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An Application of Homotopy Analysis Method for Estimation the Diaphragm Deflection in MEMS Capacitive Microphone

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Abstract: In this article, homotopy analysis method (HAM) is introduced to analyze the deflection of poly-silicon diaphragm of Micro Electro Mechanical Systems (MEMS) capacitive microphone. The residual stresses in the material used to make the diaphragm change the vibrational characteristics of the microphone diaphragm and consequently influence the microphone's first resonant frequency, cutoff frequency and sensitivity. The most successful devices use poly-silicon as a diaphragm material, because of its residual stress is controllable by high temperature annealing after ion implantation by boron or phosphorous. We prove the existence and the uniqueness of the solution of considered problem using the theory of semi-group. Series solutions of the problem under consideration are developed by means of HAM and the recurrence relations are given explicitly. The numerical examples show the rapid convergence of the series constructed by this method to the exact solution. Moreover, this technique does not require any discretization, linearization or small perturbations. Test problems have been considered to ensure that HAM is accurate and efficient compared with the variational iteration method (VIM).

Keywords: Analytical solution; Diaphragm deflection; Capacitive microphone; Semi-group theory; Homotopy analysis method.

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1 Introduction

It is well known that most of the scientific phenomena are modeled using ordinary or partial differential equations. Analytical solutions of these equations may well describe the various phenomena in science and nature, such as vibrations, solitons and propagation with a finite speed. The homotopy analysis method is an analytical technique for solving nonlinear differential equations devised by Shi-Jun Liao in 1992 [1]. This method has been successfully applied to solve many types of nonlinear problems in science and engineering by many authors ([2]-[9]), and references therein. We aim in this work to effectively employ HAM to establish the estimation of diaphragm deflection in MEMS capacitive microphone ([10]-[12]). By the presented method, numerical results can be obtained with using a few iterations [1]. Moreover, HAM contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series [13]. Therefore, HAM handles linear and nonlinear problems without any assumption and restriction.

The paper has been organized as follows. In section 2, the deflection of diaphragm in differential form is presented. In section 3, some notations and preliminaries results concerning semigroup are introduced. In section 4, the existence and the uniqueness of the solution of the boundary value problem are proved. In section 5, the basic idea of homotopy analysis method is described. In section 6, applying HAM to analyze the deflection of polysilicon diaphragm of micro-electro mechanical systems capacitive microphone is presented. In section 7, the simulation results is given. Also, in section 8, the convergence of the exact solution is illustrated. Discussion and conclusions are presented in section 9.

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2 Deflection of diaphragm in differential form

The performance of the microphone depends on the size, stress and deflection of the diaphragm. The diaphragm deflection u can be approximated by the following differential equation ([10], [11])

$$-\alpha\nabla^4 u + \beta\nabla^2 u = \rho\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where α, β are the flexural rigidity, tensile force per unit length, and ρ is the density (mass per unit area of the diaphragm), respectively. ∇^2 is the Laplacian operator and ∇^4 indicates to $\nabla^2(\nabla^2)$. For the first fundamental mode, we can assume the deflection of the square diaphragm is [10]

$$u(x, y, t) \approx A\sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{\pi y}{d}\right)e^{-2\pi t}, \tag{2}$$

where d is the diaphragm width. Substitution of Eq.(2) into Eq.(1) yields the first resonant frequency for the diaphragm [10]

$$f_{res} = \sqrt{\frac{1}{\rho}\left(\frac{\alpha\pi^2}{d^4} + \frac{\beta}{2d^2}\right)}. \tag{3}$$

3 Notations and preliminaries results concerning semigroup

One of our aims in this paper is to study the existence and the uniqueness for solution of BVP (1). We recall the basic definitions and the most important properties of semi-groups theory [14]. These recalls are only intended to fix some notations and references and are confined to what will be useful in the sequel. For what concerns the results on semi-groups we refer to [15]. We collect firstly the known results concerning the theory of semi-group in the following definitions.

