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Super Fractal Interpolation Functions

G. P. Kapoor[1], Srijanani Anurag Prasad[2] *
[1] Department of Mathematics and Statistics,Indian Institute of Technology Kanpur, India
[2] Department of Mathematics, Indian Institute of Science Education and Research Bhopal, India.
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Abstract: In the present work, the notion of Super Fractal Interpolation Function (SFIF) is introduced for finer simulation of the objects of nature or outcomes of scientific experiments that reveal one or more structures embedded in to another. In the construction of SFIF, an IFS is chosen from a pool of several IFSs at each level of iteration leading to implementation of the desired randomness and variability in fractal interpolation of the given data. Further, an expository description of our investigations on the integral, the smoothness and determination of conditions for existence of derivatives of an SFIF is given in the present work.

Keywords: Fractal; Interpolation; Super Fractals; Iteration; Attractor; Iterated Function Systems; Smoothness; Dimension

1 Introduction

Barnsley [1] introduced Fractal Interpolation Function (FIF) using the theory of Iterated Function System (IFS). Since then, a growing number of papers have been published showing relation between fractals and wavelets [2], fractal functions and Kiesswetter-like functions [3] and on fractal dimension [4, 5]. Later, Barnsley et. al. [6] extended the idea of FIF to produce more flexible interpolation functions called Hidden-variable FIF (HFIF) which were generally non-self affine. Dalla [7] found bounds on fractal dimension for the graphs of non-affine FIFs. In 1989, Barnsley and Harrington [8] constructed an IFS to show that a FIF can be integrated infinite times, giving rise to a hierarchy of smoother functions and developed results on differentiability of a FIF. Different kinds of FIFs like Hermite FIFs, Spline FIFs were constructed in [9–13] and various properties like smoothness of FIF, perturbations were discussed in [14–16]. The constructions of multivariable FIFs generated by using higher dimensional or recurrent IFSs are treated in [17–21].

Fractal Interpolation Function, constructed as attractor of a single Iterated Function System (IFS) by virtue o f self-similarity alone, is not rich enough to describe an object found in nature or output of a certain scientific experiment. The objects of nature generally reveal one or more structures embedded in to another. Similarly, the outcomes of several scientific experiments exhibit randomness and variation at various stages. Therefore, more than one IFSs are needed to model such objects. Barnsley [22–24] introduced the class of super fractal sets constructed by using multiple IFSs to simulate such objects. Massopust [25] constructed super fractal functions and V-variable fractal functions by joining pieces of fractal functions which are attractors of finite family of IFSs. However, for a data set arising from nature or a scientific experiment, a solution of fractal interpolation problem based on several IFSs has not been investigated so far. To fill this gap, the notion of Super Fractal Interpolation Function (SFIF) is introduced in the present work. The construction of SFIF requires the use of more than one IFS wherein, at each level of iteration, an IFS can be chosen from a pool of several IFSs. This approach is likely to ensure desired randomness and variability needed to facilitate better geometrical modeling of objects found in nature and results of certain scientific experiments. The construction of SFIF is followed in the present paper by the investigations of its smoothness, its integral and determination of conditions for existence of its derivatives.

The organization of the present paper is as follows: In Section 2, for a given finite set of data, the method of construction of a Super Fractal Interpolation Function (SFIF) is developed. At each level of iteration, an IFS is chosen from a pool of IFS in our construction of SFIF. For a sample interpolation data, a computational model of SFIF, illustrating the construction method given in Section 2, is presented in Section 3. The fractal dimension and average fractal distance are computed for various SFIFs constructed in this section. Finally, in Section 4, it is found that for an SFIF passing through

*Corresponding author. *E-mail address*: janani@iiserb.ac.in

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a given interpolation data, its integral is also an SFIF, albeit for a different interpolation data. An expository description of smoothness of an SFIF and conditions for the existence of derivatives of an SFIF is also given in this section.

2 Construction of an SFIF

The notion of Super Fractal Interpolation Function (SFIF) is introduced in this section via its construction based on more than one IFS.

A hyperbolic Iterated Function System (IFS), denoted by $\{X; \omega_n, n = 1, 2, \ldots, N\}$, consists of a metric space X together with a finite set of contraction mappings $\omega_n : X \to X$ with contractivity factors $s_n, 0 \leq s_n < 1$, satisfying $d(\omega_n(x), \omega_n(y)) \leq s_n d(x, y)$, $n = 1, 2, \ldots, N$ with respect to metric d on X. The contractivity factor of the IFS is defined as $s = \max\{s_n : n = 1, 2, \ldots, N\}$. It is known [26] that for a hyperbolic IFS $\{X; \omega_n, n = 1, 2 \ldots N\}$, the set valued Hutchinson map $W : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by $W(A) = \bigcup_{n=1}^{N} \omega_n(A)$ is a contraction map with contractivity factor s. Thus, by the Banach fixed point theorem, there exists a G in $\mathcal{H}(X)$ such that $W(G) = G$ which is called the attractor of IFS. For developing the notion of Super Fractal Interpolation Function (SFIF) via its construction, let $x_0 < x_1 < \ldots < x_N$ and $I = [x_0, x_N] \subset \mathbb{R}$. Consider the set $S_0 = \{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, \ldots, N\}$ of the given interpolation data. For $k = 1, \ldots, M$, $M > 1$ and $n = 1, \ldots, N$, let the functions $\omega_{n,k} : I \times \mathbb{R} \to I \times \mathbb{R}$ be defined by

$$\omega_{n,k}(x, y) = (L_n(x), G_{n,k}(x, y)) \text{ for all } (x, y) \in I \times \mathbb{R} \tag{1}$$

where, the contractive homeomorphisms $L_n : I \to I_n = [x_{n-1}, x_n]$ are given by

$$L_n(x) = a_n x + b_n = \frac{(x_n - x_{n-1})x + (x_N x_{n-1} - x_0 x_n)}{(x_N - x_0)} \tag{2}$$

and the functions $G_{n,k} : I \times \mathbb{R} \to \mathbb{R}$ defined by

$$G_{n,k}(x, y) = e_{n,k}x + \gamma_{n,k}y + f_{n,k} \tag{3}$$

satisfy the join-up conditions

$$G_{n,k}(x_0, y_0) = y_{n-1} \quad \text{and} \quad G_{n,k}(x_N, y_N) = y_n. \tag{4}$$

Here, $\gamma_{n,k}$ are free parameters chosen such that $|\gamma_{n,k}| < 1$ and $\gamma_{n,k} \neq \gamma_{n,l}$ for $k \neq l$. By (4), it is observed that $\omega_{n,k}$ are continuous functions. The Super Iterated Function System (SIFS) that is needed to construct SFIF corresponding to the set of given interpolation data $S_0 = \{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1 \ldots, N\}$ is now defined as the pool of IFS

$$\Big\{\{\mathbb{R}^2; \omega_{n,k} : n = 1, \ldots, N\},\ k = 1, \ldots, M\Big\} \tag{5}$$

where, the functions $\omega_{n,k}$ are given by (1).

