 m_5 > 0, \\ q_1 &= b_1 = m_3 > 0, \\ q_0 &= b_0 - d_0 = (m_2 + m_6 - m_4)m_3 > 0. \end{aligned}$$

Theorem 5 *If $(H1)$ holds, then the equilibrium E^* is locally asymptotically stable for any delay $\tau_1 \geq 0$ and $\tau_2 = 0$.*

Proof. By Case 1, the eigenvalues of M of system (4) satisfy $|\arg(\lambda)| > \frac{\pi}{2}$ if $(H1)$ holds. Let $s = iw = w(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2})$ is a root of (9), with $w > 0$. After substituting and separating real and imaginary parts, we have

$$\begin{cases} w^\alpha q_1 \cos\left(\frac{\alpha\pi}{2} - w\tau_1\right) + q_0 \cos w\tau_1 = & w^{2\alpha}\cos\alpha\pi + w^\alpha p_1 \cos\frac{\alpha\pi}{2} + p_0, \\ w^\alpha q_1 \sin\left(\frac{\alpha\pi}{2} - w\tau_1\right) - q_0 \sin w\tau_1 = & w^{2\alpha}\sin\alpha\pi + w^\alpha p_1 \sin\frac{\alpha\pi}{2}. \end{cases}$$

Adding the squares of both equations together gives

$$w^{4\alpha} + \eta_3 w^{3\alpha} + \eta_2 w^{2\alpha} + \eta_1 w^\alpha + \eta_0 = 0, \tag{10}$$

where

$$\begin{aligned} \eta_3 &= 2p_1 \cos\frac{\alpha\pi}{2}, \\ \eta_2 &= p_1^2 - q_1^2 + 2p_0\cos\alpha\pi, \\ \eta_1 &= 2(p_0 p_1 - q_0 q_1)\cos\frac{\alpha\pi}{2}, \\ \eta_0 &= p_0^2 - q_0^2. \end{aligned}$$

Since $\alpha \in (0, 1]$ and $p_1 > 0$, we have $\eta_3 \geq 0$. Further, since $p_0 - q_0 = a_0 - b_0 - c_0 + d_0 > 0$, $p_1 - q_1 = a_1 - b_1 - c_1 > 0$ and $p_0, q_0, q_1 > 0$, we have

$$p_0^2 - q_0^2 = (p_0 - q_0)(p_0 + q_0) > 0 \text{ and } p_0 p_1 - q_0 q_1 = p_0(p_1 - q_1) + q_1(p_0 - q_0) > 0.$$

Then $\eta_0 > 0$ and $\eta_1 \geq 0$. In addition, since $q_0 > 0$, we have

$$\begin{aligned} \eta_2 &= p_1^2 - q_1^2 + 2q_0\cos\alpha\pi \\ &\geq p_1^2 - q_1^2 - 2q_0 \\ &= (m_1 + m_2 + m_5 + m_6 - m_4)^2 - m_3^2 - 2(m_2 + m_6 - m_4)m_3 \\ &= (m_1 + m_2 + m_6 - m_4)^2 + m_5^2 - m_3^2 + 2m_1 m_5 + 2(m_2 + m_6 - m_4)(m_5 - m_3). \end{aligned}$$

From (7), we have $m_5 - m_3 > 0$ and $m_5^2 - m_3^2 > 0$ since $m_5 + m_3 > 0$, and from (8), we have $m_6 - m_4 > 0$, then $\eta_2 > 0$. Therefore the Eq. (10) has no positive real roots, which ensures that Eq. (9) has no purely imaginary roots. According to Lemma 4, we conclude that when E^* exists, it is locally asymptotically stable for any delay $\tau_1 \geq 0$ and $\tau_2 = 0$. ■

Case 3 ($\tau_1 = 0$, $\tau_2 > 0$).
Similar discussion as those in Case 2.

