ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.29(2020) No.1,pp.66-77

On Some Growth Properties of p-adic Entire Functions on the Basis of Their (p,q)-th Relative Type and (p,q)-th Relative Weak Type

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(Received 08 February 2019, accepted 15 March 2020)

Abstract: Let us consider $\mathbb{K}$ be a complete ultrametric algebraically closed field and suppose $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entire functions on $\mathbb{K}$. For any p adic entire functions $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Similarly to complex analysis, recently Biswas define the concepts of (p,q)-th relative order $\rho_g^{(p,q)}(f)$, (p,q)-th relative type $\sigma_g^{(p,q)}(f)$ and (p,q)-th relative weak type $\tau_g^{(p,q)}(f)$ of $f \in \mathcal{A}(\mathbb{K})$ where p and q are any two positive integers. In this paper we study some growth properties of p-adic entire functions on the basis of their (p,q)-th relative order, (p,q)-th relative type and (p,q)-th relative weak type.

Keywords: p-adic entire functions; growth; (p,q)-th relative order; (p,q)-th relative type; (p,q)-th relative weak type

1 Introduction

Let us consider $\mathbb{K}$ be an algebraically closed field of characteristic 0, complete with respect to a p-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$). For any $\alpha \in \mathbb{K}$ and $R \in]0,+\infty[$, the closed disk $\{x \in \mathbb{K} : |x-\alpha| \leq R\}$ and the open disk $\{x \in \mathbb{K} : |x-\alpha| < R\}$ are denoted by $d(\alpha,R)$ and $d(\alpha,R^-)$ respectively. Also $C(\alpha,r)$ denotes the circle $\{x \in \mathbb{K} : |x-a| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represent the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [12, 14, 17]. During the last several years the ideas of p-adic analysis have been studied from different aspects and many important results were gained (see [7] to [10], [15, 16]).

Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Moreover, if f is not a constant, the $|f|(r)$ is strictly increasing function of r and tends to $+\infty$ with r therefore there exists its inverse function $\widehat{|f|} : (|f(0)|,\infty) \to (0,\infty)$ with $\lim\limits_{s\to\infty} \widehat{|f|}(s) = \infty$.

For $x \in [0,\infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm. Taking this into account the (p,q)-th order and (p,q)-th lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ are define as follows:

Definition 1 *Let $f \in \mathcal{A}(\mathbb{K})$ and $p,q \in \mathbb{N}$. Then the (p,q)-th order and (p,q)-th lower order of f are respectively define as:*

$$\begin{matrix} \rho^{(p,q)}(f) \\ \lambda^{(p,q)}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r}.$$

Definition 2 avoids the restriction $p \geq q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [17] in complex context.

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IJNS.2020.03.25

When $q = 1$, we get the definitions of generalized order and generalized lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ which symbolize as $\rho^{(p)}(f)$ and $\lambda^{(p)}(f)$ respectively. If $p = 2$ and $q = 1$ then we write $\rho^{(2,1)}(f) = \rho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$ where $\rho(f)$ and $\lambda(f)$ are respectively known as order and lower order of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al. [7]. For details about the examples of p-adic entire functions with finite order with regular growth, one may see [7, 10].

In this connection we just introduce the following definition:

Definition 2 *An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have index-pair (p, q) where p and $q \in \mathbb{N}$ if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then*

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \cdots . \end{cases}$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \cdots . \end{cases}$$

An entire function $f \in \mathcal{A}(\mathbb{K})$ of index-pair (p, q) is said to be of regular (p, q)-th growth if its (p, q)-th order coincides with its (p, q)-th lower order, otherwise f is said to be of irregular (p, q)-th growth.