To introduce a SFIF associated with SIFS (5), let $\{W_k : \mathcal{H}(\mathbb{R}^2) \to \mathcal{H}(\mathbb{R}^2), k = 1, \ldots, M\}$, be a collection of continuous functions defined by $W_k(G) = \bigcup\limits_{n=1}^{N} \omega_{n,k}(G)$ where, $\omega_{n,k}(G) := \{\omega_{n,k}(x, y), (x, y) \in G\}$. Since, $h(W_k(A), W_k(B)) \leq \max\limits_{1 \leq n \leq N} |\gamma_{n,k}|\ h(A, B)$, where h is Hausdorff metric on $\mathcal{H}(\mathbb{R}^2)$, $\{\mathcal{H}(\mathbb{R}^2);\ W_1, \ldots, W_M\}$ is a hyperbolic IFS. Hence, by Banach fixed point theorem, there exists an attractor $\mathcal{A} \in \mathcal{H}(\mathcal{H}(\mathbb{R}^2))$.

Let Λ be the code space on M natural numbers $1, 2, \ldots, M$. For $\sigma = \sigma_1\sigma_2 \ldots \sigma_k \ldots \in \Lambda$, define the function $\phi : \Lambda \to \mathcal{H}(\mathbb{R}^2)$ by

$$\phi(\sigma) = \lim_{k \to \infty} W_{\sigma_k} \circ W_{\sigma_{k-1}} \circ \ldots \circ W_{\sigma_1}(G),\ G \in \mathcal{H}(\mathbb{R}^2), \tag{6}$$

where the limit is taken with respect to the Hausdorff metric. It is shown that [27] $\phi(\sigma)$ exists, belongs to $\mathcal{A}$ and is independent of $G \in \mathcal{H}(\mathbb{R}^2)$. Also, the function ϕ is onto and continuous [27]. In the construction of SFIF, for a $\sigma = \sigma_1\sigma_2 \ldots \in \Lambda$, let the action of SIFS (5) at the iteration level j be defined by $S_j = W_{\sigma_j}(S_{j-1})$, where S_0 is the set of given interpolation data. It is easily seen that the set,

$$G_\sigma \equiv \phi(\sigma) = \lim_{k \to \infty} W_{\sigma_k} \circ \ldots \circ W_{\sigma_1}(S_0) = \lim_{k \to \infty} S_k, \tag{7}$$

where the limit is taken with respect to the Hausdorff metric, is the attractor of SIFS (5) for a fixed $\sigma \in \Lambda$. The following theorem shows that G_σ is the graph of a continuous function g_σ.

Theorem 1 *Let G_σ be the attractor of SIFS (5) for $\sigma = \sigma_1\sigma_2\ldots\sigma_k\ldots \in \Lambda$. Then, G_σ is graph of a continuous function $g_\sigma : I \to \mathbb{R}$ such that $g_\sigma(x_n) = y_n$ for all $n = 0,\ldots,N$.*

Proof. Let g_0 be a function whose graph is S_0. Then, the set S_k, $k \geq 1$, is graph of the function g_{σ_k}, where $g_{\sigma_k}(x) = G_{i_k,\sigma_k}\big(L_{i_k}^{-1}(x), g_{\sigma_{k-1}}(L_{i_k}^{-1}(x))\big)$. It is easily seen that $g_{\sigma_k}(x) = G_{i_k,\sigma_k}\Bigg(L_{i_k}^{-1}(x), G_{i_{k-1},\sigma_{k-1}}\Big(.,\ldots G_{i_1,\sigma_1}(L_{i_1}^{-1}\circ\ldots\circ L_{i_k}^{-1}(x), g_0(L_{i_1}^{-1}\circ\ldots\circ L_{i_k}^{-1}(x)))\ldots\Big)\Bigg)$. Therefore, it follows by (7) that the set G_σ is graph of the function $g_\sigma = \lim_{k\to\infty} g_{\sigma_k}$.

For proving the continuity of the function g_σ, consider $\tau_1^*\tau_2^*\ldots\tau_j^*\ldots \in \Lambda$ where $\tau_j^* \neq 1$ for some $j \in \mathbb{N}$ and $\tau_i^* = 1$ for $i \in \mathbb{N}$ and $i \neq j$. We first show that G_{τ^*} is graph of a continuous function g_{τ^*}. If not, then $G_{\tau^*} = \phi(\tau^*)$ is graph of a function g_{τ^*} that is not continuous so that there exist a $\delta_1 > 0$ such that whenever $u_1, u_2 \in I$ and $|u_1 - u_2| < \delta_1$,

$$|g_{\tau^*}(u_1) - g_{\tau^*}(u_2)| > \epsilon. \tag{8}$$

It is known that [1], for $\tau = \bar{1} \in \Lambda$, $G_\tau = \phi(\tau)$, with ϕ defined by (6), is graph of a continuous function $g_\tau : I \to \mathbb{R}$ such that $g_\tau(x_n) = y_n$, $n = 0,1,\ldots,N$. Consequently, there exists a $\delta_2 > 0$ such that $|u_1 - u_2| < \delta_2$ implies $|g_\tau(u_1) - g_\tau(u_2)| < \frac{\epsilon}{3}$. Also, since ϕ is a continuous map, there exists $\delta_3 > 0$ such that, for τ and τ^* satisfying $d_c(\tau,\tau^*) = \frac{|\tau_j - \tau_j^*|}{(M+1)^j} < \delta_3$, the Hausdorff distance between $G_\tau = \phi(\tau)$ and $G_{\tau^*} = \phi(\tau^*)$ is less than $\frac{\epsilon}{3}$ which implies that $\max_{u\in I}|g_\tau(u) - g_{\tau^*}(u)| < \frac{\epsilon}{3}$. Thus, for $\delta = \min(\delta_1,\delta_2,\delta_3)$ and u_1, u_2 satisfying $|u_1 - u_2| < \delta$, $|g_{\tau^*}(u_1) - g_{\tau^*}(u_2)| \leq |g_{\tau^*}(u_1) - g_\tau(u_1)| + |g_\tau(u_1) - g_\tau(u_2)| + |g_\tau(u_2) - g_{\tau^*}(u_2)| < \epsilon$, a contradiction to (8). Hence, G_{τ^*} is graph of continuous function g_{τ^*}.

