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Galerkin Method, Single and Double Exponential Transformation of Sinc-Galerkin Methods for Solving Singular Two-point Boundary Value Problems

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Abstract: In many applicable problems, boundary value problems with singular solutions arise. Most of the numerical methods are not appropriate for solving these problems. They often cannot pass the singular point successfully. Sinc-Galerkin method is one of the best methods for overcoming on the singular points difficulties. In this paper we incorporate sinc-Galerkin method with the double exponential transformation (DE transformation), single exponential transformation (SE transformation) and Legendre Galerkin methods for these problems. Some examples are given for highlighting the accurate and the power of the proposed methods. We show that Legendre Galerkin method is not suitable method.

Keywords: Sinc method; Singular; Double exponential transformation; single exponential transformation.

1 Introduction

The sinc method is a highly efficient numerical method that has been developed by Frank Stenger, the pioneer of this field, and his colleagues [1, 2], it is widely used in various fields of numerical analysis, solution of integral, ordinary differential and partial differential equations [3–13]. Sinc-Galerkin is one of the sinc methods that used in this paper for solving boundary value problems with singular solutions. Despite most of the numerical methods, sinc-Galerkin method comprehends problems that have been in singular solutions. Conventional form of these methods is SE transformation. It is shown that in this case, error bound of the approximate solution is $O\left(\exp\left(-c\sqrt{N}\right)\right)$ with $c > 0$, where N is number of terms in the Sinc approximation.

Takahasi and Mori [9] proposed the double exponential transformation for one dimensional numerical integration in 1974. The effectiveness of the DE transformation technique in numerical integration naturally suggests that the DE transformation technique could be useful in other numerical methods. In 1997, Sugihara [10] established the "meta-optimality" of the DE formula in a mathematically rigorous manner, and since then it has turned out that the DE transformation is also useful for other various kinds of numerical methods. Indeed, it has been demonstrated in [11–14] that, the use of the Sinc method incorporated with the DE transformation gives highly efficient numerical methods for functions approximation, indefinite numerical integration, and the solution of differential equations. It has been shown the error bound of the Sinc-Galerkin method based on the DE transformation to numerical solution of boundary value problems of second order is $O\left(\exp\left(-\frac{cN}{\log N}\right)\right)$, $c > 0$ [13, 14] which converges to zero much faster than the method based on the SE transformation as N becomes large.

In this article we apply the DE and SE transformation sinc-Galerkin method to solve boundary-value problems:

$$\begin{aligned} &\mathcal{L}(y) = p(x)y'' + q(x)y' + u(x)y = f(x, y),\\ &y(a) = y(b) = 0, \end{aligned} \tag{1}$$

where $p(x), q(x), u(x)$ and $f(x, y)$, are analytic functions and $y(x)$ has singular points on (a, b). It is shown that, both of these methods overcome on the singular points difficultly, moreover the DE transformation Sinc-Galerkin method is more accurate.

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2 Sinc bases, SE and DE transformations

Sinc function is demonstrated on $-\infty < x < \infty$ by

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{2}$$

This function is translated with evenly spaced nodes are given as

$$S(k,h)(x) = sinc(\frac{x-kh}{h}),\ k = 0, \pm 1, \pm 2, \dots, h > 0. \tag{3}$$

If $f(z)$ is analytic on a strip domain

$$|Imz| < d, \tag{4}$$

in the z-plane and $|f(z)| \to 0$ as $z \to \pm\infty$ then, the series

$$\mathcal{C}(f,h) = \sum_{k=-\infty}^{\infty} f(kh) sinc(\frac{z-kh}{h}), \tag{5}$$

converges, we call it whittaker cardinal expansion.
From [10] we can write

$$f(z) = \mathcal{C}(f,h) + E_{sinc}, \qquad E_{sinc}(h) = O\left(\exp\left(-\frac{\pi d}{h}\right)\right), \tag{6}$$

where d is half width of strip domain (4).

