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Qualitative Properties of the Solutions of a Second Order Difference Equation

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Abstract: In this work, we study qualitative properties of solutions of the following nonlinear second order difference equation

$$x_{n+1} = px_{n-1} + f(x_n - x_{n-1})$$

At first we study the case of smooth f, in this case we study stability of the solutions and the existence of Neimark-Saker and period doubling (flip) bifurcation for this system by analysing the characteristic equation. Next we investigate the direction of this bifurcations by using normal form theory. In last section we consider continuous f and study the relation of attractivity and stability of equilibrium point of this equation and some related equations, furthere more we obtain sufficient conditions for the boundedness and attractivity of the solutions.

Keywords: Difference equations; Boundedness; Attractivity;Flip bifurcation; Neimark-Sacker bifurcation.

1 Introduction

Difference equations has its applications in many areas such as economics, mathematical biology and other areas, see [1-7]. We consider the second order difference equation

$$x_{n+1} = px_{n-1} + f(x_n - x_{n-1}) \tag{1}$$

where $p \in [0, 1)$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous real function with $f(0) = 0$, $f(x) \neq 0$ for $x \neq 0$ and x_0, x_{-1} are given real numbers (initial conditions).
Equation (1.1) is a special case of the difference equation introduced in open problem 5 in [5], furthere more particular cases of (1.1) have been appeared in mathematical models of macroeconomics, see [7,8]. Equations of the form

$$x_{n+1} = px_n + f(x_n - x_{n-1}) \tag{2}$$

considered and studied extensively by [5,8,9,10].
In this work we study various properties of (1.1). At first in section 2 we study the case of smooth f. In this case we study stability of the solutions and the existence of Neimark-Saker and period doubling (flip) bifurcation for this system by analysing the characteristic equation. Furthere more we investigate the direction of this bifurcations by using normal form theory. In last section we consider continuous f and study the relation of attractivity and stability of equilibrium point of this equation and some related equations, furthere more we obtain sufficient conditions for the boundedness and attractivity of the solutions.

2 Bifurcation analysis

In this section we suppose that $f \in C^1$, hence we can use linearization theorem. Let x_n be a solution of (1.1). We define the vector $y(n) = (y_1(n), y_2(n)) \in \mathbb{R}^2$, where

$$y_j(n) = x_{n+j-2}, \qquad j = 1, 2 \tag{3}$$

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Using this notation the delay equation (1.1) transformed to the following 2-dimensional system:

$$y(n+1) = g(y(n)) \tag{4}$$

where $g(y) = (y_2, py_1 + f(y_2 - y_1))$ which has the unique equilibrium point $(0, 0)$.
The Jacobian matrix of g at 0 is:

$$H = Dg\mid_0 = \begin{bmatrix} 0 & 1 \\ p - f'(0) & f'(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ p-q & q \end{bmatrix}$$

in which $q = f'(0)$. The characteristic equation of D is

$$\lambda^2 - m\lambda + n = 0 \tag{5}$$

where $m = q$, $n = q - p$. By the Jury conditions, the necessary and sufficient conditions for all eigenvalues of the characteristic equation lying inside the unit circle are:

$$\begin{cases} p < 1 \\ 2q > p - 1 \\ q < p + 1 \end{cases}$$

Hence, the fixed point is nonhyperbolic if and only if (p,q) lies on one of the following lines:

$$\begin{cases} L_1 : 2q - p + 1 = 0 \\ L_2 : -q + p + 1 = 0 \end{cases}$$

In the following lemma we characterize all possible cases for p and q.

Lemma 1 *The fixed point is nonhyperbolic if and only if (p,q) lies on one of the lines L_1 or L_2. Otherwise, the fixed point is in one of the following types:*

Conditions	*Properties*
$q = 2 - 2\sqrt{1-p}$	*Stable*
$q = 2 + 2\sqrt{1-p}$	*Unstable*
$q < \frac{p-1}{2}$	*Unstable*
$\frac{p-1}{2} < q \leq 0$	*Stable*
$0 \leq q < 2 - 2\sqrt{1-p}$	*Stable*
$2 - 2\sqrt{1-p} < q < p+1$	*Stable*
$p + 1 < q < 2 + 2\sqrt{1-p}$	*Unstable*
$q > 2 + 2\sqrt{1-p}$	*Unstable*

Proof. The eigenvalues of D are $\lambda_1 = \frac{q-\sqrt{q^2-4(q-p)}}{2}$, $\lambda_2 = \frac{q+\sqrt{q^2-4(q-p)}}{2}$. In the case that $\Delta > 0$, we have $q > 2 + 2\sqrt{1-p}$ or $q < 2 - 2\sqrt{1-p}$ and the eigenvalues λ_1, λ_2 are distinct and real. In the case that $q < 2 - 2\sqrt{1-p}$, we have $q < 2$ and $\frac{d\lambda_1}{dq} = \frac{1}{2}(1 + \frac{4-2q}{2\sqrt{q^2-4(q-p)}}) > 0$

