Wave-breaking for the Weakly Dissipative Modified Camassa-Holm Equation

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Abstract: This paper is concerned with the local well-posedness and blow-up phenomena for the weakly dissipative modified Camassa-Holm equation. Blow-up criteria are established, and a new wave-breaking mechanism for solutions with certain initial profiles is described in detail.

Keywords: the weakly dissipative modified Camassa-Holm equation; Local well-posedness; Blow-up; Wave breaking rate; Blow-up criterion

1 Introduction

The modified Camassa-Holm equation

\[ m_t + \left( (u^2 - u_x^2) m \right)_x + \gamma u_x = 0, \] (1)

is a model of nonlinear shallow water waves, where the variable \( u(t,x) \) and \( m(t,x) \) represent, respectively, the velocity of the fluid and its potential density at time \( t \geq 0 \) in the spatial \( x \) direction, in which \( \gamma \) is a constant, and

\[ m = u - u_{xx}. \] (2)

The equation (1) arises from an intrinsic (arc-length preserving) invariant planar curve flow in Euclidean geometry and it can be regarded as a Euclidean-invariant version of the Camassa-Holm equation in [1]. It has the form of a modified Camassa-Holm equation with cubic nonlinearity. By Fuchssteiner [2] and Olver and Rosenau[3], it can be derived as a new integrable system by applying the general method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified Korteweg-de Vries equation. Later, it was obtained by Qiao [4] from the two-dimensional Euler equations. In [5] it was shown that equation (1) admits a Lax pair. In [1] it can be solved by the method of inverse scattering, its scaling limit equation satisfies the short-pulse equation.

The original Camassa-Holm (CH) equation

\[ m_t + um_x + 2u_xm + \gamma u_x = 0, \] (3)

where \( m \) is as above, (2), was derived from the Korteweg-de Vries equation by tri-Hamiltonian duality. Since the Camassa-Holm (CH) equation [6,7] was originally proposed as a model for surface waves, and has been studied extensively during the last twenty years because of its many remarkable properties: infinity of conservation laws and complete integrability [6,7,8], well-posedness and breaking waves, meaning solutions that remain bounded while their slope becomes unbounded in finite time [9,10,11,12,13]. Note that the nonlinearity in CH equation is quadratic. Since the modified CH equation has a cubic nonlinearity which the CH equation is only quadratic, one expects that the modified CH equation should also have peaked solitons and wave-breaking. In [1], the author obtained blow-up criteria for strong solutions, singularities which correspond to wave breaking and a sufficient condition for wave breaking of strong solutions in finite time.

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Two integrable CH-type equations with cubic nonlinearity have been discovered: One is the equation (1), and the second is the Novikov equation [14]

\[
m_t + u^2 m_x + \frac{3}{2} (u^2)_x \ m = 0.
\]

The integrability, peaked solitons, well-posedness and blow up phenomena to the Novikov equation have been studied extensively [14, 15, 16]. An alternative modified CH equation was introduced in [17].

In general, to avoid energy dissipation mechanisms in a real world is not so easy. Ott and Sudan [18] studied how the KdV equation was modified by the presence of dissipation and the effect of such dissipation on the solitary solution of the KdV equation, and Ghidaglia [19] investigated the long time behavior of solutions to the weakly dissipative KdV equation as a finite-dimensional dynamical system. Wu, Escher and Yin have investigated the blow-up phenomena, blow-up rate of the strong solutions of the weakly dissipative CH equation [20] and DP equation [21]. Inspired by the results mentioned above, in the paper, we are going to discuss the initial-value problem associated with the generalized weakly dissipative modified Camassa-Holm equation:

\[
m_t + \left( (u^2 - u_x^2) \right)_x \ m + \gamma u_x + \lambda m = 0, \quad m = u - u_{xx}, \quad t > 0, \quad x \in \mathbb{R},
\]

where \( \lambda > 0 \) is a constant.

The difference between Eq.(5) and Eq.(1) is that Eq.(5) has not the following conservation laws:

\[
\frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2) \ dx = 0, \quad \frac{d}{dt} \int_{\mathbb{R}} \left( u^4 + 2u^2 u_x^2 - \frac{1}{3} u_x^4 + 2\gamma u_x^2 \right) \ dx = 0,
\]

for all \( t \in [0, T) \), which play an important role in the study of Eq.(1).