Definition 1 *A family of linear, continuous operators $G(t)_{t\geq 0} \subset \pounds(B, B)$, B is a Banach space, is called a semigroup if*

$$||G(t)|| \leq M(t), \qquad G(0) = 1, \qquad G(t_1 + t_2) = G(t_1).G(t_2).$$

It is said to be strongly continuous if for each $v \in B$ the function $t \to G(t)v$ is continuous in $[0, \infty]$.

Definition 2 *An operator A from B to B is said to be generator of semigroup $G(t)$ if it in the following form*

$$Av = \lim_{t\to 0}\frac{G(t)v - v}{t},$$

whose domain $D(A)$ is the set of the elements v of B which the right hand side exists.

Definition 3 *The semigroup $G(t)$ is called contraction semigroup if $||G(t)|| \ll 1$.*

Definition 4 *An operator A in a Hilbert space H is said to be an accretive (resp. dissipative) operator if*

$$Re(Av, v) \gg 0(resp. \ll 0).$$

Proposition 1 *If A is a generator of semigroup $G(t)$, then $D(A)$ is dense in B; moreover, if $v \in D(A)$, then $G(t)v \in D(A)$ for $t \geq 0$ and it is a continuously differential function. It satisfies*

$$\frac{d}{dt}(G(t)v) = A(G(t)v) = G(t)Av, \qquad \forall\ t \geq 0.$$

This Proposition shows that if $v \in D(A)$, then $G(t)v$ is a solution of the following evolution initial value problem

$$\frac{d}{dt}u(t) = Au(t), \qquad\qquad u(0) = v. \tag{4}$$

In fact, semigroups are tool for the resolution of problem of the form (4), in fact, if $G(t)$ is known, $G(t)v$ is defined for $v \in B$, then $G(t)v$ is the limit of solutions of (4) with initial value in $D(A)$. It is natural to define "generalized solution" of (4) with $u(0) = v \in B$ by $G(t)v$. It is important to have necessary and sufficient conditions for an operator A to be the generator of a semigroup G.

Theorem 2 *(Hille-Yosida) If $G(t)$ is a continuous semigroup on B, with*

$$||G(t)|| \geq Me^{\omega t},$$

then, its generator A satisfies

[i] $D(A)$ is dense in B, A is closed;

[ii] The semi-infinite interval $\lambda > \omega$, (λ real) belongs to the resolvent set of A.

Reciprocally, if A is an operator in the Banach space B, satisfying [i] and [ii], it is the generator of a continuous semigroup $G(t)$.

Theorem 3 *(Hille-Yosida) If $G(t)$ is a continuous semigroup of contractions, its generator A satisfies*

[i] $D(A)$ is dense in B, A is closed;

[ii] The semi-infinite interval $\lambda > \omega$, (λ real) belongs to the resolvent set of A. Reciprocally, if A is an operator in the Banach space B, satisfying [i] and [ii], it is the generator of contraction semigroup $G(t)$ in B.

Theorem 4 *(Lumer-Philips) Let A be a linear operator in Hilbert space H with domain $D(A)$ dense in H. Then*

[i] If A is the generator of a contraction semi-group in H, then A is dissipative and the range $R(\lambda - A)$ of $\lambda - A$ is the whole space H for all $\lambda > 0$;

[ii] If A is dissipative and there exists $\lambda > 0$ such that the range of $\lambda - A$ is the whole space H, then A is the generator of contraction semigroup in H.

An important property of semigroup

If A is the generator of continuous semigroup $G(t)$ in B one may consider the nonhomogeneous initial value problem

$$\frac{du}{dt} = Au + f, \qquad u(0) = v, \tag{5}$$

where f is a continuous function of t with values in B. Then, the unique generalization solution of (5) with $v \in B$ for $t \geq 0$ is given by

$$u(t) = G(t)v + \int_0^t G(t-s)f(s)ds, \qquad t \geq 0. \tag{6}$$

This is a continuous function of t with values in B. Moreover, if $v \in D(A)$ and $f(t)$ is continuously differentiable with values in B and $u(t) \in D(A)$ for $t \geq 0$. Then we obtain the classical solution of (5).

