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European Option Pricing of Fractional Version of the Black-Scholes Model: Approach Via Expansion in Series

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Abstract: This paper presents the decomposition method for solution of the fractional Black-Scholes equation with boundary condition for a European option pricing problem. Undoubtedly this model is the most well known model for pricing financial derivatives. This method finds the analytical solution without any discretization or additive assumption. The numerical method has been applied in the form of convergent power series with easily computable components, to solve the fractional Black-Scholes equations.

Keywords: Fractional Black-Scholes equations; European option pricing problem; Analytical solution

1 Introduction

There is an immense amount of interest and literature on the pricing of financial derivatives. A financial derivative is an instrument whose price depends on, or is derived from, the value of another asset [14]. Often, this underlying asset is a stock. The concept of financial derivatives is not new. While there remains some historical debate as to the exact date of the creation of financial derivatives, it is well accepted that the first attempt at modern derivative pricing began with the work of Charles Castelly [6] published in 1877. Castell's book was a general introduction to concepts such as hedging and speculative trading, but it laked mathematical rigor. In 1969, Fisher Black and Myron Scholes got an idea that would change the world of finance forever. The central idea of their paper revolved around the discovery that one did not need to estimate the expected return of a stock in order to price an option written on that stock.
The Black-Scholes model (BS) for pricing stock options has been applied to many different commodities and payoff structures. The Black-Scholes model for value of an option is described by the equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 V}{\partial x^2}+r(t)x\frac{\partial V}{\partial x}-r(t)V=0,(x,t)\in R^+\times(0,T) \tag{1}$$

where $V(x,t)$ is the European option price at asset price x and at time t, T is the maturity, $r(t)$ is the risk free interest rate and $\sigma(x,t)$ represents the volatility function of underlying asset. It is well-known that problem (1) has a closed-form solution obtained for the price of a European call or European put option after several changes of variables and solving certain related diffusion equations [5, 14]. We denoted the payoff functions $c(x,t)$ and $p(x,t)$ for the European call and put options, respectively. Thus

$$c(x,t)=max(x-E,0) \quad , \quad p(x,t)=max(E-x,0),$$

where E is the exercise price. During the last decades, several numerical and analytical methods have been proposed in the literature to solve the Black-Scholes model by Ankudinova and Ehrhardt [2], Gulkac [13], Jodar and et al. [16], Cen and Le [7] and Company and et al. in [8].
The fractional calculus is used in many fields of science and engineering [3, 4, 20, 25]. In the area of financial markets, fractional order models have been recently used to described the probability distributions of log-price in the log-time limit, which is useful to characterise the natural variability in prices in the log term. Meerschaert and Scalas [19] introduced a time-space fractional diffusion equation to model the CTRW scaling limit process densities when the waiting times

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and the log-returns are uncoupled (independent) and a coupled fractional diffusion equation if the waiting times and the log-returns are coupled (dependent). Lately, the fractional differential equations have been solved with converting it into an NLP problem [12]. Jafari and et al. used a modified variational iteration method for solving fractional Riccati differential equation [15]. Rostami and Karimi introduced a novel numerical method for solving fractional heat and wave-like equations in [23]. The solution of differential equation containing fractional derivatives is much involved and it's classic analytic methods are mainly integral transforms, such as laplace transform, fourier transform, Mellin transform, etc.

In recent years, many authors have paid attention to studying the solutions of linear and nonlinear fractional differential equations by the Adomian decomposition method [9–11, 21]. This method efficiently works for initial value or boundary value problems, for linear and nonlinear, ordinary or partial differential equations, and even for stochastic systems [1] and fractional equations [24]. In the present work, we apply the decomposition method for solving fractional Black-Scholes equation.

The organization of this paper is as follows: in section 2, some notation and basic definitions are presented that will be used in the later section. In section 3, the method is discussed. In section 4, we show convergence of the decomposition series. Applications have been presented in section 5 and section 6 is the conclusion.