Case 4 ($\tau_1 = \tau_2 = \tau > 0$).
In this case, the second factor of equation (5) becomes

$$-P_1(s)e^{-s\tau} + P_2(s) + d_0 e^{-2s\tau} = 0, \tag{11}$$

where

$$\begin{aligned} P_1(s) &= (b_1+c_1)s^{\alpha}+(b_0+c_0), \\ P_2(s) &= s^{2\alpha}+a_1s^{\alpha}+a_0. \end{aligned}$$

Multiplying $e^{s\tau}$ on both sides of (11), it is obvious to obtain

$$-P_1(s)+P_2(s)e^{s\tau}+d_0e^{-s\tau}=0. \tag{12}$$

Let $s=iw$ $(w>0)$ is a root of Eq. (12). Separating the real and the imaginary parts, we get

$$\begin{cases} (A_2+A_3)\cos w\tau - B_2\sin w\tau = & A_1, \\ B_2\cos w\tau + (A_2-A_3)\sin w\tau = & B_1, \end{cases}$$

where

$$\begin{aligned} A_1 &= w^{\alpha}(b_1+c_1)\cos\frac{\alpha\pi}{2}+(b_0+c_0), \\ A_2 &= w^{2\alpha}\cos\alpha\pi + w^{\alpha}a_1\cos\frac{\alpha\pi}{2}+a_0, \\ A_3 &= d_0, \\ B_1 &= w^{\alpha}(b_1+c_1)\sin\frac{\alpha\pi}{2}, \\ B_2 &= w^{2\alpha}\sin\alpha\pi + w^{\alpha}a_1\sin\frac{\alpha\pi}{2}. \end{aligned}$$

By calculation, one gets

$$\begin{aligned} \cos w\tau &= \frac{A_1(A_2-A_3)+B_1B_2}{A_2^2-A_3^2+B_2^2}, \\ \sin w\tau &= \frac{B_1(A_2+A_3)-A_1B_2}{A_2^2-A_3^2+B_2^2}. \end{aligned}$$

Squaring and adding the two above equations, we derive that

$$\omega^{8\alpha}+\xi_7\omega^{7\alpha}+\xi_6\omega^{6\alpha}+\xi_5\omega^{5\alpha}+\xi_4\omega^{4\alpha}+\xi_3\omega^{3\alpha}+\xi_2\omega^{2\alpha}+\xi_1\omega^{\alpha}+\xi_0=0, \tag{13}$$

where

$$\begin{aligned} \xi_7 &= 2f_3, \\ \xi_6 &= f_3^2+2f_2-(b_1+c_1)^2, \\ \xi_5 &= 2\left(f_1+f_2f_3-n_2n_3-h_2h_3\right), \\ \xi_4 &= 2\left(f_0+f_1f_3+f_2^2\right)-\left(n_2^2+h_2^2+2n_1n_3+2h_1h_3\right), \\ \xi_3 &= 2\left(f_0f_3+f_1f_2-n_0n_3-n_1n_2-h_1h_2\right), \\ \xi_2 &= f_1^2-n_1^2-h_1^2+2\left(f_0f_2-n_0n_2\right), \\ \xi_1 &= 2\left(f_0f_1-n_0n_1\right), \\ \xi_0 &= f_0^2-n_0^2, \end{aligned}$$

and

$$\begin{aligned}
f_3 &= 2a_1 \cos\frac{\alpha\pi}{2},\\
f_2 &= a_1^2 + 2a_0\cos\alpha\pi,\\
f_1 &= 2a_0a_1\cos\frac{\alpha\pi}{2},\\
f_0 &= a_0^2 - d_0^2,\\
n_3 &= (b_1+c_1)\cos\frac{\alpha\pi}{2},\\
n_2 &= a_1(b_1+c_1) + (b_0+c_0)\cos\alpha\pi,\\
n_1 &= [a_1(b_0+c_0) + (b_1+c_1)(a_0-d_0)]\cos\frac{\alpha\pi}{2},\\
n_0 &= (a_0-d_0)(b_0+c_0),\\
h_3 &= -(b_1+c_1)\sin\frac{\alpha\pi}{2},\\
h_2 &= -(b_0+c_0)\sin\alpha\pi,\\
h_1 &= -[a_1(b_0+c_0) - (b_1+c_1)(a_0+d_0)]\sin\frac{\alpha\pi}{2}.
\end{aligned}$$