Next, to compare the growth of entire functions on $\mathbb{K}$ having the same (p, q)-th order, we give the definitions of (p, q)-th type and (p, q)-th lower type in the following manner :

Definition 3 *Let $f \in \mathcal{A}(\mathbb{K})$ and $p, q \in \mathbb{N}$. The (p, q)-th type and the (p, q)-th lower type of f having finite positive (p, q)-th order $\rho^{(p,q)}(f)$ $(0 < \rho^{(p,q)}(f) < \infty)$ are defined as :*

$$\begin{matrix} \sigma^{(p,q)}(f) \\ \overline{\sigma}^{(p,q)}(f) \end{matrix} = \lim_{r \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} |f|(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}}.$$

Remark 1 *If $p = 2$ and $q = 1$ then we write $\sigma^{(p,q)}(f) = \sigma(f)$ where $\sigma(f)$ is known as type of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al. [7].*

Likewise, to compare the growth of entire functions on $\mathbb{K}$ having the same (p, q)-th lower order, one can also introduce the concepts of (p, q)-th weak type in the following manner :

Definition 4 *Let $f \in \mathcal{A}(\mathbb{K})$ and $p, q \in \mathbb{N}$. The (p, q)-th weak type of f having finite positive (p, q)-th lower order $\lambda^{(p,q)}(f)$ $(0 < \lambda^{(p,q)}(f) < \infty)$ is defined as :*

$$\tau^{(p,q)}(f) = \varliminf_{r \to +\infty} \frac{\log^{[p-1]} |f|(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}}$$

Similarly one may define the growth indicator $\overline{\tau}^{(p,q)}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ in the following way :

$$\overline{\tau}^{(p,q)}(f) = \varlimsup_{r \to +\infty} \frac{\log^{[p-1]} |f|(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}}, \ 0 < \lambda^{(p,q)}(f) < \infty.$$

The notion of relative order was first introduced by Bernal [2]. In order to make some progress in the study of p-adic analysis, recently Biswas [1] introduce the definition of relative order and relative lower order of entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ in the following way:

$$\begin{matrix} \rho_g(f) \\ \lambda_g(f) \end{matrix} = \lim_{r \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \widehat{|g|}(|f|(r))}{\log r}.$$

Further the function $f \in \mathcal{A}(\mathbb{K})$, for which relative order and relative lower order with respect to another function $g \in \mathcal{A}(\mathbb{K})$ are the same is called a function of regular relative growth with respect to g. Otherwise, f is said to be irregular relative growth.with respect to g.

In the case of relative order, it therefore seems reasonable to define suitably the (p,q)-th relative order of entire function belonging to $\mathcal{A}(\mathbb{K})$ and to investigate some of its properties, which we attempt in this paper. With this in view one may introduce the definition of (p,q)-th relative order $\rho_g^{(p,q)}(f)$ and (p,q)-th relative lower order $\lambda_g^{(p,q)}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$, in the light of index-pair which are as follows:

Definition 5 *Let $f, g \in \mathcal{A}(\mathbb{K})$. Also let the index-pair of f and g are (m,q) and (m,p) , respectively where $p,q,m \in \mathbb{N}$. Then the (p,q)-th relative order $\rho_g^{(p,q)}(f)$ and (p,q)-th relative lower order $\lambda_g^{(p,q)}(f)$ of f with respect to g are defined as*

$$\begin{matrix}\rho_g^{(p,q)}(f) \\ \lambda_g^{(p,q)}(f)\end{matrix} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} \widehat{||g|}(|f|(r))}{\log^{[q]} r} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} \widehat{|g|}(r)}{\log^{[q]} \widehat{|f|}(r)} .$$

In order to refine the above growth scale, now we introduce the definitions of an another growth indicator, called (p,q)-th relative type and (p,q)-th relative lower type respectively of entire function belonging to $\mathcal{A}(\mathbb{K})$ with respect to another entire function belonging to $\mathcal{A}(\mathbb{K})$ in the light of their index-pair which are as follows:

Definition 6 *Let $f, g \in \mathcal{A}(\mathbb{K})$. Also let the index-pair of f and g are (m,q) and (m,p) , respectively, where $p,q,m \in \mathbb{N}$. The (p,q)-th relative type and (p,q)-th relative lower type of f with respect to g having finite positive (p,q)-th relative order $\rho_g^{(p,q)}(f)$ $\left(0 < \rho_g^{(p,q)}(f) < \infty\right)$ are defined as :*

$$\begin{matrix}\sigma_g^{(p,q)}(f) \\ \overline{\sigma}_g^{(p,q)}(f)\end{matrix} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}},$$

It is obvious that $0 \le \overline{\sigma}_g^{(p,q)}(f) \le \sigma_g^{(p,q)}(f) \le \infty$.

