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Existence of Multiple Periodic Solutions for Plankton Allelopathy Model with Time-delays

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Abstract: The paper studies a modified Lotka-Volterra competition system with time-delays, which describes the growth of two species of plankton with competitive and allelopathic effects on each other. With the help of Mawhin's continuation theorem of coincidence degree theory, we obtain a set of sufficient conditions for the existence of multiple positive periodic solutions. Finally, we carry out a numerical example to confirm the validity of the results.

Keywords: Coincidence degree; Multiple periodic solutions; Distributed delays; Competition system; Plankton allelopathy.

1 Introduction

The dynamic relationship between predators and their preys has attracted much attention from many fields such as aquatic ecology and mathematics due to its biological and theoretical significance[1–3]. In order to describe the behaviors of predators and their preys, the famous Lotka-Volterra competition system was proposed and studied extensively[4–6]. Recently, the Lotka-Volterra competition system was modified to describe the growth of two species of plankton with competitive and allelopathic effects on each other[7], i.e., the increased population of one species of phytoplankton might affect the growth of other species by the production of allelopathic toxins or stimulator, influencing bloom, pulses, and seasonal succession[8, 9]. In the past decades, many researches have been carried out on the model of plankton allelopathy[10–12]. Under some conditions obtained by Leray-Schauder degree theory, the model has been demonstrated that the cross-diffusion can lead to an inhomogeneous stationary pattern[11]. Another researches indicated that toxic substances are harmless for the stability of the interior equilibrium point of the plankton allelopathy model with time-delays[12]. In conclusion, the dynamical analysis of the plankton allelopathy model has become a hot topic of many fields.

Many other researches focused on the existence and global attractivity of periodic solutions of ordinary differential equations [13, 14]. For instance, the reference [15] investigated the permanence of the plankton allelopathy model, and obtained sufficient conditions for the existence of at least one positive periodic solution. With the help of Mawhin's continuation theorem of coincidence degree theory, a competition system of plankton allelopathy on time scales was studied and a set of criteria were obtained for the existence of two periodic solutions[16]. Similarly, it has also been shown that a predator-prey system with Monod-Haldane non-monotonic functional response and harvesting terms has at least four positive periodic solutions under some conditions[17]. For more detailed discussion on the existence and global attractivity of periodic solutions, we refer to [18–20].

In this paper, we consider the following modified delayed differential equation model, which describes the growth of two species of plankton with competitive and allelopathic effects on each other,

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1(t) - \sum\limits_{j=1}^{2} a_{1j}(t)\int_{-T_{1j}}^{0} K_{1j}(s)x_j(t+s)ds - b_1(t)x_1(t)\int_{-\tau_2}^{0} f_2(s)x_2(t+s)ds], \\ \dot{x}_2(t) = x_2(t)[r_2(t) - \sum\limits_{j=1}^{2} a_{2j}(t)\int_{-T_{2j}}^{0} K_{2j}(s)x_j(t+s)ds - b_2(t)x_2(t)\int_{-\tau_1}^{0} f_1(s)x_1(t+s)ds], \end{cases} \tag{1.1}$$

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where $r_i(t), a_{ij}(t) > 0, b_i(t) > 0$ are continuous $\omega-$periodic functions, T_{ij}, τ_i are positive constants representing time-delays, $K_{ij} \in C([-T_{ij}, 0], (0, \infty))$ satisfying $\int_{-T_{ij}}^{0} K_{ij}(s)ds = 1$, and $f_i \in C([-\tau_i, 0], \ (0, \infty))$ satisfying $\int_{-\tau_i}^{0} f_i(s)ds = 1, \ i, j = 1, 2.$

The model has been shown to have at least one positive periodic solution[7]. But to the best of our knowledge, few researches on the existence of multiple periodic solutions of the system (1.1) have been carried out. Motivated by the studies on multiple periodic solutions of dynamical systems[16–20], this paper carries out the specific and detailed investigations on the system (1.1) by applying Mawhin's continuation theorem of coincidence degree theory.