Theorem 6 *Assume that $(H1)$ holds. If $\xi_i > 0$ $(i = 0, 1, \ldots, 6)$, then the equilibrium E^* is locally asymptotically stable for all $\tau \geq 0$.*

Now, by regarding the delay τ as a bifurcation parameter, we investigate the problem of the Hopf bifurcation for model (1). Suppose that $(H2)$ (13) has at least one positive real root. Without loss of generality, we assume that Eq. (13) has positive real roots ω_k, $k = 1, 2, \ldots, 8$. Thus, denoting

$$\begin{aligned}
\tau_k^{(j)} &= \frac{1}{\omega_k}\arccos\left\{\frac{A_1(A_2-A_3)+B_1B_2}{A_2^2-A_3^2+B_2^2}\right\} + \frac{2\pi j}{\omega_k},\\
k &= 1, 2, \ldots, 8;\ j = 0, 1, 2, \ldots,
\end{aligned}$$

then $\pm i\omega_k$ is a pair of purely imaginary root of (13) with $\tau = \tau_k^{(j)}$. Define the bifurcation point

$$\tau_0 = \min\left\{\tau_k^{(0)} : k = 1, 2, \ldots, 8\right\}.$$

Let $s(\tau) = \xi(\tau) + i\omega(\tau)$ be the root of (12) near $\tau = \tau_0$ satisfying

$$\xi(\tau_0) = 0,\ \omega(\tau_0) = \omega_0.$$

Taking the derivative of Eq. (12) with respect to τ, we obtain

$$\frac{ds}{d\tau} = \frac{-s\left(P_2(s)e^{s\tau} - d_0e^{-s\tau}\right)}{-P_1'(s) + \left(P_2'(s) + \tau P_2(s)\right)e^{s\tau} - \tau d_0 e^{-s\tau}}.$$

Then

$$Re\left[\frac{ds}{d\tau}\right]\Bigg|_{(\tau=\tau_0,\omega=\omega_0)} = Re\left(\frac{N_1 - iN_2}{N_3 + iN_4}\right) = \frac{N_1N_3 - N_2N_4}{N_3^2 + N_4^2},$$

where

$$\begin{aligned}
N_1 &= \omega_0^{2\alpha+1}\sin(\alpha\pi+\omega_0\tau_0)+\omega_0^{\alpha+1}a_1\sin(\frac{\alpha\pi}{2}+\omega_0\tau_0)+\omega_0(a_0+d_0)\sin\omega_0\tau_0,\\
N_2 &= \omega_0^{2\alpha+1}\cos(\alpha\pi+\omega_0\tau_0)+\omega_0^{\alpha+1}a_1\cos(\frac{\alpha\pi}{2}+\omega_0\tau_0)+\omega_0(a_0-d_0)\cos\omega_0\tau_0,\\
N_3 &= 2\alpha\omega_0^{2\alpha-1}\cos\left[\frac{(2\alpha-1)\pi}{2}+\omega_0\tau_0\right]+\alpha\omega_0^{\alpha-1}a_1\cos\left[\frac{(\alpha-1)\pi}{2}+\omega_0\tau_0\right]\\
&\quad -\alpha\omega_0^{\alpha-1}(b_1+c_1)\cos\frac{(\alpha-1)\pi}{2}\\
&\quad +\tau_0\left\{\omega_0^{2\alpha}\cos(\alpha\pi+\omega_0\tau_0)+\omega_0^{\alpha}a_1\cos(\frac{\alpha\pi}{2}+\omega_0\tau_0)+(a_0-d_0)\cos\omega_0\tau_0\right\},\\
N_4 &= 2\alpha\omega_0^{2\alpha-1}\sin\left[\frac{(2\alpha-1)\pi}{2}+\omega_0\tau_0\right]+\alpha\omega_0^{\alpha-1}a_1\sin\left[\frac{(\alpha-1)\pi}{2}+\omega_0\tau_0\right]\\
&\quad -\alpha\omega_0^{\alpha-1}(b_1+c_1)\sin\frac{(\alpha-1)\pi}{2}\\
&\quad +\tau_0\left\{\omega_0^{2\alpha}\sin(\alpha\pi+\omega_0\tau_0)+\omega_0^{\alpha}a_1\sin(\frac{\alpha\pi}{2}+\omega_0\tau_0)+(a_0+d_0)\sin\omega_0\tau_0\right\}.
\end{aligned}$$