On the line L_1, the fixed point is nonhyperbolic, otherwise it suffices to consider $q > \frac{p-1}{2}$ and $q < \frac{p-1}{2}$ separately. If $\frac{p-1}{2} < q < 2 - 2\sqrt{1-p}$ by the monotonicity of λ_1 we get $-1 < \lambda_1 < 1$. In particular, $\lambda_1 < 0$ (respectively,≥ 0) if $q < 0$ (respectively,≥ 0).
On the other hand

$$\lambda_2 - 1 = \frac{\sqrt{q^2-4(q-p)}-(2-q)}{2} = \frac{4p-4}{2(\sqrt{q^2-4(q-p)}+(2-q))} < 0$$

Hence $0 < \lambda_2 < 1$. Therefore when $\frac{p-1}{2} < q < 2 - 2\sqrt{1-p}$ the fixed point is stable.
If $q < \frac{p-1}{2}$ by the monotonicity of λ_1 we see that $\lambda_1 < -1$, and the fixed point is unstable.

Now we consider the case that $q > 2 + 2\sqrt{1-p}$. In this case $q > 2$, and $\frac{d\lambda_1}{dq} = \frac{1}{2}(1 + \frac{4-2q}{2\sqrt{q^2-4(q-p)}}) < 0$

$$\lim_{q\to\infty} \lambda_1 = \lim_{q\to\infty} \frac{q - \sqrt{q^2 - 4(q-p)}}{2} = \lim_{q\to\infty} \frac{4q - 4p}{2(q + \sqrt{q^2 - 4(q-p)})} = 1.$$

By the monotonicity of λ_1 and the above limit we have $\lambda_1 > 1$. On the other hand $\lambda_2 - \lambda_1 = \sqrt{q^2 - 4(q-p)} > 0$, implies that $\lambda_2 > \lambda_1 > 1$ therefore the fixed point is unstable.
In the case that $\Delta < 0$ we have $2 - 2\sqrt{1-p} < q < 2 + 2\sqrt{1-p}$, and the eigenvalues λ_1,λ_2 are conjugate complex numbers, i.e. $\lambda_2 = \overline{\lambda_1}$ and $|\lambda_1|^2 = q - p$.
At first we consider the case that $2 - 2\sqrt{1-p} < q < p + 1$. In this case $|\lambda_1| < 1$ and $|\lambda_2| = |\lambda_1| < 1$ therefore the fixed point is stable. In the case that $p + 1 < q < 2 + 2\sqrt{1-p}$, $|\lambda_2| = |\lambda_1| > 1$ and the fixed point is unstable. ■

Now we prove the existence of the Flip and Neimark-Sacker bifurcations:

Theorem 2 *For system (2.2) the following conditions holds:*
1) Flip bifurcation occurs when $(p, q) \in L_1$.
2) Neimark-Sacker bifurcation occurs when $(p, q) \in L_2$.

Proof. First, we show the existence of the Flip bifurcation. Since $(p, q) \in L_1$, we have the following characteristic equation $(\lambda + 1)(\lambda - n) = 0$ which has eigenvalues $\lambda_1 = -1$, $\lambda_2 = n = q - p$. Now $Y = (1, -1)$ is an eigenvector of H with corresponding eigenvalue $\lambda_1 = -1$, and 1 is not an eigenvalue. A straight forward calculation shows that:

$$Range(I + Dg(0))_{|2q-p+1=0} = Span(x + y, (q + 1)(x + y))^T$$

in which $(x, y) \in \mathbb{R}^2$, and

$$\frac{d}{dq} Dg(0)_{|2q-p+1=0} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

therefore

$$\frac{d}{dq} Dg(0)_{|2q-p+1=0} Y = (0, -2) \in Range(I + Dg(0))_{|2q-p+1=0}$$

and hence by [11, Th.1.4.5], Flip bifurcation occurs.
Now we show the existence of the Neimark-Sacker bifurcation. If $\lambda = e^{i\theta}$ is a root of equation (2.3), by separating the real and imaginary parts, we have the following relations:

$$\begin{cases} cos2\theta - mcos\theta = n \\ sin2\theta - msin\theta = 0 \end{cases}$$

by squaring and adding both equations, we have:

$$1 - m^2 - 2mcos\theta = n^2$$

and hence

$$cos\theta = \frac{1 - p^2 + 2pq}{2q}.$$

Since $(p, q) \in L_2$ we have $q = p + 1$, so that $cos\theta = \frac{1-p}{2}$. Therefore $\frac{1}{2} < cos\theta < 1$. Which refers $arg\lambda \neq 0, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pm\pi$. Thus $\lambda^k \neq 1$, for k=1,2,3 and 4. On the other hand we have that:

$$(\frac{d|\lambda|^2}{dq})_{|q=p+1} = (\overline{\lambda}\frac{d\lambda}{dq} + \lambda\frac{d\overline{\lambda}}{dq})_{|q=p+1} = \frac{2(1 - cos\theta)(2cos\theta - m)}{|2e^{i\theta} - m|^2}.$$