Now, consider the sequence $\sigma_n = \sigma_{1,n}\sigma_{2,n}\ldots\ldots$ with $\sigma_{j,n} = \sigma_j$ for $j \leq n$ and $\sigma_{j,n} = 1$ for $j > n$. It is easily seen that as n tends to infinity, σ_n tends to σ with respect to the metric d_c. Using the arguments of previous paragraph inductively, it follows that $G_{\sigma_n} = \phi(\sigma_n)$ is graph of a continuous function g_{σ_n} defined on I. Let $G_\sigma = \phi(\sigma)$ be graph of a function g_σ. By continuity of ϕ, G_{σ_n} tends to G_σ with respect to Hausdorff metric h as $n \to \infty$, which implies that g_{σ_n} tends to g_σ with respect to Maximum metric as $n \to \infty$. Hence, there exist an $\epsilon > 0$ such that $\max_{u\in I}|g_{\sigma_n}(u) - g_\sigma(u)| < \frac{\epsilon}{3}$. Since g_{σ_n} is continuous on I, there exits a $\delta > 0$ such that $|u_1 - u_2| < \delta$ implies $|g_{\sigma_n}(u_1) - g_{\sigma_n}(u_2)| < \frac{\epsilon}{3}$. Therefore, it is easily seen that $|g_\sigma(u_1) - g_\sigma(u_2)| \leq |g_\sigma(u_1) - g_{\sigma_n}(u_1)| + |g_{\sigma_n}(u_1) - g_{\sigma_n}(u_2)| + |g_{\sigma_n}(u_2) - g_\sigma(u_2)| < \epsilon$ for $|u_1 - u_2| < \delta$ which implies that the function g_σ is continuous on I. This establishes that the attractor G_σ of SIFS (5) is the graph of continuous function g_σ. ■

Theorem 1 is instrumental in defining a SFIF associated with SIFS (5) as follows:

Definition 1 *The **Super Fractal Interpolation Function (SFIF)** for the given interpolation data $\{(x_i,y_i) : i = 0,1,\ldots,N\}$ is defined as the continuous function g_σ whose graph G_σ is the attractor of SIFS (5).*

Remark 2 *Consider the family of continuous functions $\mathcal{G} = \{f : I \to \mathbb{R}$ such that f is continuous, $f(x_0) = y_0$ and $f(x_N) = y_N\}$ with metric $d_{\mathcal{G}}(f,g) = \max_{x\in I}|f(x) - g(x)|$. Since $\mathcal{G}$ is a complete metric space, it is easily seen that, for a fixed $\sigma \in \Lambda$, Read-Bajraktarević operator $T : \Lambda \times \mathcal{G} \to \mathcal{G}$ defined as*

$$T(\sigma,g)(x) = \lim_{k\to\infty}\Bigg\{G_{i_k,\sigma_k}\Bigg(L_{i_k}^{-1}(x), G_{i_{k-1},\sigma_{k-1}}\Big(L_{i_{k-1}}^{-1}\circ L_{i_k}^{-1}(x), G_{i_{k-2},\sigma_{k-2}}\big(.,\ldots G_{i_1,\sigma_1}(L_{i_1}^{-1}\circ\ldots\circ L_{i_k}^{-1}(x), g(L_{i_1}^{-1}\circ\ldots\circ L_{i_k}^{-1}(x)))\ldots\big)\Big)\Bigg)\Bigg\}, \tag{9}$$

is a contraction map on $\mathcal{G}$ and so it has a unique fixed point in $\mathcal{G}$. It is observed that, SFIF g_σ satisfies $g_\sigma = T(\sigma,g_\sigma)$.

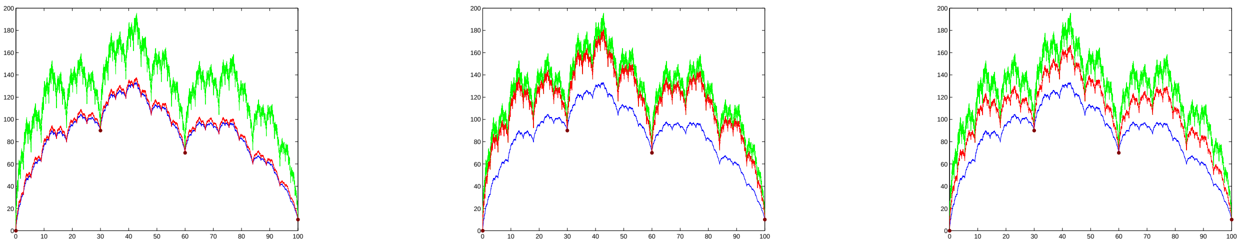
3 Computational Model of SFIF

Our method of construction developed in Section 2 is employed in the present section for generating various SFIF-s for a sample interpolation data $S_0 = \{(0,0),(30,90),(60,70),(100,10)\}$. For identifying the corresponding SIFS $\Big\{\{\mathbb{R}^2;\omega_{n,k} : n = 1,2,3\},\ k = 1,2\Big\}$, the maps $\omega_{n,k}$, $k = 1,2$ (c.f. (1)) are obtained by computing (c.f. Table 1) the values of a_i, b_i (c.f. (2)) and $e_{i,1}$, $f_{i,1}$; $e_{i,2}$ and $f_{i,2}$ (c.f. (4)) with $\gamma_{i,1} = 0.4$ and $\gamma_{i,2} = 0.6$ for $i = 1,2,3$.

Table 1: Computed Values of $a_i, b_i, e_{i,1}, f_{i,1}, e_{i,2}, f_{i,2}$, $i = 1, 2, 3$, for sample data S_0

	i=1	i=2	i=3
a	0.3	0.3	0.4
b	0	30	60
$e_{i,1}$	0.86	-0.24	-0.64
$f_{i,1}$	0	90	70
$e_{i,2}$	0.84	-0.26	-0.66
$f_{i,2}$	0	90	70