2 Existence of multiple positive periodic solutions

2.1 Notations

If $h(t)$ is a $\omega-$periodic function, we denote $\bar{h} = \frac{1}{\omega}\int_0^{\omega} h(t)dt, \ |h|_0 = \max\limits_{t\in[0,\omega]} |h(t)|$ for convenience. Obviously, $\bar{h}$ is the average of $h(t)$ over $[0, \omega]$. Similar to the reference [21], we obtain following notations.

$$\alpha_{ij} = \bar{a}_{ji}\bar{b}_i - \bar{a}_{ii}\bar{b}_j, \ \ \alpha_{ij}^{'} = \bar{a}_{ji}\bar{b}_i - \bar{a}_{ii}\bar{b}_j e^{(\bar{R}_j+\bar{r}_j)\omega}, \ \ \alpha_{ij}^{''} = (\bar{a}_{ji}\bar{b}_i e^{(\bar{R}_j+\bar{r}_j)\omega} - \bar{a}_{ii}\bar{b}_j)e^{(\bar{R}_i+\bar{r}_i)\omega},$$

$$\beta_{ij} = \bar{a}_{ii}\bar{a}_{jj} + \bar{b}_i\bar{r}_j - \bar{a}_{ij}\bar{a}_{ji} - \bar{b}_j\bar{r}_i, \beta_{ij}^{'} = \bar{a}_{ii}\bar{a}_{jj}e^{(\bar{R}_j+\bar{r}_j)\omega} + \bar{b}_i\bar{r}_j - \bar{a}_{ij}\bar{a}_{ji}e^{(\bar{R}_i+\bar{r}_i)\omega} - \bar{b}_j\bar{r}_i e^{(\bar{R}_i+\bar{r}_i+\bar{R}_j+\bar{r}_j)\omega},$$

$$\beta_{ij}^{''} = \bar{a}_{ii}\bar{a}_{jj}e^{(\bar{R}_i+\bar{r}_i)\omega} + \bar{b}_i\bar{r}_j e^{(\bar{R}_i+\bar{r}_i+\bar{R}_j+\bar{r}_j)\omega} - \bar{a}_{ij}\bar{a}_{ji}e^{(\bar{R}_j+\bar{r}_j)\omega} - \bar{b}_j\bar{r}_i, \ \gamma_{ij} = \bar{r}_i\bar{a}_{jj} - \bar{r}_j\bar{a}_{ij}$$

$$\gamma_{ij}^{'} = (\bar{r}_i\bar{a}_{jj}e^{(\bar{R}_j+\bar{r}_j)\omega} - \bar{r}_j\bar{a}_{ij})e^{(\bar{R}_i+\bar{r}_i)\omega}, \ \gamma_{ij}^{''} = \bar{r}_i\bar{a}_{jj} - \bar{r}_j\bar{a}_{ij}e^{(\bar{R}_j+\bar{r}_j)\omega},$$

where $i \neq j, \ i, j = 1, 2$. With the help of these notations, the following assumptions are proposed.

(H_1) [7] $\bar{R}_i = (1/\omega)\int_0^{\omega} |r_i(t)|dt \geq (1/\omega)\int_0^{\omega} r_i(t)dt > 0.$

(H_2) [7] $\gamma_{ij}^{''} = \bar{r}_i\bar{a}_{jj} - \bar{r}_j\bar{a}_{ij}e^{(\bar{R}_j+\bar{r}_j)\omega} > 0, \ i \neq j, \ i, j = 1, 2.$

(H_3) $\alpha_{12}^{'} > 0.$

(H_4) $\dfrac{\beta_{12}}{\alpha_{12}} > \dfrac{\beta_{12}^{'}}{\alpha_{12}^{'}}.$

With the help of the hypotheses (H_1), (H_2) and the following functions of three variables,

$$X_1(\alpha, \beta, \gamma) = \frac{\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}, \ \ X_2(\alpha, \beta, \gamma) = \frac{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \ (\alpha \neq 0, \beta^2 - 4\alpha\gamma > 0),$$

the algebraic equations

$$\begin{cases} \bar{a}_{11}X_1 + \bar{a}_{12}X_2 + \bar{b}_1X_1X_2 = \bar{r}_1, \\ \bar{a}_{21}X_1 + \bar{a}_{22}X_2 + \bar{b}_2X_1X_2 = \bar{r}_2, \end{cases} \tag{2.1}$$

satisfy that[7],

(i) If $\alpha_{12} > 0$, then equations (2.1) have two positive solutions: $(X_i(\alpha_{12}, \beta_{12}, \gamma_{12}), X_1(\alpha_{21}, \beta_{21}, \gamma_{21})), \ \ i = 1, 2;$

(ii) If $\alpha_{21} > 0$, then equations (2.1) have two positive solutions: $(X_1(\alpha_{12}, \beta_{12}, \gamma_{12}), X_i(\alpha_{21}, \beta_{21}, \gamma_{21})), \ \ i = 1, 2.$

2.2 Lemmas

In the proof of our main result, the following lemmas are also essential.