Assume that $(\frac{d|\lambda|^2}{dq})_{|q=p+1} = 0$, that is, $cos\theta = 1$ or $cos\theta = \frac{m}{2}$. In previous discussion, we obtain $\frac{1}{2} < cos\theta < 1$, hence $cos\theta \neq 1$. If $cos\theta = \frac{m}{2}$, then by $cos\theta = \frac{1-p^2+2pq}{2q}$, we have $p = -1$, which is impossible since $0 \leq p < 1$. Therefore $(\frac{d|\lambda|^2}{dq})_{|q=p+1} \neq 0$. Hence by the generic Neimark-Sacker bifurcation theorem [8,9], in this case Neimark-Sacker bifurcation occurs, that is, the system (2.2) has a unique closed invariant curve bifurcating from the equilibrium point X^*. ■

3 Direction of the bifurcations

In the previous section, we have shown that system (2.2) undergoes a Flip (period-doubling) bifurcation when $(p, q) \in L_1$ and a Neimark Sacker bifurcation when $(p, q) \in L_2$ at 0. In this section, by using normal form method for discrete systems, we study the direction of this bifurcations and stability of the bifurcating invariant curves. We can write this system as:

$$U_{n+1} = DU_n + G(U_n) \qquad\qquad U_n \in \mathbb{R}^2 \tag{6}$$

where $G(U) = O(\|U\|^3)$ is a smooth function and its Taylor expansion is

$$G(U) = \frac{1}{2}B(U,U) + \frac{1}{6}C(U,U,U) + O(\|U\|^3), \tag{7}$$

in which:

$$B(U,U) = (b_1(U,U), b_2(U,U)), \quad C(U,U,U) = (c_1(U,U,U), c_2(U,U,U))$$

and

$$\begin{cases} b_1(\phi,\psi) = 0, \\ b_2(\phi,\psi) = f''(0)(\phi_1\psi_1 - \phi_2\psi_1 + \phi_2\psi_2 - \phi_1\psi_2) \\ c_1(\phi,\psi,\eta) = 0 \\ c_2(\phi,\psi,\eta) = f'''(0)(\phi_2\psi_1\eta_1 - \phi_1\psi_1\eta_1 - \phi_1\psi_2\eta_2 + \phi_2\psi_2\eta_2 - \phi_2\psi_2\eta_1 + \phi_1\psi_2\eta_1 - \phi_2\psi_1\eta_2 + \phi_1\psi_1\eta_2) \end{cases}$$

for $\phi = (\phi_1, \phi_2) \in \mathbb{R}^2$, $\psi = (\psi_1, \psi_2) \in \mathbb{R}^2$ and $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. First we study the direction of period-doubling bifurcation and the stability of period-doubling cycle. Let $w \in \mathbb{R}^2$ be the eigenvector of D with respect to eigenvalue -1, that is, $Hw = -w$; $v \in \mathbb{R}^2$ be the adjoint eigenvector of H^T that is, $H^T v = -v$ where H^T is the transposed matrix, and $< v, w >= 1$, where $< ., . >$ is the standard scalar product in $\mathbb{R}^2$. We obtain by calculation:

$$\begin{cases} w = (1, -1) \\ v = \frac{q+1}{q+2}(1, \frac{-1}{q+1}) \end{cases}$$

Following the algorithms given in Kuznetsov [12], the critical normal form coefficient $C(0)$, that determines nondegeneracy of the period-doubling bifurcation and stability of period-doubling cycle, is given by the following formula:

$$C(0) = \frac{1}{6} < v, C(w,w,w) > -\frac{1}{2} < v, B(w,(H-I)^{-1}B(w,w)) > \tag{8}$$

From the above relations we have:

$$\begin{cases} < v, B(w,(H-I)^{-1}B(w,w)) >= 0 \\ < v, C(w,w,w) >= \frac{16f'''(0)}{p+3} \end{cases}$$

and therefore:

$$C(0) = \frac{8f'''(0)}{3(p+3)}.$$

Applying general theory for the direction of bifurcation and stability of period doubling cycle, see Kuznetsov [12] or Wiggins [13] we have the following result:

Theorem 3 *In system (2.2), period doubling bifurcation (flip bifurcation) occurs in* 0 *for* $p-1=2q$. *Furthere more if* $f'''(0)>0$ *the flip bifurcation is supercritical and if* $f'''(0)<0$ *the flip bifurcation is subcritical.*

Now, we are going to study the direction of the Neimark-Sacker bifurcation and the stability of the bifurcating invariant curve in 0. In the above section, we see that H has simple eigenvalues on the unit circle: $\lambda_{1,2}=e^{\pm i\theta_0}$, $\theta_0=arccos\frac{1-p^2+2pq}{2q}$. Let α be a complex eigenvector corresponding to $e^{i\theta}$ and β be a complex eigenvector of the transposed matrix H^T corresponding to $e^{-i\theta}$, that is:

$$H\alpha=\alpha e^{i\theta_0} \quad , \quad H^T\beta=\beta e^{-i\theta_0}$$

Then we obtain:

$$\alpha=(1,e^{i\theta_0})^T \quad , \quad \beta=(1,-e^{-i\theta_0})^T$$

Normalize α with respect to β such that $<\beta,\alpha>=1$ where $<.,.>$ means the standard scalar product in $\mathbb{C}$ defined by $<\beta,\alpha>=\overline{\beta_1}\alpha_1+\overline{\beta_2}\alpha_2$, we have:

$$\alpha=(1,e^{i\theta_0})^T \quad , \quad \beta=\frac{1}{1-e^{-2i\theta_0}}(1,-e^{-i\theta_0})^T$$

