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Starlikeness Condition for a New Integral Operator

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Abstract: For the analytical functions f and g in the open unit disk $\mathcal{U}$ we introduce a new integral operator defined by $I(z) = \frac{2}{z}\int\limits_0^z f(t)e^{g(t)}dt$. For this integral operator, using the subordinations we study some starlikeness condition.

Keywords: Analytic functions; Starlike function; Univalent; Integral. operator.

1 Introduction and definitions

Let $\mathcal{U} = \{z : |z| < 1\}$ the unit disk and $\mathcal{A}$ the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in $\mathcal{U}$ and satisfy the condition

$$f(0) = f'(0) - 1 = 0.$$

We note by $\mathcal{S}$ the class of univalent and regular functions. By $\mathcal{H}(\mathcal{U})$ we denote the class of analytic functions on $\mathcal{U}$ and we consider the set of analytic functions

$$\mathcal{A}_n = \{f \in \mathcal{H}(\mathcal{U}) : f(z) = z + a_{n+1}z^{n+1} + \dots\}$$

For $n = 1$ results that $\mathcal{A}_1 = \mathcal{A}$.
We denote by $\mathcal{S}^*$ the class of starlike functions defined by

$$\mathcal{S}^* = \{f \in \mathcal{H}(\mathcal{U}) : \mathrm{Re}\frac{zf'(z)}{f(z)} > 0, z \in \mathcal{U}\}.$$

A function $f \in \mathcal{A}$ is starlike by order $\alpha, 0 \le \alpha < 1$ iff $\mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha$.

Lemma 1 *[1] Let q the univalent function in $\mathcal{U}$ and let θ and ϕ be analytic functions in the domain $D \subset q(\mathcal{U})$ with $\phi(w) \neq 0$, when $w \in q(\mathcal{U})$.Set*

$$Q(z) = nzq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q(z)$$

and suppose that

(i) Q is starlike

(ii) $\mathrm{Re}\frac{zh'(z)}{Q(z)} = \mathrm{Re}\left[\frac{\theta'[q(z)]}{\phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right] > 0.$

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If p is analytic in $\mathcal{U}$, with

$$p(0) = q(0), p'(0) = \cdots = p^{(n-1)}(0) = 0, p(\mathcal{U}) \subset D$$

and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)]$$

then $p \prec q$, and q is the best dominant.

We introduce a new integral operator defined by:

$$I(z) = \frac{2}{z}\int_0^z f(t)e^{g(t)}dt. \tag{2}$$

2 Main results

Theorem 2 *Let*

$$h(z) = \frac{z}{1+z^2} + \frac{nz(1-z^2)}{(1+z^2)(z^2+z+1)}$$

If $f, g \in \mathcal{A}_n$ and

$$\frac{zf'(z)}{f(z)} + zg'(z) \prec h(z) \tag{3}$$

then the operator $I(z)$ defined by (2) is in $\mathcal{S}^*$.

Proof. Using (2) we obtain

$$I(z) + zI'(z) = 2f(z)e^{g(z)} \tag{4}$$

We consider

$$p(z) = \frac{zI'(z)}{I(z)}$$

and from (4) we have that

$$\frac{zp'(z)}{p(z)+1} + p(z) = z\left(\frac{f'(z)}{f(z)} + g'(z)\right).$$

Then using the relation (3) results that

$$\frac{zp'(z)}{p(z)+1} + p(z) \prec h(z).$$

To prove that $I(z) \in \mathcal{S}^*$ we use Lemma 1. Let

$$q(z) = \frac{z}{1+z^2};\;\; \theta(w) = w;\;\; \phi(w) = \frac{1}{w+1}$$

$$\theta[q(z)] = q(z);\;\; \phi[q(z)] = \frac{1}{q(z)+1} = \frac{1+z^2}{z^2+z+1}$$

$$Q(z) = nzq'(z)\phi[q(z)] = \frac{nz(1-z^2)}{(1+z^2)(z^2+z+1)}$$

$$h(z) = \theta[q(z)] + Q(z) = \frac{z}{1+z^2} + \frac{nz(1-z^2)}{(1+z^2)(z^2+z+1)}$$

Because Q is starlike and $\mathrm{Re}\phi[q(z)] > 0$, using Lemma 1 we obtain that $p \prec q$ that implies $\mathrm{Re}\frac{zI'(z)}{I(z)} > 0 \Leftrightarrow I(z) \in \mathcal{S}^*$. ■

For $n = 1$ in Theorem 2 we obtain:

Corollary 3 *Let*

$$h(z) = \frac{z}{1+z^2}\left(1 + \frac{1-z^2}{z^2+z+1}\right)$$

If $f, g \in \mathcal{A}$ *and*

$$\frac{zf'(z)}{f(z)} + zg'(z) \prec h(z)$$

then the operator $I(z)$ *defined by (2) is in* $\mathcal{S}^*$.