4 Existence and uniqueness of the solution of the boundary value problem (1)

The boundary value problem (1) can be rewritten in the following form

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{-\alpha}{\rho}\right)\nabla^4 u + \left(\frac{\beta}{\rho}\right)\nabla^2 u, \tag{7}$$

with the boundary conditions

$$u = \frac{\partial u}{\partial n} = 0, \qquad u \in \partial\Omega, \tag{8}$$

and the following initial conditions

$$u(x,y,0) = \sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{\pi y}{d}\right), \qquad \frac{\partial u(x,y,0)}{\partial t} = 0.$$

Let us introduce the following Sobolev spaces

$$H = L^2(\Omega); \quad V_0 = H_0^1(\Omega); \quad V_1 = H_0^2(\Omega),$$

and the bilinear forms

$$a_0(u,v) = \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dxdy, \quad a_1(u,v) = \int_\Omega \nabla^2 u \nabla^2 v\,dxdy, \quad a = \frac{\beta}{\rho}a_0 + a_1. \tag{9}$$

We note that all these forms verify the hypotheses of Lax-Milgram theorem [16]. The space V_0 is a Hilbert space with the scalar product $\left(\frac{\beta}{\rho}\right) a_0(u,v)$, let V denotes the space V_1 equipped with the scalar product $a = \left(\frac{\beta}{\rho}\right) a_0 + a_1$.
We have the following continuous and dense embedding

$$V \subset V_0 \subset H \equiv \acute{H} \subset \acute{V_0} \subset \acute{V}.$$

Let us define (as in first representation Riesz theorem [16]) the unbounded self-adjoint operators

$$A_0 = -\frac{\beta}{\rho}\nabla^2; \qquad A_1 = \nabla^4; \qquad A = -\frac{\beta}{\rho}\nabla^2 + \frac{\alpha}{\rho}\nabla^4,$$

associated to the forms $\frac{\beta}{\rho}a_0, \;\; a_1, \;\; a$, respectively.

Proposition 5 *The BVP (7) is equivalent to the classical equation* $\frac{d\vec{u}}{dt} = A\vec{u}$*, moreover the operator* A *is generator of a continuous semigroup in* $V \times H$.

Proof. First we suppose that $\vec{u} = (u_1, u_2)$, where $u_1 = u$, $u_2 = \frac{du}{dt}$, $A = \begin{pmatrix} 0 & \mathrm{I} \\ \text{-A} & 0 \end{pmatrix}$ then (8) can be written in the form

$$\frac{d\vec{u}}{dt} = A\vec{u}. \tag{10}$$

So, it suffices to prove that the conditions of Lumer-Philips are satisfied, that means A is generator of a contraction semigroup in $V \times H$.
1. $D(A)$ is dense in $V \times H$, on the other hand, we can write

$$Re(A\vec{u}, \vec{u}) = Re[a(u_2, u_1) + (-Au_1, u_2)_H] = 0,$$

then the operator A is dissipative.
2. The range of $(\lambda - A)$ is the all space $V \times H$ because for any given $\vec{w}$ from $V \times H$, we can find an element $\vec{u}$ from $D(A)$ verified

$$\lambda\vec{u} - A\vec{u} = \vec{w} \quad \text{for all real number}\, \lambda > 0. \tag{11}$$

Therefore, after some manipulations, we have

$$(A + \lambda^2)u_1 = w_2 + \lambda w_1. \tag{12}$$

The left hand side is an operator associated to a Hermitian and coercive form in V by the Lax-Milgram theorem, there exists u_1 from V solution of (12), then u_2 exists too. Consequently, we found $\vec{u}$ from $D(A)$ solution of (11). Problem (7) has an unique generalized solution as given in (6).
By application the Proposition 2, we are immediately in the framework of Eq.(6), then the Proposition is proved. ■

5 Basic idea of HAM

To illustrate the basic idea of HAM ([1]-[8]), we consider the following differential equation

$$N[u(\ell, t)] = 0, \tag{13}$$

where N is a linear operator for this problem, ℓ and t denote independent variables, $u(\ell, t)$ is an unknown function. For simplicity, we ignore all boundary and initial conditions, which can be treated in the similar way.