Analogously, to determine the relative growth of two entire functions belonging to $\mathcal{A}(\mathbb{K})$ and having same non zero finite (p,q)-th relative lower order with respect to another entire function belonging to $\mathcal{A}(\mathbb{K})$, one can introduce the definition of (p,q)-th relative weak type of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ of finite positive (p,q)-th relative lower order $\lambda_g^{(p,q)}(f)$ in the following way:

Definition 7 *Let $f, g \in \mathcal{A}(\mathbb{K})$. Also let the index-pair of f and g are (m,q) and (m,p) , respectively, where $p,q,m \in \mathbb{N}$. The (p,q)-th relative weak type and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of f with respect to g having finite positive (p,q)-th relative lower order $\lambda_g^{(p,q)}(f)$ $\left(0 < \lambda_g^{(p,q)}(f) < \infty\right)$ are defined as :*

$$\begin{matrix}\overline{\tau}_g^{(p,q)}(f) \\ \tau_g^{(p,q)}(f)\end{matrix} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}},$$

It is obvious that $0 \le \tau_g^{(p,q)}(f) \le \overline{\tau}_g^{(p,q)}(f) \le \infty$.

The main aim of this paper is to establish some results related to the growth rates of *p*-adic entire functions on the basis of (p,q)-th relative order, (p,q)-th relative type and (p,q)-th relative weak type where $p,q \in \mathbb{N}$. Throughout this paper, we assume that all the growth indicators are all nonzero finite.

2 Main Results

In this section we present the main results of the paper. First of all, we recall one related known property which will be needed in order to prove our results, as we see in the following theorem.

Theorem 2 *Let us consider* $f, g, h \in \mathcal{A}(\mathbb{K})$ *and* $p, q, m \in \mathbb{N}$. *If* (m,q)*-th relative order (respectively* (m,q)*-th relative lower order) of* f *with respect to* h *and* (m,p)*-th relative order (respectively* (m,p)*-th relative lower order) of* g *with respect to* h *are respectively denoted by* $\rho_h^{(m,q)}(f)$ $\left(\textit{respectively } \lambda_h^{(m,q)}(f)\right)$ *and* $\rho_h^{(m,p)}(g)$ $\left(\textit{respectively } \lambda_h^{(m,p)}(g)\right)$, *then*

$$\frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \lambda_g^{(p,q)}(f) \leq \min\left\{\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\right\}$$
$$\leq \max\left\{\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\right\} \leq \rho_g^{(p,q)}(f) \leq \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}.$$

Remark 3 *Under the same conditions of Theorem 10, one may write* $\rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}$ *and* $\lambda_g^{(p,q)}(f) = \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}$ *when* $\lambda_h^{(m,p)}(g) = \rho_h^{(m,p)}(g)$. *Similarly* $\rho_g^{(p,q)}(f) = \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}$ *and* $\lambda_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}$ *when* $\lambda_h^{(m,q)}(f) = \rho_h^{(m,q)}(f)$.

Theorem 4 *Let us consider* $f, g, h \in \mathcal{A}(\mathbb{K})$. *Also let* $0 < \lambda_h^{(m,q)}(f) \leq \rho_h^{(m,q)}(f) < \infty$ *and* $0 < \lambda_h^{(m,p)}(g) \leq \rho_h^{(m,p)}(g) < \infty$ *where* $p, q, m \in \mathbb{N}$. *Then*

$$\max\left\{\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\sigma_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \leq \sigma_g^{(p,q)}(f)$$
$$\leq \left[\frac{\sigma_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}.$$

Proof. From the definitions of $\sigma_h^{(m,q)}(f)$ and $\overline{\sigma}_h^{(m,q)}(f)$, we have for all sufficiently large values of r that

$$|f|(r) \leq |h|\left(\exp^{[m-1]}\left[\left(\sigma_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_h^{(m,q)}(f)}\right]\right), \tag{1}$$
$$|f|(r) \geq |h|\left(\exp^{[m-1]}\left[\left(\overline{\sigma}_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_h^{(m,q)}(f)}\right]\right) \tag{2}$$

and also for a sequence of values of r tending to infinity, we get that

$$|f|(r) \geq |h|\left(\exp^{[m-1]}\left[\left(\sigma_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_h^{(m,q)}(f)}\right]\right), \tag{3}$$
$$|f|(r) \leq |h|\left(\exp^{[m-1]}\left[\left(\overline{\sigma}_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_h^{(m,q)}(f)}\right]\right). \tag{4}$$