In the construction of SFIF for a $\sigma = \sigma_1\sigma_2 \ldots \in \Lambda$, the set $S_j = W_{\sigma_j}(S_{j-1}),\ j = 1, 2, \ldots$, representing the action of SIFS (5) at the iteration level j is computed. The SFIF $g_{\sigma_{(b)}}$ for $\sigma_{(b)} = \bar{1}$ (c.f. Figs. 1(a)- 1(c), blue curve) is constructed by the action of IFS $\{\mathbb{R}^2; \omega_{n,1}, n = 1, \ldots, N\}$ at every level of iteration. Similarly, SFIF $g_{\sigma_{(g)}}$ for $\sigma_{(g)} = \bar{2}$ (c.f. Figs. 1(a)- 1(c), green curve) is constructed by the action of IFS $\{\mathbb{R}^2; \omega_{n,2}, n = 1, \ldots, N\}$ at every level of iteration. The SFIF $g_{\sigma_{(r)}}$ for $\sigma_{(r)} = \overline{112}$ (c.f. Fig. 1(a), red curve) is constructed by the action of IFS $\{\mathbb{R}^2; \omega_{n,1}, n = 1, \ldots, N\}$ at j^{th} level of iteration if j is not divisible by 3 and by the action of IFS $\{\mathbb{R}^2; \omega_{n,2}, n = 1, \ldots, N\}$ if j is divisible by 3. Likewise, SFIF $g_{\sigma_{(r)}}$ for $\sigma_{(r)} = \overline{221}$ (c.f. Fig. 1(b), red curve) is constructed by the action of IFS $\{\mathbb{R}^2; \omega_{n,1}, n = 1, \ldots, N\}$ at j^{th} level of iteration if j is divisible 3 and otherwise by the action of IFS $\{\mathbb{R}^2; \omega_{n,2}, n = 1, \ldots, N\}$. Finally, SFIF $g_{\sigma_{(r)}}$ for $\sigma_{(r)} = \overline{12}$ (c.f. Fig. 1(c), red curve) is constructed by the action of IFS $\{\mathbb{R}^2; \omega_{n,1}, n = 1, \ldots, N\}$ at j^{th} level of iteration if j is not divisible by 2 and by the action of IFS $\{\mathbb{R}^2; \omega_{n,2}, n = 1, \ldots, N\}$ if j is divisible by 2.



(a) $\sigma_{(r)} = \overline{112}$ (b) $\sigma_{(r)} = \overline{221}$ (c) $\sigma_{(r)} = \overline{12}$

Blue Curve (—): $SFIF g_{\sigma_{(b)}}$ for $\sigma_{(b)} = \bar{1}$ in$(a), (b)$ and (c),

Green Curve (—): $SFIF g_{\sigma_{(g)}}$ for $\sigma_{(g)} = \bar{2}$ in$(a), (b)$ and (c),

Red Curve (—): $SFIF g_{\sigma_{(r)}}$ for $\sigma_{(r)} = \overline{112}$ in(a), for $\sigma_{(r)} = \overline{221}$ in (b) and for $\sigma_{(r)} = \overline{12}$ in (c)

Figure 1: SFIFs for $\sigma_{(b)} = \bar{1}$, $\sigma_{(g)} = \bar{2}$ and different choices of $\sigma_{(r)}$

The SFIFs $g_{\sigma_{(b)}}$ for $\sigma_{(b)} = \bar{1}$ and $g_{\sigma_{(g)}}$ for $\sigma_{(g)} = \bar{2}$ are in fact FIFs (c.f. Figs. 1(a)- 1(c), blue and green curves), since these are constructed with a single element of SIFS (5).

Heuristically, in terms of their fractal dimension [1], the graphs of SFIF $g_{\sigma_{(r)}}$ appear to fill more space in $\mathbb{R}^2$ than the graph of FIF $g_{\sigma_{(b)}}$ and less space in $\mathbb{R}^2$ than the graph of FIF $g_{\sigma_{(g)}}$. In fact, the fractal dimension of graphs of FIF $g_{\sigma_{(b)}}$ and $g_{\sigma_{(g)}}$ are computed as 1.3069 and 1.5199 respectively whereas the fractal dimension of SFIF $g_{\sigma_{(r)}}$ with $\sigma_{(r)} = \overline{112}$ (c.f. Fig. 1(a), red curve) is 1.3632, the fractal dimension of SFIF $g_{\sigma_{(r)}}$ with $\sigma_{(r)} = \overline{221}$ (c.f. Fig. 1(b), red curve) is 1.4572 and the fractal dimension of SFIF $g_{\sigma_{(r)}}$ with $\sigma_{(r)} = \overline{12}$ (c.f. Fig. 1(c), red curve) is 1.4182.

Further, for FIFs $g_{\sigma_{(b)}}$ and $g_{\sigma_{(g)}}$, the *average fractal distance* defined as $d_F(f, g) = \frac{1}{(b-a)} \left(\int_a^b |f(x) - g(x)|^2 dx \right)^{1/2}$ for the functions f and g, continuous on a closed interval $[a, b]$, is $d_F(g_{\sigma_{(b)}}, g_{\sigma_{(g)}}) = 0.297$. It is observed that (i) for

SFIF $g_{\sigma_{(r)}}$ with $\sigma_{(r)} = \overline{112}$, $d_F(g_{\sigma_{(b)}}, g_{\sigma_{(r)}}) = 0.022$ while $d_F(g_{\sigma_{(g)}}, g_{\sigma_{(r)}}) = 0.276$. So, if the data generating function is at one third average fractal distance from FIF $g_{\sigma_{(b)}}$, then SFIF $g_{\sigma_{(r)}}$ is a better approximation of the data generating function, since $g_{\sigma_{(r)}}$ is closer to $g_{\sigma_{(b)}}$ than $g_{\sigma_{(g)}}$ (c.f. Fig. 1(a)) i.e. $d_F(g_{\sigma_{(b)}}, g_{\sigma_{(r)}}) < d_F(g_{\sigma_{(g)}}, g_{\sigma_{(r)}})$. (ii) For SFIF $g_{\sigma_{(r)}}$ with $\sigma_{(r)} = \overline{221}$, $d_F(g_{\sigma_{(b)}}, g_{\sigma_{(r)}}) = 0.228$ while $d_F(g_{\sigma_{(g)}}, g_{\sigma_{(r)}}) = 0.071$. So, if the data generating function is at one third average fractal distance from FIF $g_{\sigma_{(g)}}$, then SFIF $g_{\sigma_{(r)}}$ is a better approximation of such data generating function, since $g_{\sigma_{(r)}}$ is closer to $g_{\sigma_{(g)}}$ than $g_{\sigma_{(b)}}$ (c.f. Fig. 1(b)) and (iii) for $g_{\sigma_{(r)}}$ with $\sigma_{(r)} = \overline{12}$, $d_F(g_{\sigma_{(b)}}, g_{\sigma_{(r)}}) = 0.138$ and $d_F(g_{\sigma_{(g)}}, g_{\sigma_{(r)}}) = 0.162$. So, if the data generating function lies in the middle of FIFs $g_{\sigma_{(b)}}$ and $g_{\sigma_{(g)}}$, then SFIF $g_{\sigma_{(r)}}$ (c.f. Fig. 1(c)) is a better approximation of such data generating function.

4 Integral and Derivative of SFIF

In this section, for an SFIF passing through a given interpolation data, its integral is shown to be also an SFIF, albeit for a different interpolation data. Further, in this section, the smoothness of SFIF is investigated in terms of its Lipschitz exponent and it is found that, in general, an SFIF may not be differentiable. This, as a natural follow up, led to determining in this section the conditions for existence of derivatives of an SFIF.