Lemma 1 *([21]) Assume that $(H_1) - (H_3)$ hold, then the following conclusions hold.*

(i) $\beta_{12} > 0, \ \beta_{12}^2 - 4\alpha_{12}\gamma_{12} > 0;$

(ii) $\beta_{12}^{'} > 0, \ \beta_{12}^{'2} - 4\alpha_{12}^{'}\gamma_{12}^{'} > 0.$

Lemma 2 *[16] Assume that $(H_1)-(H_4)$ hold, then the following conclusions hold.*

$$\begin{aligned} & X_1(\alpha_{12},\beta_{12}+m,\gamma_{12}-n) < X_1(\alpha_{12},\beta_{12},\gamma_{12}) < X_1(\alpha_{12}^{'},\beta_{12}^{'},\gamma_{12}^{'}) \\ < \ & X_2(\alpha_{12}^{'},\beta_{12}^{'},\gamma_{12}^{'}) < X_2(\alpha_{12},\beta_{12},\gamma_{12}) < X_2(\alpha_{12},\beta_{12}+m,\gamma_{12}-n), \end{aligned}$$

where $m=\bar{a}_{11}\bar{a}_{22}(e^{(\bar{R}_1+\bar{r}_1)\omega}-1)+\bar{b}_1\bar{r}_2(e^{(\bar{R}_1+\bar{r}_1+\bar{R}_2+\bar{r}_2)\omega}-1)>0, n=\bar{a}_{12}\bar{r}_2(e^{(\bar{R}_2+\bar{r}_2)\omega}-1)>0.$

For further convenience, we recall the following concepts and the famous Mawhin's continuation theorem from the book by Gaines and Mawhin[22].

Let Y, Z be real Banach spaces, $L: domL\subset Y\to Z$ be a Fredholm mapping of index zero (index$L$ = dim Ker$L$ - codim ImL), and let $P: Y\to Y$ and $Q: Z\to Z$ be continuous projectors such that ImP=Ker$L$, ImL=Ker$Q$=Im$(I-Q)$ and Y = Ker $L\bigoplus$ Ker P, Z = Im $L\bigoplus$ Im Q. If we define L_P : dom$L\cap$Ker$P$ $\to$ImL as the restriction $L|_{domL\cap KerP}$ of L to dom$L\cap$Ker P, then L_P is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of Y, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N:\bar{\Omega}\to Y$ is compact, i.e. continuous and such that $K_P(I-Q)N(\bar{\Omega})$ is relatively compact. Since ImQ is isomorphic to Ker$L$, J : Im$Q\to$ Ker$L$ is an isomorphism. Then one can obtain the famous Mawhin's continuation lemma.

Lemma 3 *(Continuation Theorem) [22] Let L be a Fredholm mapping of index zero and let $N:\bar{\Omega}\to Z$ be L-compact on $\bar{\Omega}$. Suppose*

(i) $Ly\neq\lambda Ny$ for every $y\in domL\cap\partial\Omega$ and every $\lambda\in(0,1)$;

(ii) $QNy\neq 0$ for every $y\in\partial\Omega\cap$ KerL, and Brouwer degree

$$deg_B(JQN,\Omega\cap KerL,0)\neq 0.$$

Then $Ly=Ny$ has at least one solution in $domL\cap\bar{\Omega}$.

2.3 Main Results

Now, we are ready to state and prove the following main result of this paper.

Theorem 4 *In addition to $(H_1)-(H_3)$, assume further that system (1.1) satisfies*
(H_5) $X_1(\alpha_{12},\beta_{12},\gamma_{12})<X_1(\alpha_{12}^{'},\beta_{12}^{'},\gamma_{12}^{'})<X_2(\alpha_{12},\beta_{12},\gamma_{12})$.
Then system (1.1) has at least two positive $\omega-$periodic solutions.