5.1 Zeroth-order deformation equation

Liao [1], constructed the so-called zeroth-order deformation equation

$$(1-q)\pounds[\phi(\ell,t;q)-u_0(\ell,t)] = q\hbar N[\phi(\ell,t;q)], \tag{14}$$

where $\pounds$ is an auxiliary linear operator, $u_0(\ell,t)$ is an initial guess, $\hbar \neq 0$ is an auxiliary parameter and $q \in [0,1]$ is the embedding parameter. Obviously, when $q=0$ and $q=1$, it holds, respectively

$$\phi(\ell,t;0) = u_0(\ell,t), \qquad \phi(\ell,t;1) = u(\ell,t). \tag{15}$$

Thus, as q increasing from 0 to 1, the solution $\phi(\ell,t;q)$ various from $u_0(\ell,t)$ to $u(\ell,t)$. Expanding $\phi(\ell,t;q)$ in Taylor series with respect to the embedding parameter q, one has

$$\phi(\ell,t;q) = u_0(\ell,t) + \sum_{m=1}^{\infty} u_m(\ell,t)q^m, \tag{16}$$

where

$$u_m(\ell,t) = \frac{1}{m!}\frac{\partial^m \phi(\ell,t;q)}{\partial q^m}\Big|_{q=0}. \tag{17}$$

Assume that the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are selected such that the series (16) is convergent at $q=1$, Then at $q=1$ and by (15), the series (16) becomes

$$u(\ell,t) = u_0(\ell,t) + \sum_{m=1}^{\infty} u_m(\ell,t). \tag{18}$$

5.2 The m^{th} order deformation equation

Define the vector

$$\vec{u}_n(\ell,t) = [u_0(\ell,t), u_1(\ell,t), ..., u_n(\ell,t)]. \tag{19}$$

Differentiating Eq.(14) m times with respect to the embedding parameter q, then setting q = 0 and dividing them by $m!$, finally using (17), we have the so-called m^{th}-order deformation equations

$$\pounds[u_m(\ell,t) - \delta_m u_{m-1}(\ell,t)] = \hbar\,\Re_m(\vec{u}_{m-1}), \tag{20}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(\ell,t;q)]}{\partial q^{m-1}}\Big|_{q=0}, \tag{21}$$

and

$$\delta_m = \begin{cases} 0, & m \leq 1; \\ 1, & m > 1. \end{cases} \tag{22}$$

Applying $\pounds^{-1}$ on both side of Eq.(20), we get

$$u_m(\ell,t) = \delta_m u_{m-1}(\ell,t) + \hbar\,\pounds^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{23}$$

In this way, it is easily to obtain u_m for $m \geq 1$, at N^{th} order, we have

$$u(\ell,t) \simeq \sum_{m=0}^{N} u_m(\ell,t). \tag{24}$$

When $N \to \infty$, we get an accurate approximation of the original Eq.(1). For the convergence of the proposed method we refer the reader to Liao ([1]-[8]). If Eq.(1) admits unique solution, then this method will produce the unique solution. If Eq.(1) has not possess unique solution, HAM will give a solution among many other (possible) solutions.

6 Applications the proposed method

We will apply HAM to estimate diaphragm deflection by solving Eq.(1) with three cases to illustrate the strength of the method and to establish the approximate solutions for these problems. We introduce a comparison with the VIM ([17]-[19]).

6.1 HAM for case study 1

The diaphragm compliance and diaphragm deflection depend on its flexural rigidity$\sim$ α, and tension, β. The flexural rigidity of the diaphragm is given by

$$\alpha = \frac{E\tau^3}{12(1-\delta^2)}, \tag{25}$$

where E is the Young's modulus of elasticity, τ is the diaphragm thickness and δ is the Poisson's ratio. We want to analyze diaphragm deflection ideally, when $E \to 0 \Rightarrow \alpha \to 0$. Eq.(1) when $\alpha \to 0$ converts to

$$\beta \nabla^2 u = \rho \frac{\partial^2 u}{\partial t^2}. \tag{26}$$

Now, to implement HAM for solving (26), we choose the linear operator

$$\pounds[\phi(x,y,t;q)] = \frac{\partial^2 \phi(x,y,t;q)}{\partial t^2}, \tag{27}$$

with the property, $\pounds[a_1 + a_2 t] = 0$, where a_1, a_2 are constants. We now define a linear operator as

$$N[\phi(x,y,t;q)] = \frac{\partial^2 \phi(x,y,t;q)}{\partial t^2} - \left(\frac{\beta}{\rho}\right) \nabla^2 \phi(x,y,t;q). \tag{28}$$