In order to study the integral of an SFIF, an SIFS

$$\Big\{\{\mathbb{R}^2;\ \omega_{n,k}(x,y) = (L_n(x), G_{n,k}(x,y)) : n = 1,\dots,N\},\ k = 1,\dots,M\Big\}, \tag{10}$$

associated with the data $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0,\dots,N\}$ is considered, where $L_n(x) = a_n x + b_n$ are given by (2) and the functions $G_{n,k}(x,y)$ defined by

$$G_{n,k}(x,y) = \gamma_{n,k} y + q_{n,k}(x), \quad n = 1,\dots,N. \tag{11}$$

satisfy the join up conditions given by (4). Here, $\gamma_{n,k}$ are free parameters chosen such that $|\gamma_{n,k}| < 1$ and $\gamma_{n,k} \neq \gamma_{n,l}$ for $k \neq l$ and $q_{n,k}(x)$ are continuous functions. Condition (4) ensures that there exits a unique attractor $G_\sigma \in \mathcal{H}(\mathbb{R}^2)$ of SIFS (10). By the arguments similar to those in the proof of Theorem 1, G_σ is graph of a continuous function g_σ.

The following notations [8] are needed in the sequel for tidy presentation of our results:

$$\left.\begin{aligned}
\hat{\gamma}_{n,k} &= a_n \gamma_{n,k} \\
\hat{y}_{N,k} &= \hat{y}_0 + \frac{\sum_{j=1}^{N} a_j \left[\int_{x_0}^{x_N} q_{j,k}(t)\, dt\right]}{1 - \sum_{j=1}^{N} a_j \gamma_{j,k}} \\
\hat{y}_{n,k} &= \hat{y}_0 + \sum_{j=1}^{n} a_j \left[\gamma_{j,k}(\hat{y}_{N,k} - \hat{y}_0) + \int_{x_0}^{x_N} q_{j,k}(t)\, dt\right] \\
\hat{q}_{n,k}(x) &= \hat{y}_{n-1,k} - a_n \gamma_{n,k} \hat{y}_0 + a_n \int_{x_0}^{x} q_{n,k}(t)\, dt
\end{aligned}\right\} \tag{12}$$

where, $\hat{y}_0$ is an arbitrary real number. To determine an interpolation data through which the integral of SFIF passes, let the functions $q_{n,k}(x)$ in (11) satisfy :

$$\frac{\sum_{j=1}^{N} a_j \int_{x_0}^{x_N} q_{j,k}}{1 - \sum_{j=1}^{N} a_j \gamma_{j,k}} = \frac{\sum_{j=1}^{N} a_j \int_{x_0}^{x_N} q_{j,l}}{1 - \sum_{j=1}^{N} a_j \gamma_{j,l}} \neq 1 \quad \text{for } k \neq l,\ k,l = 1,\dots,M. \tag{13}$$

For example, for $a_j = \frac{1}{N}$, $\gamma_{j,k} = \gamma_k$ and $q_{j,k} = (1-\gamma_k)(e_j x + f_j)$ for $j = 1,\dots,N$, the condition (13) is satisfied. Then, $\hat{y}_{i,k} = \hat{y}_{i,l} = \hat{y}_i$ for $i = 0,\dots,N; k,l = 1,\dots,M$ and $\hat{y}_N - \hat{y}_0 \neq 1$.

The SIFS associated with the data $\{(x_i, \hat{y}_i) \in \mathbb{R}^2 : i = 0,\dots N\}$ is now defined as the pool of IFS

$$\Big\{\{\mathbb{R}^2;\ \hat{\omega}_{n,k}(x,y) = (L_n(x), \hat{G}_{n,k}(x,y)) :\ n = 1,\dots,N\}, k = 1,\dots,M\Big\} \tag{14}$$

where, the functions

$$\hat{G}_{n,k}(x,y) = \hat{\gamma}_{n,k} y + \hat{q}_{n,k}(x) \tag{15}$$

satisfy the join-up conditions $\hat{G}_{n,k}(x_0, \hat{y}_0) = \hat{y}_{n-1}$ and $\hat{G}_{n,k}(x_N, \hat{y}_N) = \hat{y}_n$. These join-up conditions ensure that there exits a unique attractor $\hat{G}_\sigma \in \mathcal{H}(\mathbb{R}^2)$ of SIFS (14). The following theorem shows that the integral of SFIF is also an SFIF albeit for interpolation data $\{(x_i, \hat{y}_i) \in \mathbb{R}^2 : i = 0, 1, \ldots, N\}$.

Theorem 3 *For the interpolation data* $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, \ldots N\}$, *let* g_σ *be SFIF corresponding to SIFS* (10) *for* $\sigma \in \Lambda$. *Then, the integral*

$$\hat{g}_\sigma(x) = \hat{y}_0 + \int_{x_0}^{x} g_\sigma(t)\, dt \tag{16}$$

is SFIF associated with SIFS (14) *for the interpolation data* $\{(x_i, \hat{y}_i) : i = 0, \ldots, N\}$.

Proof. Using (16) and (9), it is observed that,

$$\begin{aligned}\hat{g}_\sigma(L_{i_k} \circ \ldots \circ L_{i_1}(x)) &= \hat{g}_\sigma(L_{i_k} \circ \ldots \circ L_{i_1}(x_0)) + \left(\prod_{j=1}^{k} a_{i_j}\gamma_{i_j,\sigma_j}\right)\Big(\hat{g}_\sigma(x) - \hat{y}_0\Big) \\ &\quad + \sum_{p=1}^{k}\left(\prod_{j=p+1}^{k} a_{i_j}\gamma_{i_j,\sigma_j}\right) a_{i_p} \int_{L_{i_{p-1}}\circ\ldots\circ L_{i_1}(x_0)}^{L_{i_{p-1}}\circ\ldots\circ L_{i_1}(x)} q_{i_p,\sigma_p}(t).\end{aligned} \tag{17}$$

Also, by (16),

$$\begin{aligned}\hat{g}_\sigma(L_{i_k} \circ \ldots \circ L_{i_1}(x_0)) &= \hat{y}_0 + \sum_{p=1}^{k}\left(\prod_{j=p+1}^{k} a_{i_j}\gamma_{i_j,\sigma_j}\right)\Bigg\{\sum_{l=1}^{i_p-1} a_l\Bigg[\gamma_{l,\sigma_p}(\hat{y}_N - \hat{y}_0) \\ &\quad + \int_{x_0}^{x_N} q_{l,\sigma_p}(t)\, dt\Bigg] + a_{i_p}\int_{x_0}^{L_{i_{p-1}}\circ\ldots\circ L_{i_1}(x_0)} q_{i_p,\sigma_p}(t)\Bigg\}.\end{aligned}$$