Proof. Since we are concerning with positive solutions of (1.1), we make the change of variables,

$$x_i(t)=\exp(y_i(t)),\quad i=1,2.$$

Then (1.1) is rewritten as

$$\dot{y}_i(t)=r_i(t)-\sum_{j=1}^{2}a_{ij}(t)\int_{-T_{ij}}^{0}K_{ij}(s)\exp(y_j(t+s))ds-b_i(t)\exp(y_i(t))\int_{-\tau_k}^{0}f_k(s)\exp(y_k(t+s))ds, \tag{2.5}$$

where $i,k=1,2,\ k\neq i$. Take

$$Y=Z=\{y(t)=(y_1(t),y_2(t))\in C(R,R^2): y(t+\omega)=y(t)\}.$$

$$\|y(t)\|=(\sum_{i=1}^{2}|y_i(t)|_0^2)^{\frac{1}{2}},\ y\in Y,\ \ (or\ z\in Z).$$

It is easy to verify that Y and Z are both Banach spaces. Define the following mappings $L: Y\to Z,\ N: Y\to Z,\ P: Y\to Y$ and $Q: Z\to Z$ as follows:

$$Nx=\begin{pmatrix} r_1(t)-\sum_{j=1}^{2}a_{1j}(t)\int_{-T_{1j}}^{0}K_{1j}(s)\exp(y_j(t+s))ds-b_1(t)\exp(y_1(t))\int_{-\tau_2}^{0}f_2(s)\exp(y_2(t+s))ds \\ r_2(t)-\sum_{j=1}^{2}a_{2j}(t)\int_{-T_{2j}}^{0}K_{2j}(s)\exp(y_j(t+s))ds-b_2(t)\exp(y_2(t))\int_{-\tau_1}^{0}f_1(s)\exp(y_1(t+s))ds \end{pmatrix},$$

$$Ly = \dot{y}, \quad Py = \frac{1}{\omega}\int_0^\omega y(t)dt = Qy, \ y \in Y \quad (or \quad y \in Z).$$

It is easy to see that Ker$L$=R^2, ImL=$\{y \in Y : \int_0^\omega y(t)dt = 0\}$ is closed in Z, and dimKer$L$=codimImL=2. Therefore, L is a Fredholm mapping of index zero. Clearly, P and Q are continuous projectors such that

$$ImP = KerL, \ KerQ = ImL = Im(I - Q).$$

On the other hand, K_p: ImL →dom $L \cap KerP$, the inverse to L, exists and is given by

$$K_p(y) = \int_0^t y(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^\eta y(s)dsd\eta.$$

Obviously, QN and $K_p(I - Q)N$ are continuous. By the Arzela-Ascoli theorem, it is not difficult to show that $\overline{K_p(I - Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Hence, N is L-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset X$.

Corresponding to the operator equation $Ly = \lambda Ny, \ \lambda \in (0, 1)$, one has

$$\dot{y}_i(t) = \lambda[r_i(t) - \sum_{j=1}^{2} a_{ij}(t)\int_{-T_{ij}}^0 K_{ij}(s)\exp(y_j(t+s))ds - b_i(t)\exp(y_i(t))\int_{-\tau_k}^0 f_k(s)\exp(y_k(t+s))ds], \quad (2.6)$$

where $i, k = 1, 2, \ k \neq i$. Suppose $y(t) = (y_1(t), y_2(t)) \in Y$ is a solution of system (2.6) for some $\lambda \in (0, 1)$. Integrating (2.6) over the interval $[0, \omega]$, we have

$$\begin{aligned} \bar{r}_i\omega &= \sum_{j=1}^{2}\int_0^\omega a_{ij}(t)\int_{-T_{ij}}^0 K_{ij}(s)\exp(y_j(t+s))dsdt \\ &+ \int_0^\omega b_i(t)\exp(y_i(t))\int_{-\tau_k}^0 f_k(s)\exp(y_k(t+s))dsdt, \end{aligned} \quad (2.7)$$

where $i, k = 1, 2, \ k \neq i$. It follows from (2.6) and (2.7) that

$$\begin{aligned} \int_0^\omega |\dot{y}_i(t)|dt &= \lambda\int_0^\omega |r_i(t) - \sum_{j=1}^{2} a_{ij}(t)\int_{-T_{ij}}^0 K_{ij}(s)\exp(y_j(t+s))ds \\ &\quad -b_i(t)\exp(y_i(t))\int_{-\tau_k}^0 f_k(s)\exp(y_k(t+s))ds|dt \\ &< \int_0^\omega |r_i(t)|dt + \sum_{j=1}^{2}\int_0^\omega a_{ij}(t)\int_{-T_{ij}}^0 K_{ij}(s)\exp(y_j(t+s))dsdt \\ &\quad +\int_0^\omega b_i(t)\exp(y_i(t))\int_{-\tau_k}^0 f_k(s)\exp(y_k(t+s))dsdt = (\bar{R}_i + \bar{r}_i)\omega, \end{aligned}$$