Theorem 4 *Let* $0 < \alpha < \frac{1}{2}$ *and*

$$h(z) = \frac{\alpha z}{1+\alpha z} + \frac{nz\alpha}{(1+\alpha z)(2\alpha z+1)}.$$

If $f \in \mathcal{A}_n$ *and* $\frac{zf'(z)}{f(z)} + zg'(z) \prec h(z)$ *then*

$$\mathrm{Re}\frac{zI'(z)}{I(z)} > \frac{\alpha}{1+\alpha}$$

where $I(z)$ *is the integral operator defined by (2).*

Proof. Using (2) we obtain

$$I(z) + zI'(z) = 2f(z)e^{g(z)} \tag{5}$$

We consider

$$p(z) = \frac{zI'(z)}{I(z)}$$

and from (5) we have that

$$\frac{zp'(z)}{p(z)+1} + p(z) = z\left(\frac{f'(z)}{f(z)} + g'(z)\right).$$

Then from the subordination relation from the hypothesis results that

$$\frac{zp'(z)}{p(z)+1} + p(z) \prec h(z).$$

To prove that $I(z) \in \mathcal{S}^*$ we use Lemma 1. Let

$$q(z) = \frac{\alpha z}{1+\alpha z};\ \ \theta(w) = w;\ \ \phi(w) = \frac{1}{w+1}; \theta[q(z)] = q(z);\ \ \phi[q(z)] = \frac{1}{q(z)+1} = \frac{1+\alpha z}{2\alpha z+1}$$

$$Q(z) = nzq'(z)\phi[q(z)] = \frac{nz\alpha}{(1+\alpha z)(2\alpha z+1)}; h(z) = \theta[q(z)] + Q(z) = \frac{\alpha z}{1+\alpha z} + \frac{nz\alpha}{(1+\alpha z)(2\alpha z+1)}$$

Because Q is starlike and $\mathrm{Re}\phi[q(z)] > 0$, from Lemma 1 we obtain that

$$p \prec q \Leftrightarrow \frac{zI'(z)}{I(z)} \prec \frac{\alpha}{1+\alpha}.$$

This implies that $\mathrm{Re}\frac{zI'(z)}{I(z)} > \frac{\alpha}{1+\alpha} \Leftrightarrow I(z) \in \mathcal{S}^*\left(\frac{\alpha}{1+\alpha}\right)$. ■

Theorem 5 *Let* $\alpha \cdot \beta \neq -1$*, for* $0 < \alpha < \frac{1}{2}$ *and*

$$h(z) = \frac{1+\alpha z}{1+\alpha\beta z} + \frac{nz\alpha(1-\beta)}{(1+\alpha\beta z)(2+\alpha z(\beta+1))}$$

If $f \in \mathcal{A}_n$ *and* $\frac{zf'(z)}{f(z)} + zg'(z) \prec h(z)$

$$\mathrm{Re}\frac{zI'(z)}{I(z)} > \frac{1+\alpha}{1+\alpha\beta}$$

where $I(z)$ *is the integral operator defined by (2).*

Proof. Using (2) we obtain

$$I(z) + zI'(z) = 2f(z)e^{g(z)} \tag{6}$$

We consider

$$p(z) = \frac{zI'(z)}{I(z)}$$

and from (6) we have that

$$\frac{zp'(z)}{p(z)+1} + p(z) = z\left(\frac{f'(z)}{f(z)} + g'(z)\right).$$

Then we obtain that

$$\frac{zp'(z)}{p(z)+1} + p(z) \prec h(z).$$

To prove that $I(z) \in \mathcal{S}^*$ we use Lemma 1. Let

$$q(z) = \frac{1+\alpha z}{1+\alpha\beta z};\ \ \theta(w) = w;\ \ \phi(w) = \frac{1}{w+1}; \theta[q(z)] = q(z);\ \ \phi[q(z)] = \frac{1}{q(z)+1} = \frac{1+\alpha\beta z}{2+\alpha z(\beta+1)}$$

$$Q(z) = nzq'(z)\phi[q(z)] = \frac{nz\alpha(1-\beta)}{(1+\alpha\beta z)(2+\alpha z(\beta+1))}$$

$$h(z) = \theta[q(z)] + Q(z) = \frac{1+\alpha z}{1+\alpha\beta z} + \frac{nz\alpha(1-\beta)}{(1+\alpha\beta z)(2+\alpha z(\beta+1))}$$

Because Q is starlike and $\mathrm{Re}\phi[q(z)] > 0$, from Lemma 1 results that

$$p \prec q \Leftrightarrow \frac{zI'(z)}{I(z)} \prec \frac{1+\alpha}{1+\alpha\beta}$$

Thus

$$\mathrm{Re}\frac{zI'(z)}{I(z)} > \frac{1+\alpha}{1+\alpha\beta} \Leftrightarrow I(z) \in \mathcal{S}^*\left(\frac{1+\alpha}{1+\alpha\beta}\right).$$

■

For $n = 1$ in Theorem 5 we obtain:

Corollary 6 *Let*

$$h(z) = \frac{1+\alpha z}{1+\beta\alpha z} + \frac{z\alpha(1-\beta)}{(1+\alpha\beta z)(2+z\alpha(\beta+1))}$$

If $f \in \mathcal{A}$ *and* $\frac{zf'(z)}{f(z)} + zg'(z) \prec h(z)$ *then*

$$\mathrm{Re}\frac{zI'(z)}{I(z)} > \frac{1+\alpha}{1+\alpha\beta}$$

where $I(z)$ *is the integral operator defined by (2).*

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