If $f(x)$ be a real function, sinc expansion (5) is defined on $-\infty < x < \infty$, while the equation that we want to solve is defined $a < x < b$, and hence we need some transformation which the given interval transform on to $-\infty < x < \infty$. In many of applications of the sinc method transformation

$$\phi(z) = log\left(\frac{z-a}{b-z}\right). \tag{7}$$

has been used. The map ϕ carries the eye-shaped region

$$D_E = \left\{z = x + iy : \left|arg\left(\frac{z-a}{b-z}\right)\right| < d < \frac{\pi}{2}\right\}, \tag{8}$$

on to

$$D_d = \{\xi = \xi + i\eta : |\eta| < d < \pi/2\}. \tag{9}$$

Define h by

$$h = \sqrt{\frac{\pi d}{\alpha N}}, \quad 0 < \alpha \leq 1. \tag{10}$$

The h is the mesh size in D_d for the uniform grids $\{kh\}$, $-\infty < k < \infty$. The base functions on (a,b) are given by

$$S(j,h)o\phi(x) = sinc\left(\frac{\phi(x) - jh}{h}\right). \tag{11}$$

The sinc grid points $z \in (a,b)$ in D_E will be denoted by x because they are real. The inverse images of the equispaced grids in the SE transformation are

$$x = \phi^{-1}(t) = \psi(t) = \frac{a + be^t}{1 + e^t}. \tag{12}$$

or

$$x=\phi^{-1}(t)=\psi(t)=\frac{b-a}{2}\tanh\left(\frac{t}{2}\right)+\frac{b+a}{2} \tag{13}$$

In the DE transformation, we can use

$$t=\phi(x)=\log(\frac{1}{\pi}\log\left(\frac{x-a}{b-x}\right)+\sqrt{(\frac{1}{\pi}\log\left(\frac{x-a}{b-x}\right))^2+1}), \tag{14}$$

$$x=\phi^{-1}(t)=\psi(t)=\frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh(t)\right)+\frac{b+a}{2}, \tag{15}$$

which Takahasi and Mori proposed for numerical integration [9]. One of the best reasons for using (15) is optimality of this transformation. It usually gives significantly faster convergence than (13) [9, 10, 15].

Similar to homogeneous boundary condition $y(a)=y(b)=0$ in (1), assume that $y(x)$ satisfies

$$y(x)=\begin{cases} O((x-a)^{\beta_-}); & x\to a(0<\beta_-),\\ O((x-a)^{\beta_+}); & x\to b(0<\beta_+),\end{cases} \tag{16}$$

in the neighborhood of the boundary points. From (15) we have

$$\begin{cases} x-a=\frac{b-a}{1+\exp(-\pi\sinh t)}\approx(b-a)\exp\left(-\frac{\pi}{2}e^{|t|}\right); & t\to-\infty\\ b-x=\frac{b-a}{1+\exp(\pi\sinh t)}\approx(b-a)\exp\left(-\frac{\pi}{2}e^{t}\right); & t\to+\infty\end{cases} \tag{17}$$

Therefore from (16) we obtain

$$y(\psi(t))=\begin{cases} O\left(\exp\left(-\frac{\pi}{2}\beta_- e^{|t|}\right)\right); & t\to-\infty,\\ O\left(\exp\left(-\frac{\pi}{2}\beta_+ e^{|t|}\right)\right); & t\to+\infty,\end{cases} \tag{18}$$

In this case we call (15) the DE transformation. Using (13), we have

$$y(\psi(t))=\begin{cases} O(\exp(-\beta_-|t|)); & t\to-\infty,\\ O(\exp(-\beta_+ t)); & t\to+\infty,\end{cases} \tag{19}$$

that is called SE transformation.

If $y(x)$ be a solution of (1) and we use sinc expansion for numerical solution over (a,b) and $y(\psi(t))$ is analytic in a strip domain $|Imt|<d$, then from (5) and (6) we have

$$y(x)=\sum_{j=-\infty}^{\infty}y(x_j)S(j,h)(\psi^{-1}(x))+E_{sinc}(h).$$
$$x_j=\psi(jh),\quad E_{sinc}(h)=O\left(\exp\left(-\frac{\pi d}{h}\right)\right). \tag{20}$$