Using above definition, we construct the zeroth-order deformation equation

$$(1-q)\pounds[\phi(x,y,t;q) - u_0(x,y,t)] = q\hbar\, N[\phi(x,y,t;q)]. \tag{29}$$

For $q = 0$ and $q = 1$, we can write

$$\phi(x,y,t;0) = u_0(x,y,t), \qquad \phi(x,y,t;1) = u(x,y,t). \tag{30}$$

Thus, we obtain the m^{th} order deformation equations

$$\pounds[u_m(x,y,t) - \delta_m u_{m-1}(x,y,t)] = \hbar\, \Re_m(\vec{u}_{m-1}),$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial^2 \phi(x,y,t;q)}{\partial t^2} - \left(\frac{\beta}{\rho}\right) \nabla^2 \phi(x,y,t;q). \tag{31}$$

Now the solution of the m^{th} order deformation equations for $m \geq 1$ becomes

$$u_m(x,y,t) = \delta_m u_{m-1}(x,y,t) + \hbar\, \pounds^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{32}$$

This in turn gives the first few components of the approximate solution. We start with initial approximation

$$u_0(x,y,t) = \sin\left(\frac{\pi x}{d}\right) \sin\left(\frac{\pi y}{d}\right). \tag{33}$$

Since $\tilde{u} = u_0 + u_1 + u_2 + \ldots$. From the above equations (32), we can obtain u_n's as follows

$$u_1(x,y,t) = \left(\frac{\hbar\beta\pi^2 t^2}{\rho d^2}\right) \sin\left(\frac{\pi x}{d}\right) \sin\left(\frac{\pi y}{d}\right),$$

$$u_2(x,y,t) = \left(\frac{\hbar\beta\pi^2 t^2}{\rho d^2} + \frac{\beta\hbar^2\pi^2 t^2}{\rho d^2} + \frac{\beta^2\hbar^2\pi^4 t^4}{6\rho^2 d^4}\right) \sin\left(\frac{\pi x}{d}\right) \sin\left(\frac{\pi y}{d}\right),$$

$$u_3(x,y,t) = \Big(\frac{\beta\hbar\pi^2 t^2}{\rho d^2} + \frac{\beta\hbar^2\pi^2 t^2}{\rho d^2} + \frac{\beta^2\hbar^2\pi^4 t^4}{6d^4\rho^2} + \frac{\hbar^2\pi^2 t^2\beta(90d^4\rho^2(1+\hbar) + 15d^2\rho(1+2\hbar)\pi^2 t^2\beta + \hbar^2\beta^2\pi^4 t^4)}{90d^6\rho^3}\Big) \sin\left(\frac{\pi x}{d}\right) \sin\left(\frac{\pi y}{d}\right),$$

other components of the approximate solution can be obtained in the same manner.

6.2 HAM for case study 2

The tension, β, is determined by the residual stress of the diaphragm material, θ and τ, is the diaphragm thickness which satisfy

$$\beta = \theta\tau. \tag{34}$$

We want to investigate diaphragm deflection ideally, when $\theta \to 0 \Rightarrow \beta \to 0$. Eq.(1) when $\beta \to 0$ converts to

$$-\alpha\nabla^4 u = \rho\frac{\partial^2 u}{\partial t^2}. \tag{35}$$

Now, to implement HAM for solving (35), we choose the same linear operator defined in (27) and define a linear operator as

$$N[\phi(x,y,t;q)] = \frac{\partial^2\phi(x,y,t;q)}{\partial t^2} + \left(\frac{\alpha}{\rho}\right)\nabla^4\phi(x,y,t;q). \tag{36}$$