2 Preliminaries

In this section, we set up notation and basic definitions and main properties of fractional calculus theory which shall be used in this paper

Definition 1 *The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [18]:*

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}.$$

Definition 2 *A real function $y(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(> \mu)$, such that $y(t) = t^p y_1(t)$, where $y_1(t) \in C[0, \infty]$, and it is said to be in the space C_μ^m iff $y^{(m)} \in C_\mu, m \in N$.*

The Riemann-Liouville fractional integral and Caputo derivative are defined as follows.

Definition 3 *The Riemann-Liouville fractional derivative of y is defined as:*

$$_{RL}D^\alpha y(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} y(\tau) d\tau,$$

for $m - 1 < \alpha \le m,\ m \in N,\ t > 0,\ y \in C_{-1}^m$.

Definition 4 *The fractional derivative of $y(t)$ in the Caputo sense is defined as:*

$$_{C}D^\alpha y(t) = J^{m-\alpha} D^m y(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} y^{(m)}(\tau) d\tau,$$

for $m - 1 < \alpha \le m,\ m \in N,\ t > 0,\ y \in C_{-1}^m$.

Definition 5 *The Riemann-Liouville fractional integral operator of order $\alpha \ge 0$, of a function $y \in C_\mu,\ \mu \ge -1$, is defined as:*

$$J^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau,\ \alpha > 0,\ t > 0,\ J^0 y(t) = y(t).$$

Note that the relation between Riemann-Liouville operator and Caputo fractional differential operator is given as follows

$$J^\alpha D_t^\alpha f(t) = D_t^{-\alpha} D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^k(0), \quad m - 1 < \alpha \le m. \tag{2}$$

Some of the most important properties of operator J^α for $y \in C_\mu,\ \mu \ge -1,\ \alpha,\ \beta \ge 0$ and $\gamma > -1$, are as follows [22]:

1. $J^\alpha J^\beta y(t) = J^{(\alpha+\beta)} y(t)$,
2. $J^\alpha J^\beta y(t) = J^\beta J^\alpha y(t)$,
3. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

3 The analysis of decomposition method

In this section, we illustrate the idea of the Adomian decomposition method. Let us consider the nonlinear fractional differential equation:

$$D_t^\alpha V(x,t) + \Re[x]V(x,t) + \mathcal{N}[x]V(x,t) = 0 \quad t > 0,\ x \in R,\ 0 < \alpha \leq 1, \tag{3}$$

$$V(x,0) = g(x),$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$, $\Re[x]$ is the linear operator, $\mathcal{N}[x]$ is the general nonlinear operator. We rewrite the fractional PDE in the form

$$D_t^\alpha V(x,t) = -\mathcal{L}[x] * V(x,t), \tag{4}$$

with $\mathcal{L}[x] = \Re[x] + \mathcal{N}[x]$.
The technique is based on the relation between Riemann-Liouville operator and Caputo fractional differential operator. By applying the operator $D_t^{-\alpha}$ to equation (3), and on taking into account of (2) and (4), we obtain

$$V(x,t) = g(x) + D_t^{-\alpha}(-\mathcal{L}[x] * V(x,t)). \tag{5}$$

We look for a solution expandable in the form

$$V(x,t) = V_0(x,t) + \sum_{k=1}^{\infty} V_k(x,t), \tag{6}$$

with $V_0(x,t) = g(x)$. By substituting (6) into (4), we are lead to the identification

$$V_{n+1}(x,t) = D_t^{-\alpha}(-\mathcal{L}[x] * V_n(x,t)). \tag{7}$$

4 Convergence

In this section, we show convergence of the decomposition series. We consider the following fractional Black-Scholes equation for option pricing:

$$V_t^\alpha + ax^2 V_{xx} + bxV_x - rV = 0, \quad 0 < \alpha \leq 1,$$

where $a = \frac{\sigma^2}{2}$, $b = r$ and with the initial condition $V(x,0) = g(x)$. We assume $g(x)$ is bounded for all x.
In view of (5) and by applying (6) and (7), we obtain

$$V_0(x,t) = g(x),$$

$$V_{n+1}(x,t) = ((-\mathcal{L}[x])^n * g(x)) D_t^{-\alpha}(\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}).$$