The above identity and (12) give

$$\begin{aligned}\hat{g}_\sigma(L_{i_k} \circ \ldots \circ L_{i_1}(x_0)) &= \left(\prod_{j=1}^{k} a_{i_j}\gamma_{i_j,\sigma_j}\right)\hat{y}_0 \\ &\quad + \sum_{p=1}^{k}\left(\prod_{j=p+1}^{k} a_{i_j}\gamma_{i_j,\sigma_j}\right)\Bigg\{\hat{y}_{i_p-1} - a_{i_p}\gamma_{i_p,\sigma_p}\,\hat{y}_0 + a_{i_p}\int_{x_0}^{L_{i_{p-1}}\circ\ldots\circ L_{i_1}(x_0)} q_{i_p,\sigma_p}(t)\Bigg\}.\end{aligned} \tag{18}$$

Now, substituting the value of $\hat{g}_\sigma(L_{i_k} \circ \ldots \circ L_{i_1}(x_0))$ from (18) in (17), it follows that

$$\hat{g}_\sigma(L_{i_k} \circ \ldots \circ L_{i_1}(x)) = \hat{G}_{i_k,\sigma_k}\Bigg(L_{i_{k-1}} \circ \ldots \circ L_{i_1}(x), \hat{G}_{i_{k-1},\sigma_{k-1}}\Big(., \ldots \hat{G}_{i_2,\sigma_2}\big(L_{i_1}(x), \hat{G}_{i_1,\sigma_1}(x, \hat{g}_\sigma(x))\big)\ldots\Big)\Bigg).$$

Thus, $\hat{g}_\sigma$ is SFIF associated with SIFS (14). ■

Remark 4 *Suppose* $\hat{y}_N$ *is given and* (12) *is defined as*

$$\left.\begin{aligned}\hat{\gamma}_{n,k} &= a_n\gamma_{n,k} \\ \hat{y}_{0,k} &= \hat{y}_N - \frac{\sum_{j=1}^{N} a_j\left[\int_{x_0}^{x_N} q_{j,k}(t)\, dt\right]}{1 - \sum_{j=1}^{N} a_j\gamma_{j,k}} \\ \hat{y}_{n,k} &= \hat{y}_N - \sum_{j=n+1}^{N} a_j\left[\gamma_{j,k}(\hat{y}_N - \hat{y}_{0,k}) + \int_{x_0}^{x_N} q_{j,k}(t)\, dt\right] \\ \hat{q}_{n,k}(x) &= \hat{y}_{n,k} - a_n\gamma_{n,k}\hat{y}_N - a_n\int_{x}^{x_N} q_{n,k}(t)\, dt.\end{aligned}\right\}$$

Then the integral of SFIF defined by $\hat{g}_\sigma(x) = \hat{y}_N - \int_x^{x_N} g_\sigma(t)\, dt$ *is also an SFIF associated with SIFS* (14) *for the interpolation data* $\{(x_i, \hat{y}_i) : i = 0, 1, \ldots, N\}$.

For investigating the smoothness of an SFIF, the following notations and definitions are needed:

$$\lambda = \min\{\lambda_{n,k} : n = 1, 2, \ldots, N,\ k = 1, 2, \ldots M\}, \text{ where } \lambda_{n,k} \text{ are real numbers satisfying } 0 < \lambda_{n,k} \le 1$$

$$C_1 = \max\{\frac{|\gamma_{n,k}|}{|I_n|^\lambda} : n = 1, 2, \ldots, N,\ k = 1, 2, \ldots, M\}, \text{ where } \gamma_{n,k} \text{ are real numbers satisfying } |\gamma_{n,k}| \le 1. \tag{19}$$

The modulus of continuity of SFIF $g_\sigma(x)$ is defined as

$$\omega(g_\sigma, t) = \max_{|h| \le t} \max_x |g_\sigma(x + h) - g_\sigma(x)|$$

and a function $f : \mathbb{R} \to \mathbb{R}$ is said to have Lipschitz exponent δ if $|f(x) - f(\bar{x})| \le K|x - \bar{x}|^\delta$, where K is any positive constant and $0 < \delta \le 1$.

The smoothness of an SFIF in terms of its Lipschitz exponent is given by the following theorem:

Theorem 5 *Let* g_σ *be a SFIF corresponding to SIFS* (10) *with* $q_{n,k} \in Lip\ \lambda_{n,k}$, $0 < \lambda_{n,k} \le 1$. *Then,*

(i) for $C_1 < 1,\ g_\sigma \in Lip\ \lambda$

(ii) for $C_1 = 1,\ \omega(g_\sigma, t) = O(|t|^\lambda \log |t|)$

(iii) for $C_1 > 1,\ g_\sigma \in Lip\ \bar{\lambda}$,

where, $\bar{\lambda} \le \max_{\substack{n=1,\ldots,N \\ k=1,\ldots,M}} \left(\frac{\log \gamma_{n,k}}{\log a_n}\right)$ *and* C_1, λ *are given by* (19).

Proof. The method of proof is similar to that in [28], wherein γ_n is replaced by γ_{n,σ_n}. ■

In general, an SFIF belonging to certain Lipschitz class, need not be differentiable. This, as a natural follow up, leads to identification of conditions for the existence of derivative of a SFIF in the following proposition:

Proposition 6 *For the interpolation data* $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, \ldots N\}$, *let* g_σ *be an SFIF corresponding to SIFS* (10) *for* $\sigma \in \Lambda$. *Then,* $\hat{g}'_\sigma$ *exists and* $\hat{g}'_\sigma(x) = g_\sigma(x)$ *if and only if* $\hat{g}_\sigma$ *is an SFIF associated with SIFS* (14) *for the interpolation data* $\{(x_i, \hat{y}_i) : i = 0, 1, \ldots, N\}$, *where* $\hat{\gamma}_{j,k} = a_j \gamma_{j,k}$ *and* $\frac{d}{dx}(\hat{q}_{j,k}(x)) = a_j q_{j,k}(x)$ *hold.*

Proof. If $\hat{g}'_\sigma(x) = g_\sigma(x)$, then $\hat{g}_\sigma(x) = \hat{y}_0 + \int_{x_0}^{x} g_\sigma(t)\, dt$, so that " if " part follows from Theorem 3. Conversely, suppose $\hat{g}_\sigma$ is an SFIF associated with SIFS (14) for the interpolation data $\{(x_i, \hat{y}_i) : i = 0, \ldots, N\}$. Then,

$$\hat{g}_\sigma(L_{i_k} \circ \ldots \circ L_{i_1}(x)) = \left(\prod_{j=1}^{k} \hat{\gamma}_{i_j,\sigma_j}\right) \hat{g}_\sigma(x) + \sum_{p=1}^{k} \left(\prod_{j=p+1}^{k} \hat{\gamma}_{i_j,\sigma_j}\right) \hat{q}_{i_p,\sigma_p}(L_{i_{p-1}} \circ \ldots \circ L_{i_1}(x)). \tag{20}$$