In (20) $x_j=\psi(jh)$, $j=0,\pm1,\pm2,\ldots$ are called sinc points. By replacing $y(x_j)$ with its approximation y_j, we obtain

$$\tilde{y}_h(x)=\sum_{j=-\infty}^{\infty}y_jS(j,h)(\psi^{-1}(x)). \tag{21}$$

In practical calculations we use finite terms of (21). Suppose that we truncate sum (21) at $j=-N_-$ on negative side and $j=N_+$ on positive side of j. Using (3) and (18), the truncation error is bounded with

$$\begin{aligned}&\exp\left(-\frac{\pi}{2}\beta_- e^{(N_-+1)h}\right)+\exp\left(-\frac{\pi}{2}\beta_- e^{(N_-+2)h}\right)+\ldots\\ &=\delta^{e^h}+\delta^{e^{2h}}+\ldots,\delta=-\frac{\pi}{2}\beta_- e^{N_-h}\\ &\quad<\delta^{1+h}+\delta^{1+2h}+\ldots=\delta\frac{\delta^h}{1-\delta^h}<\delta=\exp\left(-\frac{\pi}{2}\beta_- e^{N_-h}\right).\end{aligned} \tag{22}$$

where, δ is small then we can assume $\delta^h < 1/2$ [12]. Situation is the same when we truncate at $j = N_+$ on positive side of j. For simplicity we assume that $\beta_- = \beta_+ = \beta$ and $N_- = N_+ = N$. Therefore from (6) we have

$$\exp\left(-\frac{\pi}{2}\beta e^{Nh}\right) = \exp\left(-\frac{\pi d}{h}\right). \tag{23}$$

Then, we can obtain ([14])

$$h = \frac{1}{N}\log(\frac{2dN}{\beta}). \tag{24}$$

Using (24) on (6), we have

$$E_{DE} = O\left(\exp\left(-\frac{N}{\log(2dN/\beta)}\right)\right). \tag{25}$$

Similarly, in the case of the SE transformation by choosing $h = \sqrt{\frac{2d}{N}}$, we obtain

$$E_{SE} = O\left(\exp\left(-\sqrt{\pi^2 dN/2}\right)\right). \tag{26}$$

By comparing (25) and (26) it is observed that convergence by the DE transformation as N become large is much faster than that by SE transformation. In fact it is proved that DE sinc approximation is optimal in some sense in the approximation [15].

3 Sinc-Galerkin method

For solving problem (1) with the DE sinc-Galerkin method, we need some information.

Lemma 1 *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E to D_d Given by (21). Then*

$$\delta_{jk}^{(0)} = [S(j,h)o\phi(x)]_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{27}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)o\phi(x)]_{x=x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases} \tag{28}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)o\phi(x)]_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k, \end{cases} \tag{29}$$

Proof. Ref [1]. ■

In linear problem we have

$$\begin{aligned} &\mathcal{L}(y) = p(x)y" + q(x)y' + u(x)y = f(x), \\ &y(a) = y(b) = 0. \end{aligned} \tag{30}$$

We consider equation (30) and its approximation solution by

$$y_m(x) = \sum_{j=-N}^{N} \alpha_j S_j(x), \qquad m = 2N+1, \tag{31}$$

where $S_j(x)$ is the function $S(j,h)o\phi(x)$ for some fixed step size h. The unknown coefficients α_j is determined that

$$\langle \mathcal{L}[y_m] - f, S_n\rangle = 0, \quad n = 1,\ldots,N, \tag{32}$$

or

$$\langle p(x)y"(x), S_n\rangle + \langle q(x)y'(x), S_n\rangle + \langle u(x)y(x), S_n\rangle = \langle f(x), S_n\rangle, \qquad n = 1,\ldots,N. \tag{33}$$