Similarly from the definitions of $\sigma_h^{(m,p)}(g)$ and $\overline{\sigma}_h^{(m,p)}(g)$, it follows for all sufficiently large values of r that

$$|g|(r) \leq |h|\left(\exp^{[m-1]}\left[\left(\sigma_h^{(m,p)}(g)+\varepsilon\right)\left[\log^{[p-1]} r\right]^{\rho_h^{(m,p)}(g)}\right]\right)$$
$$i.e.,\ |h|(r) \geq |g|\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\sigma_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right) \text{ and} \tag{5}$$

$$|h|(r) \leq |g|\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\overline{\sigma}_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right). \tag{6}$$

Also for a sequence of values of r tending to infinity, we obtain that

$$|h|(r) \leq |g|\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\sigma_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right) \text{ and } \tag{7}$$

$$|h|(r) \geq |g|\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\overline{\sigma}_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right). \tag{8}$$

From the definitions of $\overline{\tau}_h^{(m,q)}(f)$ and $\tau_h^{(m,q)}(f)$, we have for all sufficiently large values of r that

$$|f|(r) \leq |h|\left(\exp^{[m-1]}\left[\left(\overline{\tau}_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\lambda_h^{(m,q)}(f)}\right]\right), \tag{9}$$

$$|f|(r) \geq |h|\left(\exp^{[m-1]}\left[\left(\tau_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]} r\right]^{\lambda_h^{(m,q)}(f)}\right]\right) \tag{10}$$

and also for a sequence of values of r tending to infinity, we get that

$$|f|(r) \geq |h|\left(\exp^{[m-1]}\left[\left(\overline{\tau}_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]} r\right]^{\lambda_h^{(m,q)}(f)}\right]\right), \tag{11}$$

$$|f|(r) \leq |h|\left(\exp^{[m-1]}\left[\left(\tau_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\lambda_h^{(m,q)}(f)}\right]\right). \tag{12}$$

Similarly from the definitions of $\overline{\tau}_h^{(m,p)}(g)$ and $\tau_h^{(m,p)}(g)$, it follows for all sufficiently large values of r that

$$|g|(r) \leq |h|\left(\exp^{[m-1]}\left[\left(\overline{\tau}_h^{(m,p)}(g)+\varepsilon\right)\left[\log^{[p-1]} r\right]^{\lambda_h^{(m,p)}(g)}\right]\right)$$

$$i.e.,\ |h|(r) \geq |g|\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\overline{\tau}_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right) \text{ and } \tag{13}$$

$$|h|(r) \leq |g|\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\tau_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right). \tag{14}$$

Also for a sequence of values of r tending to infinity, we obtain that

$$|h|(r) \leq |g|\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\overline{\tau}_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right) \text{ and } \tag{15}$$

$$|h|(r) \geq |g|\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]} r}{\left(\tau_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_g(m,p)}}\right). \tag{16}$$

Now from (3) and in view of (13), we get for a sequence of values of r tending to infinity that

$$\widehat{|g|}(|f|(r)) \geq \widehat{|g|}\left(|h|\left(\exp^{[m-1]}\left[\left(\sigma_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_h^{(m,q)}(f)}\right]\right)\right)$$

$$i.e.,\ \widehat{|g|}\left(|f|\left(r\right)\right) \geq$$

$$\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\sigma_h^{(m,q)}\left(f\right)-\varepsilon\right)\left[\log^{[q-1]}r\right]^{\rho_h^{(m,q)}(f)}\right]}{\left(\overline{\tau}_h^{(m,p)}\left(g\right)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}$$

$$i.e.,\ \log^{[p-1]}\widehat{|g|}\left(|f|\left(r\right)\right) \geq \left[\frac{\left(\sigma_h^{(m,q)}\left(f\right)-\varepsilon\right)}{\left(\overline{\tau}_h^{(m,p)}\left(g\right)+\varepsilon\right)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}} \cdot \left[\log^{[q-1]}r\right]^{\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}}.$$