Since, $\frac{d}{dx}(\hat{q}_{j,k}(x)) = a_j q_{j,k}(x)$

$$\hat{q}_{j,k}(x) = \hat{q}_{j,k}(x_0) + a_j \int_{x_0}^{x} q_{j,k}(t) dt = \hat{y}_{j-1} - a_j \gamma_{j,k} \hat{y}_0 + a_j \int_{x_0}^{x} q_{j,k}(t) dt. \tag{21}$$

Substituting (21) and $\hat{\gamma}_{j,k} = a_j \gamma_{j,k}$ in (20),

$$\hat{g}_\sigma(L_{i_k} \circ \ldots \circ L_{i_1}(x)) = \Big(\prod_{j=1}^{k} a_{i_j} \gamma_{i_j,\sigma_j}\Big) \hat{g}_\sigma(x) + \sum_{p=1}^{k} \Big(\prod_{j=p+1}^{k} a_{i_j} \gamma_{i_j,\sigma_j}\Big) \times \times \Big[\hat{y}_{i_p-1} - a_{i_p} \gamma_{i_p,\sigma_p} \hat{y}_0 + a_{i_p} \int_{x_0}^{L_{i_{p-1}} \circ \ldots \circ L_{i_1}(x)} q_{i_p,\sigma_p}(t) dt\Big]. \tag{22}$$

For a fixed $\sigma \in \Lambda$, it is easily seen that the Read-Bajraktarević operator $\hat{T}$ defined by

$$\hat{T}(\sigma, g)(x) = \lim_{k\to\infty} \Big\{ \Big(\prod_{j=1}^{k} a_{i_j}\gamma_{i_j,\sigma_j} \Big) g(L_{i_1}^{-1} \circ \ldots \circ L_{i_k}^{-1}(x)) + \sum_{p=1}^{k} \Big(\prod_{j=p+1}^{k} a_{i_j}\gamma_{i_j,\sigma_j} \Big) \times \times \Big[\hat{y}_{i_p-1} - a_{i_p}\gamma_{i_p,\sigma_p}\hat{y}_0 + a_{i_p} \int_{x_0}^{L_{i_p}^{-1}\circ\ldots\circ L_{i_k}^{-1}(x)} q_{i_p,\sigma_p}(t)dt \Big] \Big\} \tag{23}$$

is a contraction map on $\hat{\mathcal{G}} = \{f : I \to \mathbb{R}$ such that f is continuous, $f(x_0) = \hat{y}_0$ and $f(x_N) = \hat{y}_N\}$. By (22), the function $\hat{g}_\sigma$ is a fixed point of $\hat{T}$. Also, by Theorem 3, the function $h(x) = \hat{y}_0 + \int_{x_0}^{x} g_\sigma(t)\, dt$ is a SFIF associated with SIFS (14) satisfying (22). Consequently, h also is a fixed point of $\hat{T}$. Hence, by uniqueness of fixed point of Read-Bajraktarevic operator $\hat{T}$, $\hat{g}_\sigma(x) = \hat{y}_0 + \int_{x_0}^{x} g_\sigma(t)\, dt$ which implies that $\hat{g}'_\sigma$ exists and $\hat{g}'_\sigma(x) = g_\sigma(x)$, since g_σ being a SFIF corresponding to SIFS (10), is a continuous function. ■

For the investigation of n^{th} derivative of SFIF, denote

$$G_{i,k,j}(x, y) = \gamma_{i,k,j}\, y + q_{i,k,j}(x) \tag{24}$$

where, $G_{i,k,0}(x, y) = G_{i,k}(x, y)$, $q_{i,k,0}(x) = q_{i,k}(x)$, $\gamma_{i,k,0} = \gamma_{i,k}$ and $G_{i-1,k,j}(x_N, y_{N,k,j}) = G_{i,k,j}(x_0, y_{0,k,j})$, $i = 1,\ldots,N$, $k = 1,\ldots,M$ and $j = 0, 1,\ldots,n$. To determine interpolation data through which derivatives of SFIF passes, let the functions $q_{i,k,j}(x)$ in (24) satisfy:

$$\frac{\sum_{p=1}^{N} a_p \int_{x_0}^{x_N} q_{p,k,j}}{1 - \sum_{p=1}^{N} a_p\gamma_{p,k,j}} = \frac{\sum_{p=1}^{N} a_p \int_{x_0}^{x_N} q_{p,l,j}}{1 - \sum_{p=1}^{N} a_p\gamma_{p,l,j}} \neq 1, \tag{25}$$

where $\hat{y}_{0,j}, j = 0, 1,\ldots,n$ are arbitrary real numbers. For example, for $a_i = \frac{1}{N}$, $\gamma_{i,k,j} = \gamma_{k,j}$ and $q_{i,k,j} = (1-\gamma_k)\bar{q}_{i,j}(x)$, where $\bar{q}_{i,j}(x)$ are polynomials of degree $n-j$ for $i = 1,\ldots,N$, the condition (25) is satisfied. Then, $\hat{y}_{i,k,j} = \hat{y}_{i,l,j} = \hat{y}_{i,j}$ for $i = 1,\ldots,N$, $k, l = 1,\ldots,M$ and $j = 0, 1,\ldots,n$.

The SIFS associated with the interpolation data $\{(x_i, y_{i,j}) : i = 0, 1,\ldots,N\}$, $j = 0, 1,\ldots,n$, is now defined as

$$\Big\{ \{\mathbb{R}^2;\ \omega_{i,k,j}(x, y) = (L_i(x), G_{i,k,j}(x, y)) : i = 1,\ldots,N\},\ k = 1,\ldots,M \Big\}. \tag{26}$$

It is observed that SIFS (26) reduces to SIFS (10) if $j = 0$. The following theorem gives the existence of derivatives of a SFIF.

Theorem 7 *Let the functions $G_{i,k,j}(x, y)$ defined in* (24) *be such that, for some integer $n \geq 0$, $|\gamma_{i,k}| < a_i^n$, $q_{i,k} \in C^n[x_0, x_N]$, $i = 1, 2,\ldots,N$, $k = 1, 2,\ldots,M$ and g_σ be an SFIF corresponding to SIFS* (26) *for $j = 0$ and $\sigma \in \Lambda$. Then, for $j = 1, 2,\ldots,n$, $g_\sigma^{(j)}$ exists and is an SFIF associated with SIFS* (26) *for the interpolation data $\{(x_i, y_{i,j}) : i = 0, 1,\ldots,N\}$, where $\gamma_{i,k,j} = \frac{\gamma_{i,k}}{a_i^j}$ and $q_{i,k,j}(x) = \frac{q^{(1)}_{i,k,j-1}(x)}{a_i}$.*