which implies that

$$\int_0^\omega |\dot{y}_i(t)|dt < (\bar{R}_i + \bar{r}_i)\omega, \quad (2.8)$$

where $i = 1, 2$. Since $y(t) \in X$, there exist two constants ξ_i, η_i, such that

$$y_i(\xi_i) = \min_{t\in[0,\omega]} y_i(t), \ y_i(\eta_i) = \max_{t\in[0,\omega]} y_i(t), \quad (2.9)$$

where $i = 1, 2$. From (2.7), (2.9), one obtains

$$\bar{a}_{11}\exp(y_1(\eta_1)) + \bar{a}_{12}\exp(y_2(\eta_2)) + \bar{b}_1\exp(y_1(\eta_1))\exp(y_2(\eta_2)) \geq \bar{r}_1, \quad (2.10)$$

$$\bar{a}_{21}\exp(y_1(\xi_1)) + \bar{a}_{22}\exp(y_2(\xi_2)) + \bar{b}_2\exp(y_1(\xi_1))\exp(y_2(\xi_2)) \leq \bar{r}_2. \quad (2.11)$$

We can derive from (2.11) that

$$y_2(\eta_2) \leq y_2(\xi_2) + \int_0^\omega |\dot{y}_2(t)|dt < \ln\frac{\bar{r}_2 - \bar{a}_{21}\exp(y_1(\xi_1))}{\bar{a}_{22} + \bar{b}_2\exp(y_1(\xi_1))} + (\bar{R}_2 + \bar{r}_2)\omega,$$

which, together with (2.10), leads to

$$\begin{aligned} \exp(y_1(\eta_1)) &\geq \frac{\bar{r}_1 - \bar{a}_{12}\exp(y_2(\eta_2))}{\bar{a}_{11} + \bar{b}_1\exp(y_2(\eta_2))} \\ &\geq \frac{\bar{r}_1(\bar{a}_{22} + \bar{b}_2\exp(y_1(\xi_1))) - \bar{a}_{12}\exp((\bar{R}_2 + \bar{r}_2)\omega)(\bar{r}_2 - \bar{a}_{21}\exp(y_1(\xi_1)))}{\bar{a}_{11}(\bar{a}_{22} + \bar{b}_2\exp(y_1(\xi_1))) + \bar{b}_1\exp((\bar{R}_2 + \bar{r}_2)\omega)(\bar{r}_2 - \bar{a}_{21}\exp(y_1(\xi_1)))}. \end{aligned} \quad (2.12)$$

From (2.8), we have

$$y_1(\xi_1) \geq y_1(\eta_1) - \int_0^{\omega} |y_1^{'}(t)|dt > y_1(\eta_1) - (\bar{R}_1 + \bar{r}_1)\omega.$$

This is

$$\exp(y_1(\xi_1)) > \exp(y_1(\eta_1)) \cdot \exp(-(\bar{R}_1 + \bar{r}_1)\omega),$$

which, together with (2.12), leads to

$$\begin{array}{ll} & \exp((\bar{R}_1 + \bar{r}_1)\omega)\exp(y_1(\xi_1)) \\ > & \dfrac{\bar{r}_1(\bar{a}_{22} + \bar{b}_2\exp(y_1(\xi_1))) - \bar{a}_{12}\exp((\bar{R}_2 + \bar{r}_2)\omega)(\bar{r}_2 - \bar{a}_{21}\exp(y_1(\xi_1)))}{\bar{a}_{11}(\bar{a}_{22} + \bar{b}_2\exp(y_1(\xi_1))) + \bar{b}_1\exp((\bar{R}_2 + \bar{r}_2)\omega)(\bar{r}_2 - \bar{a}_{21}\exp(y_1(\xi_1)))}. \end{array}$$

Therefore, we have

$$\alpha_{12}^{''}\exp(2y_1(\xi_1)) - \beta_{12}^{''}\exp(y_1(\xi_1)) + \gamma_{12}^{''} < 0.$$

So from (2.3), one obtains

$$\alpha_{12}\exp(2y_1(\xi_1)) - (\beta_{12} + m)\exp(y_1(\xi_1)) + \gamma_{12} - n < 0,$$

where $m = \bar{a}_{11}\bar{a}_{22}(e^{(\bar{R}_1+\bar{r}_1)\omega} - 1) + \bar{b}_1\bar{r}_2(e^{(\bar{R}_1+\bar{r}_1+\bar{R}_2+\bar{r}_2)\omega} - 1) > 0$, $n = \bar{a}_{12}\bar{r}_2(e^{(\bar{R}_2+\bar{r}_2)\omega} - 1) > 0$. According to (i) of Lemma 3, we obtain

$$X_1(\alpha_{12}, \beta_{12} + m, \gamma_{12} - n) < \exp(y_1(\xi_1)) < X_2(\alpha_{12}, \beta_{12} + m, \gamma_{12} - n). \tag{2.13}$$