Thus, to obtain the conditions of bifurcation, we introduce the following hypothesis.

$(H3)$ $N_1N_3 - N_2N_4 \neq 0$.

Therefore, we have the following result.

Theorem 7 *Suppose that $(H1)$-$(H3)$ hold, for system (1).*

(i) *If $\tau \in [0,\tau_0)$, then the positive equilibrium E^* is locally asymptotically stable.*

(ii) *If $\tau > \tau_0$, then the positive equilibrium E^* is unstable. Furthermore, system (1) undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$.*

Case 5 ($\tau_1 > 0, \tau_2 \in [0,\tau_{20})$ and $\tau_1 \neq \tau_2$).

In this case, we consider system (1) with τ_2 in its stable interval $[0,\tau_{20})$ and regard τ_1 as a parameter. Let $s = iw$ is a root of the second factor of Eq. (5), with $w > 0$. Then we have

$$\begin{cases} E_2\cos w\tau_1 + F_2\sin w\tau_1 = & E_1,\\ F_2\cos w\tau_1 - E_2\sin w\tau_1 = & F_1, \end{cases} \tag{14}$$

where

$$\begin{aligned}
E_1 &= -\left(w^{2\alpha}\cos\alpha\pi + w^{\alpha}a_1\cos\frac{\alpha\pi}{2}+a_0\right)+\left(w^{\alpha}c_1\cos\frac{\alpha\pi}{2}+c_0\right)\cos\omega\tau_2\\
&\quad +\left(w^{\alpha}c_1\sin\frac{\alpha\pi}{2}\right)\sin\omega\tau_2,\\
F_1 &= -\left(w^{2\alpha}\sin\alpha\pi + w^{\alpha}a_1\sin\frac{\alpha\pi}{2}\right)-\left(w^{\alpha}c_1\cos\frac{\alpha\pi}{2}+c_0\right)\sin\omega\tau_2\\
&\quad +\left(w^{\alpha}c_1\sin\frac{\alpha\pi}{2}\right)\cos\omega\tau_2,\\
E_2 &= -\left(w^{\alpha}b_1\cos\frac{\alpha\pi}{2}+b_0\right)+d_0\cos\omega\tau_2,\\
F_2 &= -w^{\alpha}b_1\sin\frac{\alpha\pi}{2}-d_0\sin\omega\tau_2.
\end{aligned}$$