Following the algorithms given in Kuznetsov [12], the critical normal form coefficient $a(0)$, that determines nondegeneracy of the Neimark - Sacker bifurcation and allows us to predict the stability of bifurcating invariant curve, is given by the following formula:

$$\begin{aligned} a(0)=\frac{1}{2}Re(e^{-i\theta_0}[<\beta,C(\alpha,\alpha,\overline{\alpha})> &+ 2<\beta,B(\alpha,(I-H)^{-1}B(\alpha,\overline{\alpha}))> \\ &+ <\beta,B(\overline{\alpha},(e^{2i\theta_0}I-H)^{-1})B(\alpha,\alpha)>]). \end{aligned}$$

Furtheremore in this case we have:

$$<\beta,C(\alpha,\alpha,\overline{\alpha})>=\frac{f'''(0)(-3e^{i\theta_0}+3e^{2i\theta_0}+e^{3i\theta_0}+1)}{1-e^{2i\theta_0}}$$

$$<\beta,B(\alpha,(I-H)^{-1}B(\alpha,\overline{\alpha}))>=0$$

$$<\beta,B(\overline{\alpha},(e^{2i\theta_0}I-H)^{-1})B(\alpha,\alpha)>=\frac{-f''(0)(3e^{i\theta_0}+e^{3i\theta_0}-1-2e^{2i\theta_0}-e^{5i\theta_0})}{e^{2i\theta_0}(e^{4i\theta_0}-e^{2i\theta_0}+1)}$$

which yields the following formula for $a(0)$:

$$a(0)=\frac{f'''(0)A_1+f''(0)B_1}{A_0+B_0}$$

in which
$A_0=(2cos6\theta_0-cos8\theta_0-2cos4\theta_0+cos2\theta_0)^2$
$B_0=(2sin6\theta_0-sin8\theta_0-2sin4\theta_0+sin2\theta_0)^2$
$A_1=8-16cos2\theta_0+8cos4\theta_0-2cos\theta_0+4cos3\theta_0-3cos5\theta_0-2cos6\theta_0+cos7\theta_0$
$B_1=-4-8cos6\theta_0-6cos6\theta_0+7cos4\theta_0+4cos\theta_0cos6\theta_0+cos2\theta_0-4cos3\theta_0-4cos\theta_0cos4\theta_0$
$+2cos\theta_0+2cos\theta_0cos2\theta_0+3cos8\theta_0-2cos\theta_0cos8\theta_0+2cos5\theta_0+6sin4\theta_0-3sin2\theta_0+3sin8\theta_0$

From the theory of the direction of Neimark Sacker bifurcation and stability of the bifurcating invariant curve, we have the following theorem:

Theorem 4 *In system (2.2), if $q = p + 1$ hold, then $a(0) < 0$(respectively, $a(0) > 0$) implies that a unique and stable (respectively, unstable) closed invariant curve bifurcates from 0, and the Neimark - Sacker bifurcation is supercritical (respectively, subcritical).*

4 Furthere properties

In this section we study relation of (1.1) with some related equations and then obtain sufficient conditions for boundedness, global attractivity and stability of the equilibrium point. Equation (1.1) can be transformed to another form which has equivalent properties. Let

$$u_n = x_n - x_{n-1} \tag{9}$$

Then (1.1) reduced to

$$u_{n+1} = pu_{n-1} + f(u_n) - f(u_{n-1}) \tag{10}$$

which has the unique equilibrium $\overline{u} = 0$.

First we show the following result.

Theorem 5 *The equilibrium point $\overline{x} = 0$ is global attractive (respectively asymptotically stable) in (1.1) if and only if $\overline{u} = 0$ is global attractive (respectively asymptotically stable) in (4.2).*

Proof. Equation (1.1) can be written as:

$$x_{n+1} = px_{n-1} + f(u_n), forn = 0, 1, 2, ... \tag{11}$$

Hence we see that$x_1 = px_{-1} + f(u_0)$, $x_2 = px_0 + f(u_1)$, $x_3 = px_1 + f(u_2) = p^2x_{-1} + pf(u_0) + f(u_2)$, $x_4 = px_2 + f(u_3) = p^2x_0 + pf(u_1) + f(u_3)$. Using induction we have that, if n is even then:

$$x_n = p^{\frac{n}{2}}x_0 + \sum_{i=1}^{\frac{n}{2}} p^{\frac{n}{2}-i}f(u_{2i-1}), \tag{12}$$

And if n is odd then:

$$x_n = p^{\frac{n+1}{2}}x_{-1} + \sum_{i=1}^{\frac{n+1}{2}} p^{\frac{n+1}{2}-i}f(u_{2i-2}), \tag{13}$$

Now if n is even, let $\overline{u}_n = \sum_{i=1}^{\frac{n}{2}} p^{\frac{n}{2}-i} \mid f(u_{2i-1}) \mid$ and if n is odd, $\overline{v}_n = \sum_{i=1}^{\frac{n+1}{2}} p^{\frac{n+1}{2}-i} \mid f(u_{2i-2}) \mid$ we prove that

$$\lim_{n\to\infty} \overline{u}_n = 0$$

We distinguish two cases.