In a similar way as the above proof, one can conclude from

$$\bar{a}_{21}\exp(y_1(\eta_1)) + \bar{a}_{22}\exp(y_2(\eta_2)) + \bar{b}_2\exp(y_1(\eta_1))\exp(y_2(\eta_2)) \geq \bar{r}_2,$$

$$\bar{a}_{11}\exp(y_1(\xi_1)) + \bar{a}_{12}\exp(y_2(\xi_2)) + \bar{b}_1\exp(y_1(\xi_1))\exp(y_2(\xi_2)) \leq \bar{r}_1,$$

that

$$\alpha_{12}^{'}\exp(2x_1(\eta_1)) - \beta_{12}^{'}\exp(y_1(\eta_1)) + \gamma_{12}^{'} > 0.$$

According to (ii) of Lemma 3, one has

$$\exp(y_1(\eta_1)) > X_2(\alpha_{12}^{'}, \beta_{12}^{'}, \gamma_{12}^{'}), \quad or \quad \exp(y_1(\eta_1)) < X_1(\alpha_{12}^{'}, \beta_{12}^{'}, \gamma_{12}^{'}). \tag{2.14}$$

It follows from (2.8) and (2.13) that

$$\begin{array}{lll} y_1(\eta_1) & \leq & y_1(\xi_1) + \int_0^{\omega} |\dot{y}_1(t)|dt \\ & < & \ln y_2(\alpha_{12}, \beta_{12} + m, \gamma_{12} - n) + (\bar{R}_1 + \bar{r}_1)\omega := H. \end{array} \tag{2.15}$$

On the other hand, it follows from (2.7) and (2.9) that

$$\bar{a}_{ii}\omega\exp(y_i(\xi_i)) \leq \int_0^{\omega} a_{ii}(t)\int_{-T_{ii}}^0 K_{ii}(s)\exp(y_i(t+s))dsdt < \bar{r}_i\omega,$$

that is

$$y_i(\xi_i) < \ln\frac{\bar{r}_i}{\bar{a}_{ii}}, \quad i = 1, 2. \tag{2.16}$$

From (2.8) and (2.16), one obtains

$$y_i(t) \leq y_i(\xi_i) + \int_0^{\omega} |\dot{y}_i(t)|dt < \ln\frac{\bar{r}_i}{\bar{a}_{ii}} + (\bar{R}_i + \bar{r}_i)\omega, \quad i = 1, 2. \tag{2.17}$$

It follows from (2.7) and (2.9) that

$$\begin{array}{lll} \bar{r}_2\omega & = & \sum\limits_{j=1}^{2}\int_0^{\omega} a_{2j}(t)\int_{-T_{2j}}^0 K_{2j}(s)\exp(y_j(t+s))dsdt \\ & + & \int_0^{\omega} b_2(t)\exp(y_2(t))\int_{-\tau_1}^0 f_1(s)\exp(y_1(t+s))dsdt \\ & \leq & \sum\limits_{j=1}^{2}\bar{a}_{2j}\omega\exp(y_j(\eta_j)) + \bar{b}_2\omega\exp(y_1(\eta_1))\exp(y_2(\eta_2)), \end{array}$$

which implies that

$$\exp(y_2(\eta_2)) \geq \frac{\bar{r}_2 - \bar{a}_{21}\exp(y_1(\eta_1))}{\bar{a}_{22} + \bar{b}_2\exp(y_1(\eta_1))}. \tag{2.18}$$

From (2.17) and (2.18), we have

$$y_2(\eta_2)) \geq \ln\frac{\bar{a}_{11}\bar{r}_2 - \bar{a}_{21}\bar{r}_1\exp((\bar{R}_1+\bar{r}_1)\omega)}{\bar{a}_{11}\bar{a}_{22} + \bar{b}_2\bar{r}_1\exp((\bar{R}_1+\bar{r}_1)\omega)} := M,$$

which leads to

$$y_2(t) \geq x_2(\eta_2) - \int_0^{\omega}|\dot{x}_2(t)|dt > M - (\bar{R}_2+\bar{r}_2)\omega. \tag{2.19}$$