Squaring and adding the two equations in (14), we obtain

$$\begin{aligned}
&w^{4\alpha}+\zeta_3w^{3\alpha}+\zeta_2w^{2\alpha}+\zeta_1w^{\alpha}+\zeta_0-2\left(\sigma_3w^{3\alpha}+\sigma_2w^{2\alpha}+\sigma_1w^{\alpha}+\sigma_0\right)\cos\omega\tau_2\\
&-2(\upsilon_3w^{3\alpha}+\upsilon_2w^{2\alpha}+\upsilon_1w^{\alpha})\sin\omega\tau_2=0,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
\zeta_3 &= 2a_1 \cos\frac{\alpha\pi}{2}, \\
\zeta_2 &= a_1^2 - b_1^2 + c_1^2 + 2a_0 \cos\alpha\pi, \\
\zeta_1 &= 2\left(a_0a_1 - b_0b_1 + c_0c_1\right)\cos\frac{\alpha\pi}{2}, \\
\zeta_0 &= a_0^2 - b_0^2 + c_0^2 - d_0^2, \\
\sigma_3 &= c_1 \cos\frac{\alpha\pi}{2}, \\
\sigma_2 &= a_1c_1 + c_0 \cos\alpha\pi, \\
\sigma_1 &= \left(a_1c_0 + a_0c_1 - b_1d_0\right)\cos\frac{\alpha\pi}{2}, \\
\sigma_0 &= a_0c_0 - b_0d_0, \\
\upsilon_3 &= -c_1 \sin\frac{\alpha\pi}{2}, \\
\upsilon_2 &= -c_0 \sin\alpha\pi, \\
\upsilon_1 &= \left(a_0c_1 - a_1c_0 + b_1d_0\right)\sin\frac{\alpha\pi}{2}.
\end{aligned}$$

Suppose that $(H4)$ (15) has finite positive real roots $\omega_1^{(1)}, \omega_1^{(2)}, \ldots, \omega_1^{(k)}$. For every fixed $\omega_1^{(i)}$, $i = 1, 2, \ldots, k$, there exists a sequence $\tau_{1i}^{(j)}$ such that (14) holds, where

$$\begin{aligned}
\tau_{1i}^{(j)} &= \frac{1}{\omega_1^{(i)}} \arccos\left\{\frac{E_1E_2 + F_1F_2}{E_2^2 + F_2^2}\right\} + \frac{2\pi j}{\omega_1^{(i)}}, \\
i &= 1, 2, \ldots, k;\ j = 0, 1, 2, \ldots.
\end{aligned}$$

Let $\tau^* = \min\left\{\tau_{1i}^{(0)} : i = 1, 2, \ldots, k\right\}$. When $\tau_1 = \tau^*$, Eq. (5) has a pair of purely imaginary roots $\pm i\omega^*$ for $\tau_2 \in [0, \tau_{20})$.

Differentiating the second factor of Eq. (5) with respect to τ_1, we get that

$$\frac{ds}{d\tau_1} = \frac{-s\left(Q(s) - d_0e^{-s\tau_2}\right)e^{-s\tau_1}}{P'(s) - \left(Q'(s) - \tau_1Q(s)\right)e^{-s\tau_1} - \left(R'(s) - \tau_2R(s)\right)e^{-s\tau_2} + (\tau_1 + \tau_2)d_0e^{-s(\tau_1+\tau_2)}},$$

where

$$\begin{aligned}
P(s) &= s^{2\alpha} + a_1s^{\alpha} + a_0, \\
Q(s) &= b_1s^{\alpha} + b_0, \\
R(s) &= c_1s^{\alpha} + c_0.
\end{aligned}$$

Then

$$Re\left[\frac{ds}{d\tau_1}\right]\Bigg|_{(\tau_1=\tau^*, s=i\omega^*)} = Re\left(\frac{\Omega_1 - i\Omega_2}{\Omega_3 + i\Omega_4}\right) = \frac{\Omega_1\Omega_3 - \Omega_2\Omega_4}{\Omega_3^2 + \Omega_4^2},$$