Case 1:($\sum_{i=1}^{\infty} \mid f(u_{2i-1}) \mid /p^i < \infty$). In this case

$$\lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} p^{\frac{n}{2}} \sum_{i=1}^{\frac{n}{2}} \frac{\mid f(u_{2i-1}) \mid}{p^i} = 0$$

Case 2 :($\sum_{i=1}^{\infty} \mid f(u_{2i-1}) \mid /p^i = \infty$). In this case by using Stolz Theorem and since n is even, we have that:

$$\begin{aligned}
\lim_{n\to\infty} \overline{u}_n &= \lim_{n\to\infty} \frac{\sum_{i=1}^{\frac{n}{2}} \frac{|f(u_{2i-1})|}{p^i}}{\frac{1}{p^{\frac{n}{2}}}} \\
&= \lim_{n\to\infty} \frac{\sum_{i=1}^{\frac{n+2}{2}} \frac{|f(u_{2i-1})|}{p^i} - \sum_{i=1}^{\frac{n}{2}} \frac{|f(u_{2i-1})|}{p^i}}{\frac{1}{p^{\frac{n+2}{2}}} - \frac{1}{p^{\frac{n}{2}}}} \\
&= \lim_{n\to\infty} \frac{\mid f(u_{n+1}) \mid}{1-p}
\end{aligned}$$

Using continuity of f we see that if

$$\lim_{n\to\infty} u_n = 0$$

then

$$\lim_{n\to\infty} \overline{u}_n = 0$$

(and by a similar argument)

$$\lim_{n\to\infty} \overline{v}_n = 0$$

which implies that

$$\lim_{n\to\infty} x_n = 0$$

■

Now we study the relation of the delay equation (1.1) with the associated nondelay equation:

$$x_{n+1} = f(x_n) \quad n \geq -1 \tag{14}$$

First we prove the following lemma which will be used in the next results.

Lemma 6 *Let $y(n)$ be a solution of (2.2). The following statements are true:*
For odd $n+j$ with $n+j \geq 3$:

$$| y_j(n) | \leq p^{\frac{n+j-1}{2}} | y_1(0) | + \sum_{i=1}^{\frac{n+j-1}{2}} p^{\frac{n+j-1}{2}-i} | f(y_2(2i-2) - y_1(2i-2)) | \tag{15}$$

And for even $n+j$ with $n+j \geq 3$:

$$| y_j(n) | \leq p^{\frac{n+j-2}{2}} | y_2(0) | + \sum_{i=1}^{\frac{n+j-2}{2}} p^{\frac{n+j-2}{2}-i} | f(y_2(2i-1) - y_1(2i-1)) | \tag{16}$$

Furtheremore for $0 \leq n+j \leq 2$:

$$y_j(n) = y_{j+n}(0) \tag{17}$$

Proof. From (2.1) we have that for $j = 1, 2$:

$$y_j(n) = x_{n+j-2} = x_{(n-1)+(j+1)-2} = y_{j+1}(n-1) \tag{18}$$

$$y_j(n) = x_{n+j-2} = x_{0+(n+j)-2} = y_{n+j}(0) \tag{19}$$

Now by using these relations and induction we see that, if n is odd then:

$$y_2(n) = p^{\frac{n+1}{2}} y_1(0) + \sum_{i=1}^{\frac{n+1}{2}} p^{\frac{n+1}{2}-i} f(y_2(2i-2) - y_1(2i-2)) \tag{20}$$

And if n is even then:

$$y_2(n) = p^{\frac{n}{2}} y_2(0) + \sum_{i=1}^{\frac{n}{2}} p^{\frac{n}{2}-i} f(y_2(2i-1) - y_1(2i-1)) \tag{21}$$

Furthere more:

$$y_j(n) = x_{n+j-2} = x_{(n+j-2)+2-2} = y_2(n+j-2) \tag{22}$$

By using this relation we have that, if $n+j$ is odd then:

$$y_j(n) = y_2(n+j-2) = p^{\frac{n+j-1}{2}} y_1(0) + \sum_{i=1}^{\frac{n+j-1}{2}} p^{\frac{n+j-1}{2}-i} f(y_2(2i-2) - y_1(2i-2))$$

And if $n+j$ is even then:

$$y_j(n) = y_2(n+j-2) = p^{\frac{n+j-2}{2}} y_2(0) + \sum_{i=1}^{\frac{n+j-2}{2}} p^{\frac{n+j-2}{2}-i} f(y_2(2i-1) - y_1(2i-1))$$

■

Theorem 7 *Assume that f satisfies:*

$$| f(x+y) | \leq | f(x) | + | f(y) | \tag{23}$$

for all $x, y \in \mathbb{R}$. If the equilibrium point of (4.6) is stable, then the equilibrium point of (1.1) is also stable.