By (2.17) and (2.19), we obtain that

$$|y_2(t)| < \max\{|\ln\frac{\bar{r}_2}{\bar{a}_{22}} + (\bar{R}_2+\bar{r}_2)\omega)|, |M - (\bar{R}_2+\bar{r}_2)\omega)|\} := A. \tag{2.20}$$

Now, let's consider QNy with $y=(y_1,y_2)\in R^2$. Note that

$$QN(y_1,y_2) = \begin{pmatrix} \bar{r}_1 - \bar{a}_{11}\exp(y_1) - \bar{a}_{12}\exp(y_2) - \bar{b}_1\exp(y_1)\exp(y_2) \\ \\ \bar{r}_2 - \bar{a}_{21}\exp(y_1) - \bar{a}_{22}\exp(y_2) - \bar{b}_2\exp(y_1)\exp(y_2) \end{pmatrix}.$$

According to Lemma 2, we can show that $QNy=0$ has two distinct solutions

$$\hat{y}^i = (\ln X_i(\alpha_{12},\beta_{12},\gamma_{12}), \ln X_1(\alpha_{21},\beta_{21},\gamma_{21})),\ i=1,2.$$

Choose $C>0$ such that

$$C > |\ln X_1(\alpha_{21},\beta_{21},\gamma_{21})|. \tag{2.21}$$

Let

$$\Omega_1 = \left\{ y\in Y \;\middle|\; \begin{array}{l} y_1(t)\in(\ln X_1(\alpha_{12},\beta_{12}+m,\gamma_{12}-n), \ln X_1(\alpha'_{12},\beta'_{12},\gamma'_{12})), \\ |y_2(t)| < A+C. \end{array} \right\},$$

$$\Omega_2 = \left\{ x\in X \;\middle|\; \begin{array}{l} \min\limits_{t\in[0,\omega]} y_1(t)\in(\ln X_1(\alpha_{12},\beta_{12}+m,\gamma_{12}-n), \ln X_2(\alpha_{12},\beta_{12}+m,\gamma_{12}-n), \\ \max\limits_{t\in[0,\omega]} y_1(t)\in(\min\{\ln X_2(\alpha_{12},\beta_{12},\gamma_{12}), \ln X_2(\alpha'_{12},\beta'_{12},\gamma'_{12})\}-\delta, H), \\ |y_2(t)| < A+C. \end{array} \right\},$$

where δ is a constant such that

$$\min\{\ln X_2(\alpha_{12},\beta_{12},\gamma_{12}), \ln X_2(\alpha'_{12},\beta'_{12},\gamma'_{12})\} - \ln X_1(\alpha'_{12},\beta'_{12},\gamma'_{12}) > \delta > 0.$$

Then both Ω_1 and Ω_2 are bounded open subsets of Y. It follows from Lemma 2 and (2.21) that $\hat{y}^i\in\Omega_i, i=1,2$. With the help of (2.13), (2.14), (2.15), (2.20) and (H_5), it is easy to see that $\bar{\Omega}_1\bigcap\bar{\Omega}_2=\phi$ and Ω_i satisfies the requirement (i) in Lemma 1 for i=1,2. Moreover, $QNy\neq 0$ for $y\in\partial\Omega_i\bigcap KerL\ (i=1,2)$. A direct computation gives

$$deg_B(JQN, \Omega_i\cap KerL, 0)\neq 0.$$

Here J is taken as the identity mapping since $ImQ=KerL$. So far we have proved that Ω_i satisfies all the assumptions in Lemma 1. Hence (2.5) has at least two different ω-periodic solutions $\breve{y}^i(t)\in DomL\bigcap\bar{\Omega}_i,\ i=1,2$. Then $\breve{x}^i_j(t) = \exp(\breve{y}^i_j(t)),\ i,j=1,2$, are two different positive $\omega-$periodic solutions of (1.1). ■

Remark For fixed $a_{ij}, r_i(i,j=1,2)$ and b_2, it is easy to show that

$$\lim_{\bar{b}_1\to\infty}\beta_{12}=\infty,\quad \lim_{\bar{b}_1\to\infty}\beta'_{12}=\infty,$$

which, together with (2.2) and (2.3), leads to

$$\lim_{\bar{b}_1\to\infty}\frac{X_1(\alpha_{12},\beta_{12},\gamma_{12})}{X_1(\alpha'_{12},\beta'_{12},\gamma'_{12})} = \lim_{\bar{b}_1\to\infty}\left(\frac{\beta'_{12}+\sqrt{\beta'^2_{12}-4\alpha'_{12}\gamma'_{12}}}{\beta_{12}+\sqrt{\beta^2_{12}-4\alpha_{12}\gamma_{12}}}\cdot\frac{\gamma_{12}}{\gamma'_{12}}\right) = \frac{\gamma_{12}}{\gamma'_{12}} < 1,$$