where

$$\begin{aligned}
\Omega_1 &= (\omega^*)^{\alpha+1} b_1 \sin\left(\frac{\alpha\pi}{2} - \omega^*\tau^*\right) + \omega^* d_0 \sin\omega^*(\tau_2 - \tau^*) - \omega^* b_0 \sin\omega^*\tau^*, \\
\Omega_2 &= (\omega^*)^{\alpha+1} b_1 \cos\left(\frac{\alpha\pi}{2} - \omega^*\tau^*\right) + \omega^* d_0 \cos\omega^*(\tau_2 - \tau^*) + \omega^* b_0 \cos\omega^*\tau^*, \\
\Omega_3 &= 2\alpha(\omega^*)^{2\alpha-1} \cos\frac{(2\alpha-1)\pi}{2} + \alpha(\omega^*)^{\alpha-1} a_1 \cos\frac{(\alpha-1)\pi}{2} \\
&\quad -\alpha(\omega^*)^{\alpha-1} b_1 \cos\left[\frac{(\alpha-1)\pi}{2} - \omega^*\tau^*\right] + \tau^*(\omega^*)^{\alpha} b_1 \cos\left(\frac{\alpha\pi}{2} - \omega^*\tau^*\right) + \tau^* b_0 \cos\omega^*\tau^* \\
&\quad -\alpha(\omega^*)^{\alpha-1} c_1 \cos\left[\frac{(\alpha-1)\pi}{2} - \omega^*\tau_2\right] + \tau_2(\omega^*)^{\alpha} c_1 \cos\left(\frac{\alpha\pi}{2} - \omega^*\tau_2\right) + \tau_2 c_0 \cos\omega^*\tau_2 \\
&\quad + d_0(\tau^* + \tau_2)\cos\omega^*(\tau^* + \tau_2), \\
\Omega_4 &= 2\alpha(\omega^*)^{2\alpha-1} \sin\frac{(2\alpha-1)\pi}{2} + \alpha(\omega^*)^{\alpha-1} a_1 \sin\frac{(\alpha-1)\pi}{2} \\
&\quad -\alpha(\omega^*)^{\alpha-1} b_1 \sin\left[\frac{(\alpha-1)\pi}{2} - \omega^*\tau^*\right] + \tau^*(\omega^*)^{\alpha} b_1 \sin\left(\frac{\alpha\pi}{2} - \omega^*\tau^*\right) - \tau^* b_0 \sin\omega^*\tau^* \\
&\quad -\alpha(\omega^*)^{\alpha-1} c_1 \sin\left[\frac{(\alpha-1)\pi}{2} - \omega^*\tau_2\right] + \tau^*(\omega^*)^{\alpha} c_1 \sin\left(\frac{\alpha\pi}{2} - \omega^*\tau_2\right) - \tau_2 c_0 \sin\omega^*\tau^* \\
&\quad - d_0(\tau^* + \tau_2)\sin\omega^*(\tau^* + \tau_2).
\end{aligned}$$

Therefore, the condition $Re\left[\frac{ds}{d\tau_1}\right]\Big|_{(\tau_1=\tau^*, s=i\omega^*)} \neq 0$ is satisfied provided,

$(H5)$ $\Omega_1\Omega_3 - \Omega_2\Omega_4 \neq 0$.

Theorem 8 *Suppose that $(H1)$, $(H4)$, $(H5)$ and $\tau_2 \in [0, \tau_{20})$ hold, for system (1).*

(i) *If $\tau_1 \in [0, \tau^*)$, then the positive equilibrium E^* is locally asymptotically stable.*

(ii) *If $\tau_1 > \tau^*$, then the positive equilibrium E^* is unstable. Furthermore, system (1) undergoes a Hopf bifurcation at E^* when $\tau_1 = \tau^*$.*

Case 6 ($\tau_1 \in [0, \tau_{10}), \tau_2 > 0$ and $\tau_1 \neq \tau_2$).
Similar discussion as those in previous case 5.

4 Conclusions

In this paper, a delayed fractional order SIS epidemic model with double diseases and Beddington-DeAngelis functional response is first proposed and the problems of stability and Hopf bifurcation for the newly-established model are addressed. With the help of the stability theory of fractional order differential equations with delay, sufficient conditions, which can guarantee the local asymptotical stability of the system, are developed. Furthermore, taking the time delay as the bifurcation parameter, the occurrence of Hopf bifurcation and the critical value of the time delay for Hopf bifurcation are determined.

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