Proof. It is sufficient to prove the stability of the equilibrium of (2.2) because of the equivalence of (1.1) and (2.2). Let $\varepsilon > 0$ be arbitrary. Since the equilibrium point of (4.6) is stable, there exists $\delta_1 > 0$ such that $| x_{-1} | < \delta_1$ implies $| x_n | < \frac{(1-p)\varepsilon}{2}$ for all $n \geq -1$. Choose $\delta = min(\delta_1, \frac{(1-p)\varepsilon}{2})$, since $y(0) = (y_1(0), y_2(0)) = (x_{-1}, x_0)$, we have that

$$\| y(0) \| = max(| y_1(0) |, | y_2(0) |) = max(| x_{-1} |, | x_0 |) \leq \delta \leq \delta_1$$

Now for $n \geq -1$:

$$| x_n | \leq \frac{(1-p)\varepsilon}{2} \tag{24}$$

which implies that:

$$| f(x_n) | \leq \frac{(1-p)\varepsilon}{2} \tag{25}$$

for all $n \geq -1$. Therefore, for $n \geq 0$:

$$| f(y_2(n)) | \leq \frac{(1-p)\varepsilon}{2} \qquad | f(y_1(n)) | \leq \frac{(1-p)\varepsilon}{2} \tag{26}$$

And hence

$$| f(y_2(n) - y_1(n)) | < | f(y_2(n)) | + | f(y_1(n)) | < \frac{(1-p)\varepsilon}{2} + \frac{(1-p)\varepsilon}{2} = (1-p)\varepsilon$$

Now $\| y(0) \| \leq \delta$ implies that $| y_j(0) | < \delta < \frac{(1-p)\varepsilon}{2} < \varepsilon$ for $j = 1, 2$. Hence

$$| y_j(n) | = | y_{j+n}(0) | < \varepsilon, \quad for \quad 0 \leq n \leq 2-j$$

and from the previous lemma, if $n + j$ is odd then:

$$\begin{aligned} | y_j(n) | &\leq p^{\frac{n+j-1}{2}} | y_1(0) | + \sum_{i=1}^{\frac{n+j-1}{2}} p^{\frac{n+j-1}{2}-i} | f(y_2(2i-2) - y_1(2i-2)) | \\ &\leq \varepsilon p^{\frac{n+j-1}{2}} + (1-p)\varepsilon p^{\frac{n+j-1}{2}} \sum_{i=1}^{\frac{n+j-1}{2}} p^{-i} = \varepsilon \end{aligned}$$

and if $n + j$ is even then:

$$\begin{aligned} | y_j(n) | &\leq p^{\frac{n+j-2}{2}} | y_2(0) | + \sum_{i=1}^{\frac{n+j-2}{2}} p^{\frac{n+j-2}{2}-i} | f(y_2(2i-1) - y_1(2i-1)) | \\ &\leq \varepsilon p^{\frac{n+j-2}{2}} + (1-p)\varepsilon p^{\frac{n+j-2}{2}} \sum_{i=1}^{\frac{n+j-2}{2}} p^{-i} = \varepsilon. \end{aligned}$$

Therefore, for arbitrary $\varepsilon > 0$, there exists $\delta > 0$, such that $\| y(0) \| < \delta$ implies that $\| y(n) \| < \varepsilon$ for $n \geq 0$, and hence the equilibrium point of (1.1) is stable. ■

Theorem 8 *Assume that (4.15) holds. If there exists $m > 0$ such that $G(m) = \{x \in \mathbb{R} : | x | < m\}$ is a subset of the attractive region of the equilibrium point of (4.6), then $G(m)$ is also contained in the attractive region of the equilibrium point of (1.1).*

Proof. Let $\varepsilon > 0$ be arbitrary. Since $G(m)$ is a subset of attractive region of (2.2), there exists T_1 such that $| x_{-1} | < m$ implies that $| x_n | < \varepsilon$ for $n > T_1$. Assume that $\| y(0) \| < m$, then we have $| x_{-1} | < m$. So there exists $T_2 \geq T_1$ such that $| x(n) | < (1-p)\varepsilon/4$ for $n \geq T_2$, which implies that:

$$\begin{aligned} | f(y_2(n) - y_1(n)) | &\leq | f(y_2(n)) | + | f(y_1(n)) | \\ &< \frac{(1-p)\varepsilon}{4} + \frac{(1-p)\varepsilon}{4} \leq \frac{(1-p)\varepsilon}{2} \end{aligned}$$

for all $n \geq T_2 + 1$. Let $j = 1, 2$. If $n + j$ is odd we have:

$$| y_j(n) | \leq p^{\frac{n+j-1}{2}} | y_1(0) | + \sum_{i=1}^{\frac{n+j-1}{2}} p^{\frac{n+j-1}{2}-i} | f(y_2(2i-2) - y_1(2i-2)) |$$
$$< mp^{\frac{n+j-1}{2}} + \sum_{i=1}^{T_2+1} p^{\frac{n+j-1}{2}-i} | f(y_2(2i-2) - y_1(2i-2)) | + \frac{\varepsilon}{2}$$

provided that $n \geq 3 - j$. If $n + j$ is even we have:

$$| y_j(n) | \leq p^{\frac{n+j-2}{2}} | y_2(0) | + \sum_{i=1}^{\frac{n+j-2}{2}} p^{\frac{n+j-2}{2}-i} | f(y_2(2i-1) - y_1(2i-1)) |$$
$$< mp^{\frac{n+j-2}{2}} + \sum_{i=1}^{T_2+1} p^{\frac{n+j-2}{2}-i} | f(y_2(2i-1) - y_1(2i-1)) | + \frac{\varepsilon}{2}$$