Using above definition, we construct the zeroth-order deformation equation

$$(1-q)\pounds[\phi(x,y,t;q) - u_0(x,y,t)] = q\hbar\, N[\phi(x,y,t;q)]. \tag{37}$$

For $q = 0$ and $q = 1$, we can write

$$\phi(x,y,t;0) = u_0(x,y,t), \qquad \phi(x,y,t;1) = u(x,y,t). \tag{38}$$

Thus, we obtain the m^{th} order deformation equations

$$\pounds[u_m(x,y,t) - \delta_m u_{m-1}(x,y,t)] = \hbar\,\Re_m(\vec{u}_{m-1}),$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial^2\phi(x,y,t;q)}{\partial t^2} + \left(\frac{\alpha}{\rho}\right)\nabla^4\phi(x,y,t;q). \tag{39}$$

Now the solution of the m^{th} order deformation equations for $m \geq 1$ becomes

$$u_m(x,y,t) = \delta_m u_{m-1}(x,y,t) + \hbar\,\pounds^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{40}$$

This in turn gives the first few components of the approximate solution. We use arbitrary an initial approximation that satisfies the initial conditions as (33). Substituting Eq.(33) into Eq.(40) and summarize it, we have

$$u_1(x,y,t) = \left(\frac{2\hbar\alpha\pi^4t^2}{\rho d^4}\right)\sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{\pi y}{d}\right),$$

$$u_2(x,y,t) = \left(\frac{2\hbar\alpha\pi^4t^2}{\rho d^4} + \frac{2\hbar^2\alpha^2\pi^8t^4}{3\rho^2d^8} + \frac{2\hbar\alpha\pi^4t^2}{\rho d^4}\right)\sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{\pi y}{d}\right),$$

$$u_3(x,y,t) = \Big(\frac{2\hbar\alpha\pi^4t^2}{\rho d^4} + \frac{2\hbar^2\alpha^2\pi^8t^4}{3\rho^2d^8} + \frac{2\hbar\alpha\pi^4t^2}{\rho d^4} + \frac{2\alpha\hbar^2\pi^4t^2(45d^8\rho^2(1+\hbar) + 15d^4\rho\alpha(1+2\hbar)\pi^4t^2 + 2\alpha^2\hbar\pi^8t^4)}{45\rho^3d^{12}}\Big)\sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{\pi y}{d}\right),$$

other components of the approximate solution can be obtained in the same manner.

6.3 HAM for case study 3

In this subsection, we estimate the diaphragm deflection at this practical substrates in the general case ($\alpha \neq \beta \neq 0$).
Now, to implement HAM for solving (1), we choose the same linear operator (27). We now define a linear operator as

$$N[\phi(x,y,t;q)] = \frac{\partial^2\phi(x,y,t;q)}{\partial t^2} + \left(\frac{\alpha}{\rho}\right)\nabla^4\phi(x,y,t;q) - \left(\frac{\beta}{\rho}\right)\nabla^2\phi(x,y,t;q). \tag{41}$$

Using above definition, we construct the zeroth-order deformation equation

$$(1-q)\pounds[\phi(x,y,t;q)-u_0(x,y,t)] = q\hbar N[\phi(x,y,t;q)]. \tag{42}$$

For $q = 0$ and $q = 1$, we can write

$$\phi(x,y,t;0) = u_0(x,y,t), \qquad \phi(x,y,t;1) = u(x,y,t). \tag{43}$$

Thus, we obtain the m^{th} order deformation equations

$$\pounds[u_m(x,y,t) - \delta_m u_{m-1}(x,y,t)] = \hbar\,\Re_m(\vec{u}_{m-1}),$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial^2\phi(x,y,t;q)}{\partial t^2} + \left(\frac{\alpha}{\rho}\right)\nabla^4\phi(x,y,t;q) - \left(\frac{\beta}{\rho}\right)\nabla^2\phi(x,y,t;q). \tag{44}$$

Now the solution of the m^{th} order deformation equations for $m \geq 1$ becomes

$$u_m(x,y,t) = \delta_m u_{m-1}(x,y,t) + \hbar\,\pounds^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{45}$$