Since in view of Theorem 13, $\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} \geq \rho_g^{(p,q)}\left(f\right)$ and as $\varepsilon\left(>0\right)$ is arbitrary, therefore it follows from above that

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p-1]}\widehat{|g|}\left(|f|\left(r\right)\right)}{\left[\log^{[q-1]}r\right]^{\rho_g^{(p,q)}(f)}} \geq \left[\frac{\sigma_h^{(m,q)}\left(f\right)}{\overline{\tau}_h^{(m,p)}\left(g\right)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}$$

$$i.e.,\ \sigma_g^{(p,q)}\left(f\right) \geq \left[\frac{\sigma_h^{(m,q)}\left(f\right)}{\overline{\tau}_h^{(m,p)}\left(g\right)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}. \tag{17}$$

Similarly from (2) and in view of (16), it follows for a sequence of values of r tending to infinity that

$$\widehat{|g|}\left(|f|\left(r\right)\right) \geq \widehat{|g|}\left(|h|\left(\exp^{[m-1]}\left[\left(\overline{\sigma}_h^{(m,q)}\left(f\right)-\varepsilon\right)\left[\log^{[q-1]}r\right]^{\rho_h^{(m,q)}(f)}\right]\right)\right)$$

$$i.e.,\ |\widehat{|g|}\left(|f|\left(r\right)\right) \geq$$

$$\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\overline{\sigma}_h^{(m,q)}\left(f\right)-\varepsilon\right)\left[\log^{[q-1]}r\right]^{\rho_h^{(m,q)}(f)}\right]}{\left(\tau_h^{(m,p)}\left(g\right)+\varepsilon\right)}\right)^{\frac{1}{\lambda_g(m,p)}}$$

$$i.e.,\ \log^{[p-1]}\widehat{|g|}\left(|f|\left(r\right)\right) \geq \left[\frac{\left(\overline{\sigma}_h^{(m,q)}\left(f\right)-\varepsilon\right)}{\left(\tau_h^{(m,p)}\left(g\right)+\varepsilon\right)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}} \cdot \left[\log^{[q-1]}r\right]^{\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}}.$$

Since in view of Theorem 1, it follows that $\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} \geq \rho_g^{(p,q)}\left(f\right)$. Also $\varepsilon\left(>0\right)$ is arbitrary, so we get from above that

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p-1]}\widehat{|g|}\left(|f|\left(r\right)\right)}{\left[\log^{[q-1]}r\right]^{\rho_g^{(p,q)}(f)}} \geq \left[\frac{\overline{\sigma}_h^{(m,q)}\left(f\right)}{\tau_h^{(m,p)}\left(g\right)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}$$

$$i.e.,\ \sigma_g^{(p,q)}\left(f\right) \geq \left[\frac{\overline{\sigma}_h^{(m,q)}\left(f\right)}{\tau_h^{(m,p)}\left(g\right)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}. \tag{18}$$

Again in view of (6), we have from (1) for all sufficiently large values of r that

$$\widehat{|g|}\left(|f|\left(r\right)\right) \leq \widehat{|g|}\left(|h|\left(\exp^{[m-1]}\left[\left(\sigma_h^{(m,q)}\left(f\right)+\varepsilon\right)\left[\log^{[q-1]}r\right]^{\rho_h^{(m,q)}(f)}\right]\right)\right)$$

$$i.e.,\ \widehat{|g|}\left(|f|(r)\right)\le$$

$$\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\sigma_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]}r\right]^{\rho_h^{(m,q)}(f)}\right]}{\left(\overline{\sigma}_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

$$i.e.,\ \log^{[p-1]}\widehat{|g|}\left(|f|(r)\right)\le$$

$$\left[\frac{\left(\sigma_h^{(m,q)}(f)+\varepsilon\right)}{\left(\overline{\sigma}_h^{(m,p)}(g)-\varepsilon\right)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\cdot\left[\log^{[q-1]}r\right]^{\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}}. \tag{19}$$