Now:

$$| f(y_2(i) - y_1(i)) | \leq | f(x_i) | + | f(x_{i-1}) | \leq | f^{i+2}(x_{-1}) | + | f^{i+1}(x_{-1}) | \tag{27}$$

The continuity of f implies there exists $L > 0$ such that $| f^{i+2}(x_{-1}) | < L$ and $| f^{i+1}(x_{-1}) | < L$. Now if $n + j$ is odd we have for $n \geq T_2 + 2$:

$$| y_j(n) | < mp^{\frac{n+j-1}{2}} + \sum_{i=1}^{T_2+1} p^{\frac{n+j-1}{2}-i} | f(y_2(2i-2) - y_1(2i-2)) | + \frac{\varepsilon}{2}$$
$$< \frac{\varepsilon}{2} + (m + \frac{2L}{1-p})p^{\frac{n+j-2T_2-3}{2}}$$

Now choose T_3 such that:

$$(m + \frac{2L}{1-p})p^{\frac{n+j-2T_2-3}{2}} \leq \frac{\varepsilon}{2} \tag{28}$$

holds for $n \geq T_3$.
If $n + j$ is even we have:

$$| y_j(n) | < mp^{\frac{n+j-2}{2}} + \sum_{i=1}^{T_2+1} p^{\frac{n+j-2}{2}-i} | f(y_2(2i-1) - y_1(2i-1)) | + \frac{\varepsilon}{2}$$
$$< \frac{\varepsilon}{2} + (m + \frac{2L}{1-p})p^{\frac{n+j-2T_2-4}{2}}$$

Now choose T_4 such that:

$$(m + \frac{2L}{1-p})p^{\frac{n+j-2T_2-4}{2}} \leq \frac{\varepsilon}{2} \tag{29}$$

holds for $n \geq T_4$. We consider $T_5 = min(T_3, T_4)$, Then $\| y(0) \| < m$ implies that $\| y(n) \| < \varepsilon$ for $n \geq T_5$. Hence $G(m)$ is also subset of attractive region of the equilibrium of (1.1). ■

Now we study other dynamical properties of (1.1). At first we obtain sufficient conditions for boundedness of solutions.

Theorem 9 *Assume that there are constants $B > 0$ and $a \in [0, 1)$ such that*

$$| f(t) - at | \leq B \qquad for \quad all \quad t. \tag{30}$$

Then (1.1) has an absorbing interval.

Proof. Assume that (1.1) holds and let y_n be any solution of (1.1). We show that there is constant $M > 0$ independent of y_0, y_{-1} such that $|y_n| \leq M$ for all sufficiently large $n \geq 1$, i.e., $[-M, M]$ is an absorbing interval. Define $r_n = f(y_n - y_{n-1}) - a(y_n - y_{n-1})$ and note that $| r_n | \leq B$ for all n. Furthere more we consider the non-homogeneous linear equation:

$$x_{n+1} - ax_n + (a - p)x_{n-1} = r_n \tag{31}$$

and note that $\{y_n\}$ is a solution of this equation. Hence $y_n = x_n^{(h)} + x_n^{(p)}$, in which $(x_n^{(h)})$ is the solution of the homogeneous equation corresponding to (4.23) and $(x_n^{(p)})$ is a particular solution of (4.23). There are two possible cases:
Case 1: $a^2 - 4(a-p) \neq 0$; in this case:

$$x_n^{(h)} = C_1\lambda_1^n + C_2\lambda_2^n$$

in which $\lambda_{1,2} = \frac{1}{2}[a \pm \sqrt{a^2 - 4(a-p)}]$ are distinct (real or complex) eigenvalues and the constants C_1, C_2 are determined by the initial conditions y_0, y_{-1}. Using the method of variation of constants, the particular solution has the following form:

$$x_n^p = \frac{1}{\lambda_2 - \lambda_1}\sum_{j=0}^{n-1} r_{n-(j+1)}(\lambda_2^j - \lambda_1^j).$$

Now we have that for all n:

$$|y_n| \leq |C_1\lambda_1|^n + |C_2\lambda_2|^n + \frac{B}{|\lambda_2 - \lambda_1|}[\sum_{j=0}^{n-1}|\lambda_2|^j + \sum_{j=0}^{n-1}|\lambda_1|^j]$$

Hence, for sufficiently large n:

$$|y_n| \leq 1 + \frac{B}{|\lambda_2 - \lambda_1|}[\sum_{j=0}^{\infty}|\lambda_2|^j + \sum_{j=0}^{\infty}|\lambda_1|^j] = 1 + \frac{B}{|\lambda_2 - \lambda_1|(1-|\lambda_2|)(1-|\lambda_1|)} = M.$$

Case 2: $a^2 - 4(a-p) = 0$, in this case:

$$x_n^{(h)} = C_1\lambda^n + C2n\lambda^n, \lambda = \frac{a}{2} \in (0,1)$$

and

$$x_n^p = \sum_{k=0}^{n-1} r_k(n-k-1)\lambda^{n-k-2} = \sum_{j=1}^{n-1} r_{n-j-1} j\lambda^{j-1}$$

therefore

$$|y_n| \leq |C_1|\lambda^n + |C_2|n\lambda^n + B\sum_{j=1}^{n-1} j\lambda^j$$

Thus, for sufficiently large n we have:

$$|y_n| \leq 1 + B\sum_{k=1}^{\infty} j\lambda^j = 1 + B\frac{d}{d|\lambda|}\sum_{k=0}^{\infty}\lambda^j = 1 + \frac{B}{(1-|\lambda|)^2} = M.$$

This completes the proof. ■

By using weak contractions introduced in [10] we obtain the following sufficient conditions for attractivity of solutions of (1.1).