In this paper, we will present the local well-posedness and blow-up phenomena for the weakly dissipative modified Camassa-Holm equation. The remainder of the paper is organized as follows. In section 2, we give the local posedness of Eq.(5). In section 3, its detailed blow-up criteria for strong solutions are established. It is shown that the solutions to the modified CH equation can only have singularities which correspond to wave breaking.

## 2 Local well-posedness

In this section, we will study the local posedness for the weakly dissipative modified Camassa-Holm equation on the entire line:

\[
\begin{cases}
    m_t + \left( (u^2 - u_x^2) \right)_x \ m + \gamma u_x + \lambda m = 0, \quad m = u - u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \quad \lambda > 0, \\
    u(0, x) = u_0(x), \quad x \in \mathbb{R}, \\
    u, m \to 0, \quad |x| \to \infty.
\end{cases}
\]

Substituting the formula for \( m \) in terms of \( u \) into the partial differential equation (1) results in the following fully nonlinear partial differential equation:

\[
u_t + u^2 u_x = -\left( 1 - \partial_x^2 \right)^{-1} \left( 2u^2 u_x + 2uu_x u_{xx} + \gamma u_x + \frac{4}{3} u_x^3 \right) - \lambda u + \frac{1}{3} u_x^3.
\]

Recall that

\[
u = \left( 1 - \partial_x^2 \right)^{-1} m = p \ast m, \quad p(x) = \frac{1}{2} e^{-|x|}
\]

and \( \ast \) denotes the convolution product on \( \mathbb{R} \), defined by

\[
(f \ast g)(x) = \int_{\mathbb{R}} f(y) g(x-y) \ dy.
\]

Using this identity, we can rewrite (1) as follows

\[
\begin{cases}
    u_t + u^2 u_x = -p \ast \left( 2u^2 u_x + 2uu_x u_{xx} + \gamma u_x + \frac{4}{3} u_x^3 \right) - \lambda u + \frac{1}{3} u_x^3, \\
    u(0, x) = u_0(x), \quad x \in \mathbb{R}.
\end{cases}
\]
Consider the abstract quasi-linear evolution equation:

\[
\frac{du}{dt} + Au = f(u), \quad t > 0; \quad u(0) = u_0.
\] (6)

**Proposition 1** *(See [22].)* Given the evolution equation (6), assume that the conditions (a), (b), and (c) hold. For a fixed \( v_0 \in Y \), there is a maximal \( T > 0 \) depending only on \( \|v_0\|_Y \) and a unique solution \( v \) to the abstract quasi-linear evolution equation (6) such that

\[
v = v(\cdot, v_0) \in C([0, T]; Y) \cap C^1([0, T]; X).
\]

Moreover, the map \( v_0 \rightarrow v(\cdot, v_0) \) is continuous from \( Y \) to

\[
C([0, T]; Y) \cap C^1([0, T]; X).
\]

The local well-posedness of Cauchy problem (5) with initial data \( u_0 \in H^s, s > \frac{3}{2} \), by applying Kato’s semigroup theory [9]. We can obtain the following local well-posedness result, as was done for the Camassa-Holm equation in [1,23].

**Theorem 2** Let \( m_0 = (1 - \partial_x^2) u_0 \in H^s(R) \) with \( s > \frac{1}{2} \), then there exists a time \( T > 0 \) such that the initial-value problem (1) has a unique strong solution \( m \in C([0, T]; H^s) \cap C'([0, T]; H^{s-1}) \) and the map \( m_0 \rightarrow m \) is continuous from a neighborhood of \( m_0 \) in \( H^s \) into \( C([0, T]; H^s) \cap C'([0, T]; H^{s-1}) \).

### 3 Wave-breaking criteria

In this section, we will establish criteria for the blow up of solutions to the Cauchy problem for the modified CH equation (1). We first introduce some 1-D Moser-type estimates, [24].