As in view of Theorem 1, it follows that $\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\le\rho_g^{(p,q)}(f)$. Since $\varepsilon\,(>0)$ is arbitrary, we get from (19) that

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p-1]}\widehat{|g|}\left(|f|(r)\right)}{\left[\log^{[q-1]}r\right]^{\rho_g^{(p,q)}(f)}}\le\left[\frac{\sigma_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

$$i.e.,\ \sigma_g^{(p,q)}(f)\le\left[\frac{\sigma_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}. \tag{20}$$

Thus the theorem follows from (17), (18) and (20). ■

The conclusion of the following corollary can be carried out from (6) and (9); (9) and (14) respectively after applying the same technique of Theorem 4 and with the help of Theorem 2. Therefore its proof is omitted.

Corollary 5 *Let us suppose that* $f,\,g,\,h\in\mathcal{A}(\mathbb{K})$. *Also let* $0<\lambda_h^{(m,q)}(f)\le\rho_h^{(m,q)}(f)<\infty$ *and* $0<\lambda_h^{(m,p)}(g)\le\rho_h^{(m,p)}(g)<\infty$ *where* $p,\,q,\,m\in\mathbb{N}$. *Then*

$$\sigma_g^{(p,q)}(f)\le\min\left\{\left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}},\left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}.$$

Similarly in the line of Theorem 4 and with the help of Theorem 3, one may easily carried out the following theorem from pairwise inequalities numbers (10) and (13); (7) and (9); (6) and (12) respectively and therefore its proofs is omitted:

Theorem 6 *Let us consider* $f,\,g,\,h\in\mathcal{A}(\mathbb{K})$. *Also let* $0<\lambda_h^{(m,q)}(f)\le\rho_h^{(m,q)}(f)<\infty$ *and* $0<\lambda_h^{(m,p)}(g)\le\rho_h^{(m,p)}(g)<\infty$ *where* $p,\,q,\,m\in\mathbb{N}$. *Then*

$$\left[\frac{\tau_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\le\tau_g^{(p,q)}(f)\le$$

$$\min\left\{\left[\frac{\tau_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}},\left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}.$$

Corollary 7 *Let us suppose that* $f,\,g,\,h\in\mathcal{A}(\mathbb{K})$. *Also let* $0<\lambda_h^{(m,q)}(f)\le\rho_h^{(m,q)}(f)<\infty$ *and* $0<\lambda_h^{(m,p)}(g)\le\rho_h^{(m,p)}(g)<\infty$ *where* $p,\,q,\,m\in\mathbb{N}$. *Then*

$$\tau_g^{(p,q)}(f)\ge\max\left\{\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}},\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}.$$

With the help of Theorem 1, the conclusion of the above corollary can be carry out from (2), (5) and (2), (13) respectively after applying the same technique of Theorem 4 and therefore its proof is omitted.

Theorem 8 *Let us consider* $f,\ g,\ h\in\mathcal{A}(\mathbb{K})$. *Also let* $0<\lambda_h^{(m,q)}(f)\le\rho_h^{(m,q)}(f)<\infty$ *and* $0<\lambda_h^{(m,p)}(g)\le\rho_h^{(m,p)}(g)<\infty$ *where* $p,q,m\in\mathbb{N}$.

$$\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\le\overline{\sigma}_g^{(p,q)}(f)\le \min\left\{\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}},\left[\frac{\sigma_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}.$$

Proof. From (2) and in view of (13), we get for all sufficiently large values of r that

$$\widehat{|g|}(|f|(r))\ge\widehat{|g|}\left(|h|\left(\exp^{[m-1]}\left[\left(\overline{\sigma}_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]}r\right]^{\rho_h^{(m,q)}(f)}\right]\right)\right)$$

$$i.e.,\ \widehat{|g|}(|f|(r))\ge$$
$$\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\overline{\sigma}_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]}r\right]^{\rho_h^{(m,q)}(f)}\right]}{\left(\overline{\tau}_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}$$

$$i.e.,\ \log^{[p-1]}\widehat{|g|}(|f|(r))\ge\left[\frac{\left(\overline{\sigma}_h^{(m,q)}(f)-\varepsilon\right)}{\left(\overline{\tau}_h^{(m,p)}(g)+\varepsilon\right)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\cdot\left[\log^{[q-1]}r\right]^{\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}}.$$