Proof. The equation $G_{1,k,j}(x_0, y_{0,j}) = y_{0,j}$ gives $y_{0,j} = \frac{\gamma_{1,k}}{a_1^j} y_{0,j} + \frac{q^{(j)}_{1,k}(x_0)}{a_1^j}$ which implies $y_{0,j} = \frac{q^{(j)}_{1,k}(x_0)}{(a_1^j - \gamma_{1,k})}$. Similarly, $G_{N,k,j}(x_N, y_{N,j}) = y_{N,j}$ gives $y_{N,j} = \frac{q^{(j)}_{N,k}(x_N)}{(a_N^j - \gamma_{N,k})}$. By Proposition 6, it now follows that, for $j = 1, 2,\ldots,n$, $g_\sigma^{(j)}(x)$ is the SFIF associated with SIFS $\Big\{ \{\mathbb{R}^2;\ \omega_{i,k,j}(x, y) = (L_i(x), G_{i,k,j}(x, y)) : i = 1, 2,\ldots,N\},\ k = 1, 2,\ldots,M \Big\}$. ■

5 Conclusions

In the present work, the notion of Super Fractal Interpolation Function (SFIF) is introduced for finer simulation of the objects of the nature or outcomes of scientific experiments that reveal one or more structures embedded in to another.

Since, in the construction of SFIF, at each level of iteration, an IFSs can be chosen from a pool of several IFS, the desired randomness and variability can be implemented in fractal interpolation of the given data. Thus, SFIF may be used as a tool for better geometrical modeling of objects found in nature and results of certain scientific experiments. Also, a description of investigations on the integral, the smoothness and determination of conditions for existence of derivatives of an SFIF is given in the present work. It is proved that, for an SFIF passing through a given interpolation data, its integral is also an SFIF, albeit for a different interpolation data. The smoothness of an SFIF is given in terms of its Lipschitz exponent. An SFIF g_σ, for $C_1 \neq 1$, belongs to a Lipschitz class and, for $C_1 = 1$, $\omega(g_\sigma, t) = O(|t|^\lambda \log |t|)$. It is seen that the smoothness of SFIF depend on free variables $\gamma_{n,k}$ as well as on the smoothness of functions $q_{n,k}(x)$ occurring in its definition. Further, sufficient conditions for existence of derivatives of an SFIF are derived in the present paper. Our results on SFIF found here are likely to have wide applications in areas like pattern-forming alloy solidification in chemistry, blood vessel patterns in biology, signal processing, fragmentation of thin plates in engineering, stock markets in finance, wherein significant randomness and variability is observed in simulation of various processes.

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References

[1] M. F. Barnsley. Fractal functions and interpolation. *Constructive Approximation*, 2(1986):303–329.

[2] G. C. Donovan et al. Construction of orthogonal wavelets using fractal interpolation function. *SIAM Journal on Mathematical Analysis*, 27(1996)(4):1158–1192.

[3] T. Yong and Y. Guangjun. Construction of some kiesswetter-like functions-the continuous but non-different-iable function defined by quinary decimal. *Analysis in Theory and Applications*, 20(2004)(1):58–68.

[4] T. J. Bedford. Dimension and dynamics for fractal recurrent sets. *Journal of the London Mathematical Society*, 2(1986)(33):89–100.

[5] P. R. Massopust. Vector valued fractal interpolation functions and their box dimension. *Aequationes Mathematicae*, 42(1991):1–22.

[6] M. F. Barnsley et al. Hidden variable fractal interpolation functions. *SIAM Journal of Mathematical Analysis*, 20(1989)(5):1218–1242.

[7] L. Dalla , V. Drakopoulos and M. Prodromou. On the box dimension for a class of non-affine fractal interpolation functions. *Analysis in Theory and Applications*, 19(2003)(3):220–233.

[8] M. F. Barnsley and A. N. Harrington. The calculus of fractal interpolation functions. *Journal of Approximation Theory*, 57(1989):14–34.

[9] L. Dalla and V. Drakopoulos. On the parameter identification problem in the plane and the polar fractal interpolation functions. *Journal of Approximation Theory*, 101(1999):289–302.

[10] M. A. Navascués. Fractal polynomial interpolation. *Zeitschrift fu Analysis und ihre Anwendungen*, 204(2004):401–418.

[11] A. K. B. Chand and G. P. Kapoor. Generalized cubic spline fractal interpolation functions. *SIAM Journal of Numerical Analysis*, 44(2006)(2):655–676.

[12] A. K. B. Chand and P. Viswanathan. A constructive approach to cubic hermite fractal interpolation function and its constrained aspects. *BIT Numerical Mathematics*, 53(2013):841–865.

[13] A. K. B. Chand and P. Viswanathan. Fractal rational functions and their approximation properties. *Journal of Approximation Theory*, 185(2014):31–50.

[14] P. Massopust. Fractal interpolation and approximation. *Chaos, Solitons and Fractals*, 8(1997):171–190.

[15] M. A. Navascués and M. V. Sebastián. Smooth fractal interpolation. *Journal of Inequalities and Applications*.

[16] M. A. Navascués. Affine fractal functions as bases of continuous functions. *Quaestiones Mathematicae*, (2014):1–14.

[17] P. R. Massopust. Fractal surfaces. *Journal of Mathematical Analysis and Applications*, 151(1990):275–290.

[18] J. S. Geronimo and D. P. Hardin. Fractal interpolation surfaces and a related 2-D multiresolution analysis. *Journal of Mathematical Analysis and Applications*, 176(1993):561–586.

[19] H. Xie and H. Sun. The study on bivariate fractal interpolation functions and creation of fractal interpolated surfaces. *Fractals*, 5(1997)(4):625–634.

[20] L. Dalla. Bivariate fractal interpolation functions on grids. *Fractals*, 10(2002)(1):53–58.

[21] A. K. B. Chand and G. P. Kapoor. Hidden variable bivariate fractal interpolation surfaces. *Fractals*, 11(2003)(3):227–288.

[22] M. F. Barnsley, J. E. Hutchinson and O. Stenflo. A fractal valued random iteration algorithm and fractal hierarchy. *Fractals*, 13(2005)(2):111–146.

[23] M. F. Barnsley. Super Fractals. Cambridge University Press. 2006.

[24] M. F. Barnsley, J. E. Hutchinson and O. Stenflo. V-variable fractals: Fractals with partial self similarity. *Advances in Mathematics*, 218(2008):2051–2088.

[25] P. Massopust. Interpolation and Approximation with Splines and Fractals. Oxford University Press. 2010.

[26] J. E. Hutchinson. Fractals and self-similarity. *Indiana University Mathematics Journal*, 30(1981):713–747.

[27] M. F. Barnsley. Fractals Everywhere. Academic Press, Orlando, Florida. 1988.

[28] H. Y. Wang and J. S. Yu. Fractal interpolation functions with variable parameters and their analytical properties. *Journal of Approximation Theory*, 175(2013):1–18.