$$\lim_{\bar{b}_1\to\infty}\frac{X_1(\alpha'_{12},\beta'_{12},\gamma'_{12})}{X_2(\alpha_{12},\beta_{12},\gamma_{12})} = \lim_{\bar{b}_1\to\infty}\left(\frac{2\gamma'_{12}}{\beta'_{12}+\sqrt{\beta'^2_{12}-4\alpha'_{12}\gamma'_{12}}}\cdot\frac{2\alpha_{12}}{\beta_{12}+\sqrt{\beta^2_{12}-4\alpha_{12}\gamma_{12}}}\right) = 0.$$

So Assumption (H_5) can be satisfied when $\bar{b}_1$ is large enough.

2.4 Corollaries

Similar to the proof of Theorem 4, we can prove the following three corollaries.

Corollary 1. *Under the assumptions* $(H_1)-(H_4)$*, the system (1.1) has at least two positive* $\omega-$*periodic solutions.*

Corollary 2. *In addition to* (H_1) *and* (H_2)*, assume further that the system (1.1) satisfies*
$(H_3)^{'}$ $\alpha_{21}^{'}>0$;
$(H_5)^{'}$ $X_1(\alpha_{21},\beta_{21},\gamma_{21})<X_1(\alpha_{21}^{'},\beta_{21}^{'},\gamma_{21}^{'})<X_2(\alpha_{21},\beta_{21},\gamma_{21})$,
the system (1.1) has at least two positive $\omega-$*periodic solutions.*

Corollary 3. *In addition to* $(H_1),(H_2)$ *and* $(H_3)^{'}$*, assume further that the system (1.1) satisfies*
$(H_4)^{'}$ $\dfrac{\beta_{21}}{\alpha_{21}}>\dfrac{\beta_{21}^{'}}{\alpha_{21}^{'}}$,
the system (1.1) has at least two positive $\omega-$*periodic solutions.*

3 Numerical Examples

As an application of Corollary 1, we consider the following system

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\Big[2+\cos(400\pi t)-(1+\cos(400\pi t)/2)\textstyle\int_{-T_{11}}^{0}K_{11}(s)x_1(t+s)ds\\ &- (0.1+\cos(400\pi t)/20)\textstyle\int_{-T_{12}}^{0}K_{12}(s)x_2(t+s)ds-(2+\cos(400\pi t))x_1(t)\int_{-\tau_2}^{0}f_2(s)x_2(t+s)ds\Big],\end{aligned}$$

$$\begin{aligned}\dot{x}_2(t) &= x_2(t)\Big[4.1+\cos(400\pi t)-(2+\cos(400\pi t))\textstyle\int_{-T_{11}}^{0}K_{11}(s)x_1(t+s)ds\\ &- (1+\cos(400\pi t)/2)\textstyle\int_{-T_{12}}^{0}K_{12}(s)x_2(t+s)ds-(2+\cos(400\pi t))x_2(t)\int_{-\tau_1}^{0}f_1(s)x_1(t+s)ds\Big],\end{aligned}$$

where

$$K_{ij}\in C([-T_{ij},0],(0,\infty)),\ \int_{-T_{ij}}^{0}K_{ij}(s)ds=1,\ f_i\in C([-\tau_i,0],(0,\infty)),\ \int_{-\tau_i}^{0}f_i(s)ds=1,$$

and T_{ij},τ_i are positive constants, $i,j=1,2,\ \ j\neq i.$

A direct computation gives that

$$\bar{r}_1=2=\bar{R}_1,\ \bar{r}_2=4.1=\bar{R}_2,\ \bar{a}_{11}=\bar{a}_{22}=1,\ \bar{a}_{12}=0.1,\ \bar{a}_{21}=2,$$

$$\bar{b}_1=\bar{b}_2=2,\omega=0.005,$$

and

$$\alpha_{12}^{'}>1.91629,\ \ \gamma_{12}^{''}>1.57284,\ \ \gamma_{21}^{''}>0.01919,\ \ \alpha_{12}^{'}\beta_{12}-\alpha_{12}\beta_{12}^{'}>0.00904.$$

According to Corollary 1, the system mentioned above has at least two positive 0.005-periodic solutions.

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