Now in view of Theorem 1, it follows that $\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}\ge\rho_g^{(p,q)}(f)$. Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\underline{\lim}_{r\to+\infty}\frac{\log^{[p-1]}\widehat{|g|}(|f|(r))}{\left[\log^{[q-1]}r\right]^{\rho_g^{(p,q)}(f)}}\ge\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}$$
$$i.e.,\ \overline{\sigma}_g^{(p,q)}(f)\ge\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}. \tag{21}$$

Further in view of (7), we get from (1) for a sequence of values of r tending to infinity that

$$\widehat{|g|}(|f|(r))\le\widehat{|g|}\left(|h|\left(\exp^{[m-1]}\left[\left(\sigma_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]}r\right]^{\rho_h^{(m,q)}(f)}\right]\right)\right)$$

$$i.e.,\ \widehat{|g|}(|f|(r))\le$$
$$\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\sigma_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]}r\right]^{\rho_h^{(m,q)}(f)}\right]}{\left(\sigma_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

$$i.e.,\ \log^{[p-1]}\widehat{|g|}(|f|(r))\le$$

$$\left[\frac{\left(\sigma_h^{(m,q)}(f)+\varepsilon\right)}{\left(\sigma_h^{(m,p)}(g)-\varepsilon\right)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\cdot\left[\log^{[q-1]} r\right]^{\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}}. \quad (22)$$

Again as in view of Theorem 1, $\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \rho_g^{(p,q)}(f)$ and $\varepsilon\,(>0)$ is arbitrary, therefore we get from (22) that

$$\varliminf_{r\to+\infty}\frac{\log^{[p-1]}\widehat{|g|}(|f|(r))}{\left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)}} \leq \left[\frac{\sigma_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

$$i.e.,\ \overline{\sigma}_g^{(p,q)}(f) \leq \left[\frac{\sigma_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}. \quad (23)$$

Likewise from (4) and in view of (6), it follows for a sequence of values of r tending to infinity that

$$\widehat{|g|}(|f|(r)) \leq \widehat{|g|}\left(|h|\left(\exp^{[m-1]}\left[\left(\overline{\sigma}_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_h^{(m,q)}(f)}\right]\right)\right)$$

$$i.e.,\ \widehat{|g|}(|f|(r)) \leq$$

$$\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\overline{\sigma}_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_h^{(m,q)}(f)}\right]}{\left(\overline{\sigma}_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

$$i.e.,\ \log^{[p-1]}\widehat{|g|}(|f|(r)) \leq$$

$$\left[\frac{\left(\overline{\sigma}_h^{(m,q)}(f)+\varepsilon\right)}{\left(\overline{\sigma}_h^{(m,p)}(g)-\varepsilon\right)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\cdot\left[\log^{[q-1]} r\right]^{\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}}. \quad (24)$$

Analogously, we get from (24) that

$$\varliminf_{r\to+\infty}\frac{\log^{[p-1]}\widehat{|g|}(|f|(r))}{\left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)}} \leq \left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

$$i.e.,\ \overline{\sigma}_g^{(p,q)}(f) \leq \left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \quad (25)$$

since in view of Theorem 1, $\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \rho_g^{(p,q)}(f)$ and $\varepsilon\,(>0)$ is arbitrary.

Thus the theorem follows from (21), (23) and (25). ■

Corollary 9 *Let us consider* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$. *Also let* $0 < \lambda_h^{(m,q)}(f) \leq \rho_h^{(m,q)}(f) < \infty$ *and* $0 < \lambda_h^{(m,p)}(g) \leq \rho_h^{(m,p)}(g) < \infty$ *where* $p, q, m \in \mathbb{N}$. *Then*

$$\overline{\sigma}_g^{(p,q)}(f) \leq$$

$$\min\left\{\begin{array}{c}\left[\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}},\left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}},\\ \left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}},\left[\frac{\tau_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\end{array}\right\}.$$

The conclusion of the above corollary can be carried out from pairwise inequalities no (6) and (12) ; (7) and (9) ; (12) and (14); (9) and (15) respectively after applying the same technique of Theorem 8 and with the help of Theorem 1. Therefore its proof is omitted.