The used inner product is defined by

$$\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)w(x)dx, \tag{34}$$

where $w(x) = \frac{1}{\phi'(x)}$ [1].
For solving achieved system we can use following theorem.
Theorem 1: The following relations hold

$$\langle p(x)y'', S_k\rangle \approx h \sum_{j=-N}^{N}\sum_{i=0}^{2} \frac{y(x_j)}{\phi'(x_j)h^i}\delta_{kj}^{(i)} g_{2,i}, \tag{35}$$

$$\langle q(x)y', S_k\rangle \approx -h \sum_{j=-N}^{N}\sum_{i=0}^{1} \frac{y(x_j)}{\phi'(x_j)h^i}\delta_{kj}^{(i)} g_{1,i}, \tag{36}$$

and

$$\langle G, S_k\rangle \approx h\frac{G(x_k)w(x_k)}{\phi'(x_k)}, \tag{37}$$

where

$$g_{2,2} = (pw)(\phi')^2, \qquad g_{2,1} = (pw)\phi'' + 2(pw)'\phi', \qquad g_{2,0} = (pw)'',$$
$$g_{1,1} = (qw)\phi', \qquad g_{1,0} = (qw)'.$$

and

$$G = u(x)y \qquad \text{or} \qquad \text{G = f(x)}.$$

Proof. Ref [16]. ■

If use theorem 1 for replacing in inner product (32) we obtain following theorem:

Theorem 2 *If the assumed approximate solution of the boundary-value problem (30) is (31), then the discrete Sinc-Galerkin system for the determination of the unknown coefficients α_j is given by*

$$\begin{aligned}&\sum_{j=-N}^{N}\left(\sum_{i=0}^{2}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2,i}(x_j)}{\phi'(x_j)}\alpha_j - \sum_{i=0}^{1}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{1,i}(x_j)}{\phi'(x_j)}\alpha_j\right) + \frac{u(x)w(x_k)}{\phi'(x_k)}\alpha_k \\ &= \frac{f(x_k)w(x_k)}{\phi'(x_k)}, \qquad -N \le k \le N.\end{aligned} \tag{38}$$

We can rewrite (38) in following system:

$$A\alpha = D(\frac{wf}{\phi'})1, \tag{39}$$

where

$$A = \sum_{i=0}^{2}\frac{1}{h^i}I^{(i)}D(a_i), \qquad D(g(x))_{ij} = \begin{cases} g(x_i) & i = j \\ 0 & i \neq j \end{cases}, I^{(i)}, 0 \le i \le 2,$$

the $m \times m$ matrices whose $jk - th$ entry is given by (16)-(18), α be the m-vector with i-th component given by α_i, 1 be the m-vector each of whose components is 1 and the functions $a_i(x)$, $0 \le i \le 2$ are given by:

$$a_0 = \frac{g_{2,0} - g_{1,0} + uw}{\phi'},$$
$$a_1 = \frac{g_{2,1} - g_{1,1} + uw}{\phi'},$$
$$a_2 = \frac{g_{2,2}}{\phi'}.$$

Proof. Ref [3, 16]. ■

By solving the obtained algebraic linear system, the vector α and so the approximation solution is determined. Now we consider nonlinear boundary value problem

$$\begin{aligned}&\mathcal{L}(y) = p(x)y" + q(x)y' + u(x)y + r(x)y^n = f(x),\\ &y(a) = y(b) = 0.\end{aligned} \tag{40}$$

If (31) be a approximation solution of (40), the unknown coefficients α_j is determined that

$$\begin{aligned}&\langle p(x)y'', S_n\rangle + \langle q(x)y', S_n\rangle + \langle u(x)y, S_n\rangle + \langle r(x)y^n, S_n\rangle = \langle f(x), S_n\rangle\\ &n = 1, \ldots, N.\end{aligned} \tag{41}$$

Lemma 2: we have

$$\langle ry^n, S_n\rangle \approx \frac{hw(x_k)y^n(x_k)r(x_k)}{\phi'(x_k)}$$

Proof. Ref [3, 16]. ■

If use theorem 1 and lemma 2 for replacing in inner product (41) we obtain following theorem:

Theorem 3 *If the assumed approximate solution of the boundary-value problem (40) is (31), then the discrete Sinc-Galerkin system for the determination of the unknown coefficients α_j is given by*

$$\begin{aligned}&\sum_{j=-N}^{N}\left(\sum_{i=0}^{2}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2,i}(x_j)}{\phi'(x_j)}\alpha_j - \sum_{i=0}^{1}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{1,i}}{kj}\frac{g_{1,i}(x_j)}{\phi'(x_j)}\alpha_j\right)\\ &+\frac{u(x)w(x_k)}{\phi'(x_k)}\alpha_k + \frac{w(x_k)r(x_k)}{\phi'(x_k)}\alpha_k^n = \frac{f(x_k)w(x_k)}{\phi'(x_k)}, \qquad -N \le k \le N.\end{aligned} \tag{42}$$

We can rewrite (42) in following system:

$$A\alpha + E\alpha^n = F \tag{43}$$

where

$$E = D\left(\frac{rw}{\phi'}\right), \qquad F = D\left(\frac{wf}{\phi'}\right)1,$$

and A is given by (39). For solving nonlinear system (43), we can use Newton s method.