Thus, we have

$$V(x,t) = \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(1+k\alpha)} (-1)^k (\mathcal{L}^k[x] * g(x)), \tag{8}$$

with $\mathcal{L}[x] = ax^2 D_{xx} + bxD_x - r$.
Using the definition (2.1), we have

$$V(x,t) = E_\alpha(-t^\alpha \mathcal{L}[x] * g(x)).$$

Because the Mittag-Leffler function is convergent for real and positive α, thus we proved the convergence of the power series for the fractional Black-scholes equation.

5 Numerical examples

In the section, we use our proposed algorithm and investigate its accuracy through the following numerical examples.

Example 1 *Consider the following fractional Black-Scholes option pricing equation [13, 17]*

$$\frac{\partial^\alpha V}{\partial t^\alpha} = \frac{\partial^2 V}{\partial x^2} + (k-1)\frac{\partial V}{\partial x} - kV, \tag{9}$$

where $0 < \alpha \leq 1$, *with initial condition* $V(x, 0) = max(e^x - 1, 0)$.
Note that this system of equations contains just two dimensionless parameters $k = \frac{2r}{\sigma^2}$, *where k represents the balance between the rate of interests and the variability of stock returns and the dimensionless time to expiry* $\frac{\sigma^2 T}{2}$, *even though there are four dimensional parameters,* E, T, σ^2, *and* r, *in the original statements of the problem.*
Now, we can use decomposition procedure to solve this equation. We have

$$\frac{\partial^\alpha V}{\partial t^\alpha} = \mathcal{L}[x] * V(x,t), \tag{10}$$

with $\mathcal{L}[x] = (D_{xx} + (k-1)D_x - k)$.
Applying the inverse operator of D_t^α *on both sides in (9) and using the relation (2), we obtain*

$$V(x,t) = max(e^x - 1, 0) + D_t^{-\alpha}(\mathcal{L}[x] * V(x,t)). \tag{11}$$

We look for a solution in the form

$$V(x,t) = V_0(x,t) + \sum_{k=1}^{\infty} V_k(x,t), \tag{12}$$

with

$$V_0(x,t) = max(e^x - 1, 0), \quad V_{n+1}(x,t) = D_t^{-\alpha}(\mathcal{L}[x] * V_n(x,t)), \tag{13}$$

so we have

$$\begin{aligned} V_1(x,t) &= D_t^{-\alpha}(\mathcal{L}[x] * max(e^x - 1, 0)) = \frac{t^\alpha}{\Gamma(1+\alpha)}[\mathcal{L}[x] * max(e^x - 1, 0)] \\ &= \frac{t^\alpha}{\Gamma(1+\alpha)}[k\ max(e^x, 0) - k\ max(e^x - 1, 0)], \end{aligned} \tag{14}$$

$$\begin{aligned} V_2(x,t) &= [\mathcal{L}^2[x] * max(e^x - 1, 0)]\ D_t^{-\alpha}(\frac{t^\alpha}{\Gamma(1+\alpha)}) \\ &= \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}[k^2\ max(e^x - 1, 0) - k^2\ max(e^x, 0)], \end{aligned} \tag{15}$$

$$\vdots$$

$$\begin{aligned} V_n(x,t) &= [\mathcal{L}^n[x] * max(e^x - 1, 0)]\ D_t^{-\alpha}(\frac{t^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}) \\ &= \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}[(-k)^n\ max(e^x - 1, 0) - (-k)^n\ max(e^x, 0)]. \end{aligned} \tag{16}$$