**Proposition 3** For \( s \geq 0 \), the following estimates hold:

\[
\|fg\|_{H^s(R)} \leq C \left( \|f\|_{H^s(R)} \|g\|_{L^\infty(R)} + \|f\|_{L^\infty(R)} \|g\|_{H^s(R)} \right),
\]

\[
\|f\partial_x g\|_{H^s(R)} \leq C \left( \|f\|_{H^{s+1}(R)} \|g\|_{L^\infty(R)} + \|f\|_{L^\infty(R)} \|\partial_x g\|_{H^s(R)} \right),
\]

where the \( C \)'s are constants independent of \( f \) and \( g \).

The following estimates for solutions to the one-dimensional transport equation have been needed in [24, 25].

**Lemma 4** *(See [2].)* Consider the one-dimensional linear transport equation

\[
\partial_x f + v \partial_x f = g, \quad f|_{t=0} = f_0.
\] (2)

Let \( 0 \leq \sigma < 1 \), and suppose that

\[
f_0 \in H^\sigma(R), \quad g \in L^1([0, T]; H^\sigma(R));
\]

\[
v_x \in L^1([0, T]; L^\infty(R)), \quad f \in L^1([0, T]; H^\sigma(R)) \cap C([0, T]; S'(R)),
\]

then \( f \in C([0, T]; H^\sigma(R)). \) More precisely, there is a constant \( C \) depending only on \( \sigma \) such that, for every \( 0 < t \leq T \),

\[
\|f(t)\|_{H^\sigma} \leq \|f_0\|_{H^\sigma} + C \int_0^t \|g(\tau)\|_{H^\sigma} d\tau + C \int_0^t \|f(\tau)\|_{H^\sigma} V'(\tau) d\tau
\]

(3)

and hence,

\[
\|f(t)\|_{H^\sigma} \leq e^{CV(t)} \left( \|f_0\|_{H^\sigma} + C \int_0^t \|g(\tau)\|_{H^\sigma} d\tau \right), V(t) = \int_0^t |\partial_x v(\tau)|_{L^\infty} d\tau.
\]

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Let us rewrite the modified CH equation (1) as a transport equation in terms of $m$ with the flow generated by $u^2 - u_x^2$

\[ m_t + (u^2 - u_x^2) m_x = -2u_xm^2 - \gamma u_x - \lambda m \]  

the transport equation theory makes it certain that, if the slope

\[ (u^2 - u_x^2)_x = 2u_xm \]

is bounded, the solution will remain regular, and can’t blow up in finite time.

In [1], the following blow-up criterion was obtained (with a slight modification).

**Theorem 5** Let $m_0 = (1 - \partial_x^2) u_0 \in H^s (\mathbb{R})$ be as in Theorem 2 with $s > \frac{1}{2}$. Let $m$ be the corresponding solution to (1). Assume $T_{m_0}^* > 0$ is the maximum time of existence. Then

\[ T_{m_0}^* < \infty \Rightarrow \int_0^{T_{m_0}^*} \| m (\tau) \|_{L^\infty}^2 d\tau = \infty. \]  

**Remark 6** For a strong solution $m = u - u_{xx}$ in Theorem 2, that is

\[ \frac{d}{dt} \int_\mathbb{R} (u^2 + u_x^2) dx = -2\int_\mathbb{R} \nabla u \nabla (u^2 + u_x^2) dx, \]

\[ \| u \|_{H^1}^2 \int_\mathbb{R} (u^2 + u_x^2) dx = \| u_0 \|_{H^1}^2 e^{-2\lambda T_{m_0}^*} \leq \| u_0 \|_{H^1}^2. \]  

The following blow-up criterion shows that the wave-breaking depends only on the infimum of $mu_x$:

**Theorem 7** let $m_0 \in H^s (\mathbb{R})$ be as in Theorem 2 with $s > \frac{1}{2}$. Then the corresponding solution $m$ to (1) blows up infinite time $T_{m_0}^* > 0$ if and only if

\[ \lim_{t \to T_{m_0}^*} \inf_{x \in \mathbb{R}} \{ (mu_x) (t, x) \} = -\infty. \]  

**Proof.** Since the existence time $T_{m_0}^*$ is independent of $s$, we only need to consider the case $s = 3$, which relies on a simple density argument. 