This in turn gives the first few components of the approximate solution. We start with initial approximation (33). Since $\tilde{u} = u_0 + u_1 + ...$ From the above equations (45), we can obtain u_n's as follows

$$u_1(x,y,t) = \frac{\hbar}{\rho d^4}\left(2\alpha\pi^4 t^2 + \pi^2 t^2\beta\right)\sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{\pi y}{d}\right),$$

$$u_2(x,y,t) = \Big(\frac{\hbar}{\rho d^4}(2\alpha\pi^4 t^2 + \pi^2 t^2\beta) + \frac{\hbar^2}{6\rho^2 d^8}(r^2 t^2(2D0r^2 + a1^2T) \times (6d^4\rho + 2\alpha\pi^4 t^2 + d^2\pi^2 t^2\beta)\Big)\sin(\frac{\pi x}{d})\sin(\frac{\pi y}{d}),$$

other components of the approximate solution can be obtained in the same manner.

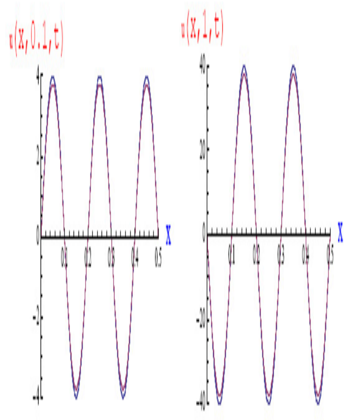
7 Simulation results

In order to comparison diaphragm deflection u which obtained by HAM with three conditions with exact solution in (1), we have numerical examples in $2D$. For each three described equations we have three examples which made by $y = 0.1,\ y = 1,\ y = 10$. In order to comparison exact solution with HAM, magnitude of exponential function in (1) considered one, $|e^{-12\pi t}| = 1$. In all examples we have some constant parameters $d = 0.1,\ \rho = 2,\ t = 0.5$.
Simulation 2d dimensional ($\alpha = 0$):

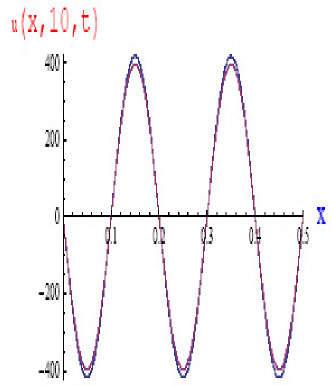
Fig. 1 presents the behavior of the approximate solution with $\hbar = 1.66$ and the exact solution at $y = 0.1,\ 1,\ 10$ at the final time $t = 0.5$. From this figure we can conclude that the solution by using the proposed method and the exact are in excellent agreement.

It is noted that our approximate solutions converges at $\hbar = 1.66$ and $\hbar = -2.425$. The explicit, analytic expression given by Eq.(32) contains the auxiliary parameter $\hbar$, which gives the convergence region and rate of approximation for HAM. However, the errors can be further be reduced by calculating higher order approximations. This proves that HAM is a very useful analytic method to get accurate analytic solutions to linear and strongly nonlinear problems.

Simulation 2d dimensional ($\beta = 0$):
Figure 2 shows comparison between the exact solution and HAM results, analysis of diaphragm deflection over x-axis where $\alpha = 1,\ A = -2.2 \times 10^{13}$.

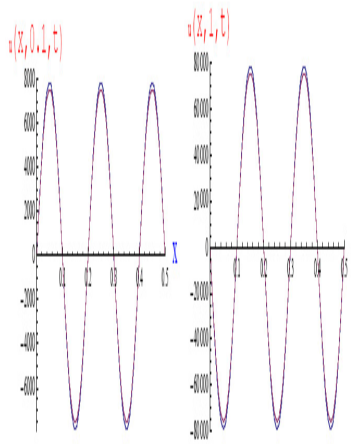


(a)

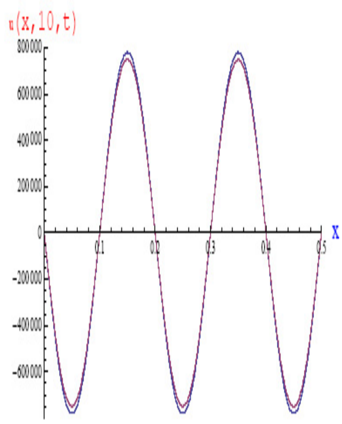


(b)

Figure 1: Comparison between the exact solution and HAM results when $\alpha = 0,\ \beta = 100,\ A = -1 \times 10^{10}$ and $d = 0.1,\ \rho = 2,\ t = 0.5$.