Thus the solution of this problem in a series form is given by

$$V(x,t) = \sum_{k=0}^{\infty} V_k(x,t) = max(e^x - 1, 0)E_\alpha(-kt^\alpha) + max(e^x, 0)(1 - E_\alpha(-kt^\alpha)), \tag{17}$$

where $E_\alpha(z)$ *is Mittag-Leffler function in one parameter. The analytical solution of this problem is consistent with the result obtained by Kumar and et al. [17]. For case* $\alpha = 1$, *we have*

$$V(x,t) = max(e^x - 1, 0)e^{-kt} + max(e^x, 0)(1 - e^{-kt}). \tag{18}$$

Example 2 *Consider the following generalized fractional Black-Scholes equation as follows [7]:*

$$\frac{\partial^\alpha V}{\partial t^\alpha} + 0.08(2 + sin(x))^2 x^2 \frac{\partial^2 V}{\partial x^2} + 0.06x \frac{\partial V}{\partial x} - 0.06V = 0, \tag{19}$$

with initial condition $V(x, 0) = max(x - 25e^{-0.06}, 0)$.
In this example, we have

$$\begin{aligned} &\frac{\partial^\alpha V}{\partial t^\alpha} = \mathcal{L}[x] * V(x,t), \\ &\mathcal{L}[x] = (-0.08(2 + sin(x))^2 x^2 D_{xx} - 0.06x D_x + 0.06). \end{aligned} \tag{20}$$

By applying $D_t^{-\alpha}$ *on both sides in (19), we obtain*

$$V(x,t) = max(x - 25e^{-0.06}, 0) + D_t^{-\alpha}(\mathcal{L}[x] * V(x,t)). \tag{21}$$

The decomposition method leads to

$$V_0(x,t) = max(x - 25e^{-0.06}, 0), \quad V_{n+1}(x,t) = D_t^{-\alpha}(\mathcal{L}[x] * V_n(x,t)), \tag{22}$$

thus, we will have

$$\begin{aligned} V_1(x,t) &= D_t^{-\alpha}(\mathcal{L}[x] * max(x - 25e^{-0.06}, 0)) \\ &= \frac{t^\alpha}{\Gamma(1+\alpha)}[\mathcal{L}[x] * max(x - 25e^{-0.06}, 0)] \\ &= \frac{t^\alpha}{\Gamma(1+\alpha)}[-0.06x + 0.06\, max(x - 25e^{-0.06}, 0)], \end{aligned} \tag{23}$$

$$\begin{aligned} V_2(x,t) &= D_t^{-\alpha}(\mathcal{L}[x] * V_1(x,t)) \\ &= [\mathcal{L}^2[x] * max(x - 25e^{-0.06}, 0)]\ D_t^{-\alpha}(\frac{t^\alpha}{\Gamma(1+\alpha)}) \\ &= \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}[-(0.06)^2 x + (0.06)^2\ max(x - 25e^{-0.06}, 0)], \end{aligned} \tag{24}$$

$$\vdots$$

$$\begin{aligned} V_n(x,t) &= D_t^{-\alpha}(\mathcal{L}[x] * V_{n-1}(x,t)) \\ &= [\mathcal{L}^n[x] * max(x - 25e^{-0.06}, 0)]\ D_t^{-\alpha}(\frac{t^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}) \\ &= \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}[-(0.06)^n x + (0.06)^n\ max(x - 25e^{-0.06}, 0)], \end{aligned} \tag{25}$$

so that the solution $V(x,t)$ *of the problem is given by:*

$$V(x,t) = \sum_{k=0}^{\infty} V_k(x,t) = max(x - 25e^{-0.06}, 0)E_\alpha(0.06t^\alpha) + x(1 - E_\alpha(0.06t^\alpha)), \tag{26}$$

which is the exact solution of the given fractional Black-Scholes equation, for pricing the European option.
The exact solution of the given option pricing equation for $\alpha = 1$ *is*

$$V(x,t) = max(x - 25e^{-0.06}, 0)e^{0.06t} + x(1 - e^{0.06t}). \tag{27}$$

6 Conclusion

In this paper, the application of decomposition method was extended to express the analytical solutions of the fractional Black-Scholes equation. This scheme was clearly very efficient and powerful technique in finding the solutions of the proposed equations. The main advantage of this method is to overcome the deficiency that is caused by unsatisfied conditions.

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