4 Numerical examples

Here we present some examples that are solved by the DE and the SE Sinc-Galerkin methods and Legendre Galerkin method. These examples have singular point in solutions. Comparisons show that Legendre Galerkin method is not appropriate method for solving these problems. Sinc-Galerkin methods are suitable methods for solving problems with singular solutions. The DE Sinc-Galerkin method gives better results than SE Sinc-Galerkin method. The problems are solved with Matlab on a personal computer.

In these examples, the maximum absolute error at sinc points is taken as

$$\|E_{SE}\| = \max_{-N\le i\le N}|y_{exact(x_i)} - y_{N,SE\ sinc-Galerkin(x_i)}|.$$

where

$$x_k = \frac{a + be^{kh}}{1 + e^{kh}},$$

with $d = \frac{\pi}{2}$, $\alpha = 1/2$ and

$$\|E_{DE}\| = \max_{-N\le i\le N}|y_{exact(x_i)} - y_{N,DE\ sinc-Galerkin(x_i)}|.$$

where

$$x_k = \frac{b-a}{2}\tanh(\frac{\pi}{2}\sinh(kh)) + \frac{b+a}{2}.$$

In Tables 1, 3 and 5 we give the absolute errors in some points using proposed methods. In tables 2, 4 and 6 we present maximum error in the SE sinc-Galerkin method and the DE sinc-Galerkin method in all sinc points. We use $N = 125$ in the SE sinc-Galerkin method and $N = 50$ in the DE sinc-Galerkin method. It is observed that although N in DE transformation is smaller but it's better than the SE transformation. In the Legendre Galerkin method our bases functions are Legendre polynomials and we cannot choose a high value N. Thus, we choice in our examples is $N = 30$.

Example 1: Consider the equation

$$\begin{aligned}&(2x+1)^3 y" + 2(2x+1)^3 y = 2(x^3 + 2x^2 + 2x + 2 + 4x^5 + 4x^4),\\ &y(-1) = y(1) = 0\end{aligned}$$

with exact solution

$$y = (x^3 - x)/(2x+1).$$

In this problem the singular point is $x = \frac{-1}{2}$.

Table 1. The error of solving example 1.

x_i in DE trasformation	Error in DE Sinc-Galerkin method	x_i in SE transformation And Galerkin method	Error in SE Sinc-Galerkin method	Error in Galerkin method
-0.96324	5.42 e-014	-0.96162	6.73 e-008	4.004496
-0.60813	7.8 e-013	-0.60594	6.58 e-007	38.77032
-0.28663	1.17 e-012	-0.27382	9.58 e-007	36.43418
0	1.07 e-012	0	9.72 e-007	37.09013
0.286634	9.47 e-013	0.273823	8.42 e-007	32.15094
0.608126	6.34 e-013	0.605941	5.20 e-007	19.88929
0.96324	6.01 e-014	0.961616	5.34 e-008	2.036606

Table 2. Maximum error in SE sinc-Galerkin method and DE sinc-Galerkin method method

N	$\|E_{SE}\|$	$\|E_{DE}\|$
25	0.0855	8.61 e-008
50	6.85 e-004	1.15 e-012
75	2.76 e-004	1.193 e-011
100	1.16 e-005	3.41 e-011
125	9.84 e-007	2.90 e-011

In this example we solve a linear problem with singular solution. In tables 1 and 2 we show that the Galerkin method is not applicable and the DE sinc-Galerkin method is better than the SE sinc-Galerkin method.