Proposition 10 *(1) If* $| f(t) | \leq a | t |$ *for all t and* $0 < a < \frac{1-p}{2}$*, then origin is globally attracting in (1.1).*
(2) If $0 < f(t) \leq a | t |$ *for all t and* $0 < a < 1-p$*, then every positive solution of (1.1) converges to zero.*

Proof. For the proof of (1), define $F(y_1, y_2) = py_1 + f(y_2 - y_1)$ and notice that

$$| F(y_1, y_2) | \leq p | y_1 | + a | y_2 - y_1 | \leq (p + 2a)max\{| y_1 |, | y_2 |\}$$

since $p + 2a < 1$, it follows that F is a weak contraction on the entire space and therefore by [10], the origin is globally attracting.
(2). For $y_1, y_2 \geq 0$ notice that:

$$F(y_1, y_2) \leq py_1 + amax\{y_1, y_2\} \leq (p + a)max\{y_1, y_2\}$$

Now since $p+a < 1$, it follows that F is a weak contraction on $[0,\infty)^2$, and since $[0,\infty)^2$ is invariant under $V_F(y_1, y_2) = (g(y_1, y_2), y_1)$, [10] implies that the origin is exponentially stable relative to $[0,\infty)^2$, hence every positive solution of (1.1) converges to zero. ■

5 Numerical Simulations

In this section, we give numerical simulations to illustrate our theoretical analysis.

Example 1 Let $p = \frac{1}{2}$ and $f(t) = -\frac{1}{4}t + t^2$. In this case $q = -\frac{1}{4}$ and $(p, q) \in L_1$, therefore by theorem 2.2 flip bifurcation occurs. Figure 1 shows bifurcation diagram.

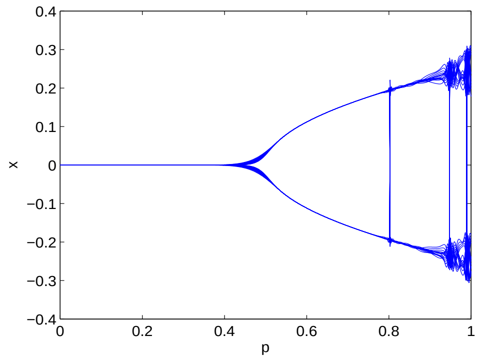

Figure 1: Bifurcation diagram in Ex.1.

Example 2 Let $f(t) = \frac{3}{2}t + t^2$. In this case $q = \frac{3}{2}$, and when $p = \frac{1}{2}$ we have, $(p, q) \in L_2$, therefore by theorem 2.2 Neimark-Sacker bifurcation occurs. Figures 2, 3 shows orbits of system when $p = \frac{1}{2}$, and $p = \frac{51}{100}$ respectively.

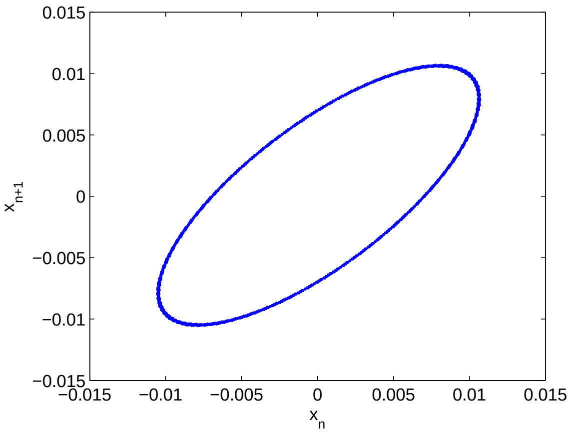

Figure 2: Orbit of system in Ex.2. with $p = \frac{1}{2}$.

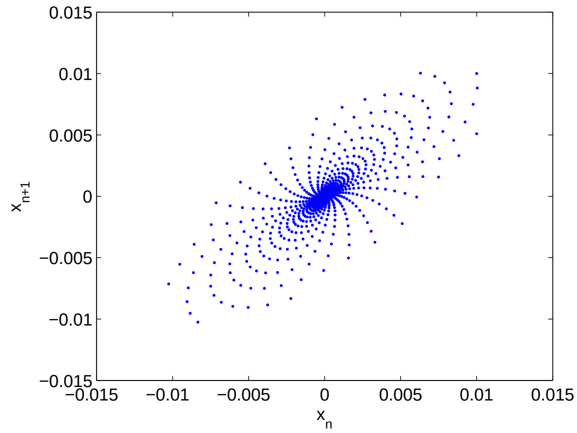

Figure 3: Orbit of system in Ex.2. with $p = \frac{51}{100}$.

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