Similarly in the line of Theorem 4 and with the help of Theorem 1, one may easily carried out the following theorem from pairwise inequalities no (11) and (13) ; (10) and (16); (6) and (9) respectively and therefore its proofs is omitted:

Theorem 10 *Let us suppose that* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$ *. Also let* $0 < \lambda_h^{(m,q)}(f) \leq \rho_h^{(m,q)}(f) < \infty$ *and* $0 < \lambda_h^{(m,p)}(g) \leq \rho_h^{(m,p)}(g) < \infty$ *where* $p, q, m \in \mathbb{N}$*. Then*

$$\max\left\{\left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_g(m,p)}}, \left[\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \leq \overline{\tau}_g^{(p,q)}(f)$$
$$\leq \left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}.$$

Corollary 11 *Let us consider* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$ *. Also let* $0 < \lambda_h^{(m,q)}(f) \leq \rho_h^{(m,q)}(f) < \infty$ *and* $0 < \lambda_h^{(m,p)}(g) \leq \rho_h^{(m,p)}(g) < \infty$ *where* $p, q, m \in \mathbb{N}$*. Then*

$$\overline{\tau}_g^{(p,q)}(f) \geq$$
$$\max\left\{\begin{array}{c}\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\sigma_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \\ \left[\frac{\sigma_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\end{array}\right\}.$$

The conclusion of the above corollary can be carried out from pairwise inequalities no (3) and (5) ; (2) and (8) ; (3) and (13); (2) and (16) respectively after applying the same technique of Theorem 8 and with the help of Theorem 1. Therefore its proof is omitted.

Now we state the following two theorems without their proofs as because they can be derived easily using the same technique or with some easy reasoning by the help of with the help of Remark **??** and therefore left to the readers.

Theorem 12 *Let us suppose that* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$ *. Also let* $0 < \lambda_h^{(m,q)}(f) \leq \rho_h^{(m,q)}(f) < \infty$ *and* $\lambda_h^{(m,p)}(g) = \rho_h^{(m,p)}(g)$ *where* $p, q, m \in \mathbb{N}$*. Then*

$$\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}} \leq \overline{\sigma}_g^{(p,q)}(f)$$
$$\leq \min\left\{\left[\frac{\sigma_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\sigma_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}$$
$$\leq \max\left\{\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\sigma_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}$$
$$\leq \sigma_g^{(p,q)}(f) \leq \left[\frac{\sigma_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

and

$$\begin{aligned}\left[\frac{\tau_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}} &\leq \tau_g^{(p,q)}(f) \\ &\leq \min\left\{\left[\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_g\left({}_h^{(m,p)}(g)m,p\right)}}, \left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \\ &\leq \max\left\{\left[\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_g\left({}_h^{(m,p)}(g)m,p\right)}}, \left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \\ &\leq \overline{\tau}_g^{(p,q)}(f) \leq \left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}.\end{aligned}$$

Theorem 13 *Let us consider* $f, g, h \in \mathcal{A}(\mathbb{K})$. *Also let* $\lambda_h^{(m,q)}(f) = \rho_h^{(m,q)}(f)$ *and* $0 < \lambda_h^{(m,p)}(g) \leq \rho_h^{(m,p)}(g) < \infty$ *where* $p, q, m \in \mathbb{N}$. *Then*

$$\begin{aligned}\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}} &\leq \tau_g^{(p,q)}(f) \\ &\leq \min\left\{\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\sigma_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\} \\ &\leq \max\left\{\left[\frac{\overline{\sigma}_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\sigma_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\} \\ &\leq \overline{\tau}_g^{(p,q)}(f) \leq \left[\frac{\sigma_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\end{aligned}$$

and

$$\begin{aligned}\left[\frac{\tau_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}} &\leq \overline{\sigma}_g^{(p,q)}(f) \\ &\leq \min\left\{\left[\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \\ &\leq \max\left\{\left[\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \\ &\leq \sigma_g^{(p,q)}(f) \leq \left[\frac{\overline{\tau}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}.\end{aligned}$$

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