Example 2: Consider the equation

$$\begin{aligned}&(10x-9)^3 y'' - 2(10x-9)^3 y' - (10x-9)^3 y = (1000x^4 + 200x^3 - 5590x^2 + 7040x - 223),\\ &y(-1) = y(1) = 0\end{aligned}$$

with exact solution

$$y = -(x^2 - 1)/(x - 0.9)$$

In this problem the singular point is $x = 0.9$.

Table 3. The error of solving example 2.

x_i in DE trasformation	Error in DE Sinc-Galerkin method	x_i in SE transformation And Galerkin method	Error in SE Sinc-Galerkin method	Error in Galerkin method
-0.999949158093507	1.31 E-16	-0.999938937287810	7.83 E-11	1.91 E-06
-0.672078625326399	2.30 E-13	-0.687383065290916	5.70 E-7	7.80 E-2
-0.537116659907527	3.92 E-13	-0.509448868030365	1.12 E-6	0.1286
0	1.91 E-12	0	4.80 E-5	0.4333
0.286634259852092	3.76 E-12	0.273823495469893	9.62 E-6	0.9274
0.672078625326399	1.01 E-11	0.687383065290916	2.66 E-5	3.8825
0.857755805879923	1.41 E-11	0.852290848820743	3.98 E-5	9.6077
0.999994677382579	3.32 E-15	0.999995130475963	3.33 E-9	1.35 E-2

Table 4. Maximum error in SE sinc-Galerkin method and DE sinc-Galerkin method method

N	$\|E_{SE}\|$	$\|E_{DE}\|$
25	5.2519	2.27 e-006
50	0.65403	1.88 e-011
75	7.02 e-003	6.08 e-011
100	7.28 e-004	4.08 e-010
125	4.29 e-005	4.39 e-010

In this example we solve a linear problem with singular solution. In tables 3 and 4 we show that the Galerkin method is not applicable and the DE sinc-Galerkin method is better than the SE sinc-Galerkin method.

Example 3: Consider the nonlinear equation

$$(3x-2)^4y'' - (3x-2)^4y - 3(3x-2)^4y^4 = (-3x^8 + 12x^6 - 27x^5 + 36x^4 - 9x^3 - 34x^2 + 6x + 9)$$
$$y(-1) = y(1) = 0$$

with exact solution

$$y = (x^2-1)/(3x-2)$$

In this problem the singular point is $x = \frac{2}{3}$.

Table 5. The error of solving example 3.

x_i in DE trasformation	Error in DE Sinc-Galerkin method	x_i in SE transformation And Galerkin method	Error in SE Sinc-Galerkin method
-0.999999986180038	6.27 E-17	-0.999999982347064	1.05 E -14
-0.821345508641236	2.38 E-14	-0.808945514479661	2.39 E -08
-0.375635017328184	8.71 E-14	-0.398183994844089	7.96 E -08
-0.193494262754776	2.13 E-13	-0.139579107197183	1.22 E -08
0.286634259852092	2.52 E-13	0.273823495469893	2.36 E -07
0.672078625326399	9.59 E-13	0.687383065290916	6.17 E -07
0.888259468294954	1.50 E-13	0.886419699493427	2.41 E -07
0.999814417834297	2.73 E-16	0.999858140267242	3.75 E -10

Table 6. Maximum error in SE sinc-Galerkin method and DE sinc-Galerkin method method

N	$\|E_{SE}\|$	$\|E_{DE}\|$
25	1.1645	6.60 e-007
50	0.0209	3.06 e-012
75	2.7825 e-004	2.05 e-008
100	8.9257 e-006	7.99 e-008
125	8.8722 e-007	8.60 e-008

In this example we solve a nonlinear problem with singular solution. In tables 5 and 6 we show that the DE sinc-Galerkin method is better than the SE sinc-Galerkin method.

5 Conclusion

In this paper we compared the DE Sinc-Galerkin and SE Sinc-Galerkin and Legendre Galerkin methods for solving boundary value problems with singular point in solutions. The DE Sinc-Galerkin and SE Sinc-Galerkin are appropriate and the Galerkin method is not suitable for these problems. It was observed that DE Sinc-Galerkin with small N gives better results than SE Sinc-Galerkin with bigger N. These results highlight the accuracy and potency of DE transformation.

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