(a)

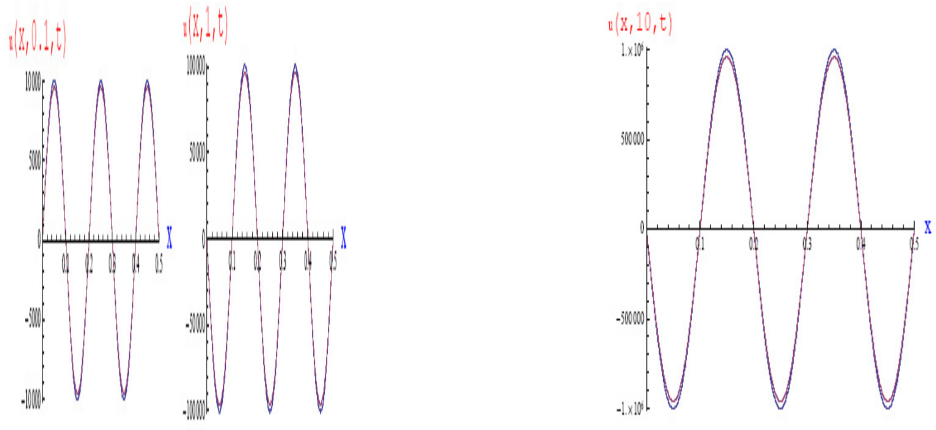


(b)

Figure 2: Comparison between the exact solution and HAM results when $\beta = 0,\ \alpha = 1,\ A = -2.2 \times 10^{13}$ and $d = 0.1,\ \rho = 2,\ t = 0.5$.

Simulation 2d dimensional (general case):

Figure 3 shows comparison between the exact solution and HAM results, analysis of diaphragm deflection over x-axis where $\alpha = 1,\ \beta = 100,\ A = -4.3 \times 10^{13}$.



(a) (b)

Figure 3: Comparison between the exact solution and HAM results when $\alpha = 1,\ \beta = 100,\ A = -4.3 \times 10^{13}$ and $d = 0.1,\ \rho = 2,\ t = 0.5$.

8 Convergence of the exact solution

Liao [1] showed that whatever a solution series converges it will be one of the solutions of considered problem. Liao ([1]-[8]) presented to be controlled by the auxiliary parameter the rate of convergence $\hbar$ the approximate solutions obtained by HAM. HAM and the VIM [11] solutions of two-dimensional diaphragm deflection in MEMS capacitive microphone when $\hbar = 1.66$ and $h = -2.55$.

9 Conclusions

In this Letter, we used HAM for obtaining the diaphragm deflection of MEMS capacitive microphone in first fundamental mode and using the PC-based Mathematica package for illustrated examples. By this method a rapid convergent series is produced. The results show that this method provides excellent approximations to the solution of related equation to diaphragm deflection with high accuracy and impression of influenced parameter in diaphragm deflection will be more sensible. Finally, it has been attempted to show the capabilities and facile applications of HAM in comparison with the exact solution. Results show a good agreement between the results obtained using HAM and VIM.

The numerical results showed that this method has very accuracy and reductions of the size of calculations compared with the VIM ([17]-[20]) and the homotopy perturbation method [21]. It may be concluded that this methodology is very powerful and efficient technique in finding exact solutions for wide class of problems. It is also worth noting to point out that the advantage of this methodology shows a fast convergence of the solutions by means of the auxiliary parameter, $\hbar = 1.66$, $\hbar = -2.55$. HAM is very easy applied to both differential equations and linear or nonlinear differential systems. The approximate solutions were almost identical to analytic solutions of the nonlinear evolution equations.

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