Multiplying equation (4) by $m$ and integrating over $\mathbb{R}$ with respect to $x$, and then integrating by parts, produces

\[ \frac{1}{2} \frac{d}{dt} \int_\mathbb{R} m^2 dx = -\int_\mathbb{R} (u^2 - u_x^2) mm_x dx - 2\int_\mathbb{R} u_xm^3 dx - \gamma \int_\mathbb{R} u_xm^3 dx - \lambda \int_\mathbb{R} m^2 dx \]

\[ = -\int_\mathbb{R} (mu_x + \lambda) m^2 dx \]

\[ = -\int_\mathbb{R} u_xm^3 dx - \lambda \int_\mathbb{R} m^2 dx. \]

By differentiating (4) once with respect to $x$, we have

\[ m_{xt} + (u^2 - u_x^2) m_{xx} = -3u_x (m^2)_x - 2u_{xx}m^2 - \gamma u_{xx} - \lambda m_x \]

\[ m_{xt} = -2u_{xx}m^2 - 6u_xmm_x - (u^2 - u_x^2) m_{xx} - \gamma u_{xx} - \lambda m_x \]

\[ = -2u^2 + 2m^2 - 6u_xmm_x - (u^2 - u_x^2) m_{xx} - \gamma u_{xx} - \lambda m_x. \]

Multiplying by $m_x$ and integrating over $\mathbb{R}$ with respect to $x$, leads to

\[ \frac{1}{2} \frac{d}{dt} \int_\mathbb{R} m^2 dx \]

\[ = -\int_\mathbb{R} (u^2 - u_x^2) m_x m_{xx} dx - 2\int_\mathbb{R} u_xm^2 m_x dx - 6\int_\mathbb{R} m_mm_x^2 dx + 2\int_\mathbb{R} m_x^3 m_x dx \]

\[ - \gamma \int_\mathbb{R} u_xm^2 m_x dx - \lambda \int_\mathbb{R} m^2 m_x^2 dx \]

\[ = \frac{1}{2} \int_\mathbb{R} (u^2 - u_x^2) m_x^2 dx - \frac{2}{3} \int_\mathbb{R} u_xm^3 m_x dx - 6\int_\mathbb{R} m_mm_x^2 dx - \gamma \int_\mathbb{R} (u^2 - u_x^2) m_x^2 dx - \frac{7}{2} \int_\mathbb{R} m_x m_{xx}^2 dx \]

\[ = -5\int_\mathbb{R} u_xm^2 m_x dx + \frac{2}{3} \int_\mathbb{R} u_xm^3 m_x dx - \lambda \int_\mathbb{R} m^2 m_x^2 dx, \]

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therefore,
\[ \frac{d}{dt} \int_{R} (m^2 + m_x^2) dx = -10 \int_{R} u_x mm^2 dx - \frac{2}{3} \int_{R} u_x m^3 dx - \lambda \int_{R} (m^2 + m_x^2) dx. \]

If \( mu_x \) is bounded from below on \([0, T^*_m) \times R\), i.e., there exists a positive constant \( C_1 > 0 \) such that \( mu_x \geq -C_1 \) on \([0, T^*_m) \times R\), then the above estimate implies
\[ \frac{d}{dt} \int_{R} (m^2 + m_x^2) dx \leq -10 \int_{R} (m^2 + m_x^2) dx - 2 \int_{R} (m^2 + m_x^2) dx = (10C_1 - 2\lambda) \int_{R} (m^2 + m_x^2) dx. \]

Applying Gronwall’s inequality then yields, for \( t \in [0, T^*_m) \),
\[ \| m(t) \|_{H^1}^2 \leq \int_{R} (m^2 + m_x^2) dx \leq e^{(10C_1 - 2\lambda)t} \| m_0 \|_{H^1}^2, \]
which ensures that the solution \( m(t, x) \) does not blow up in finite time.

On the other hand, if
\[ \liminf_{t \to T^*_m} \left[ \inf_{x \in R} (mu_x)(t, x) \right] = -\infty, \]
by theorem 5 for the existence of local strong solutions and the Sobolev embedding theorem, we infer that the solution will blow-up in finite time. ■

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