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On the Partition Satisfying the Strengthened van der Waerden Property

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Abstract: The aim of this paper is to measure the size of the sets of partitions of $\mathbb{N}$ satisfying the strengthened van der Waerden properties (the SWPA and the SWPB). The remain result, roughly speaking, states that the set of partitions without the SWPA is small in the sense of category and Hausdorff dimension; while the set of partitions with the SWPB is small in the sense of category but has full Hausdorff dimension.

Keywords: strengthened van der Waerden property; Hausdorff dimension; category

1 Introduction

The famous theorem of van der Waerden on arithmetic progressions (see [1–3]) is a classical result in combinatorial number theory and Ramsey theory (see [6]). A formulation of this theorem is stated in the following way (see [11]):

If the natural number $\mathbb{N}$ are partitioned into finitely many sets, then one of these sets contains finite arithmetic progressions of arbitrary length.

Erdös and Turán observed that the property stated in van der Waerden's theorem is translation invariant. They further conjectured in 1936 that the property holds for every set the size of which is large enough in some sense. More precisely, they conjectured that positive Banach density is sufficient. Let A be a subset of $\mathbb{N}$, the upper Banach density D^* of A is defined by

$$D^*(A) = \limsup_{N-M\to\infty} \frac{\#\big(A\cap\{M+1, M+2, \dots, N\}\big)}{N-M}, \tag{1}$$

where $\#(\cdot)$ denotes the cardinal number of a set. Szemerédi [9] proved the conjecture of Erdös and Turán in 1974:

Any set of positive upper Banach density in $\mathbb{N}$ contains finite arithmetic progressions of arbitrary length.

Szemerédi's theorem clearly implies van der Waerden's theorem, for if we partition the natural number $\mathbb{N}$ into finitely many sets, then at least one of them will have positive Banach density.

This paper focus on the partitions satisfying the strengthened van der Waerden properties.

Definition 1 *A finite partition of $\mathbb{N}$ is called satisfying* the strengthened van der Waerden property A (SWPA) *if every set of the partition contains finite arithmetic progressions of arbitrary length.*

A finite partition of $\mathbb{N}$ is called satisfying the strengthened van der Waerden property B (SWPB) *if one set of the partition contains infinite arithmetic progressions.*

It is not difficult to construct a partition of $\mathbb{N}$ which does not satisfy SWPA or SWPB.

Example 1 *Let $A = \{1, 4, 9, \dots, n^2, \dots\}$ be the set of all square numbers and $B = \mathbb{N}\setminus A$; then $\mathbb{N} = A\cup B$ is a partition of $\mathbb{N}$. It is clear that the set A does not contain any arithmetic progressions of length 3. Thus, this partition does not satisfy the SWPA.*

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Example 2 *Let*

$$A = \bigcup_{n\geq 1} \{(2n-1)^2, (2n-1)^2+1, \ldots, 4n^2-2, 4n^2-1\}$$

and

$$B = \bigcup_{n\geq 1} \{4n^2, 4n^2+1, \ldots, (2n+1)^2-2, (2n+1)^2-1\}.$$

Then $\mathbb{N} = A \cup B$ is a partition of $\mathbb{N}$. But neither A nor B contains infinite arithmetic progressions. Thus, this partition does not satisfy the SWPB.

Example 1 and 2 show that not all partitions of $\mathbb{N}$ satisfy the SWPA (or the SWPB). It is of interesting to measure the size of the set of partitions satisfying the SWPA (or the SWPB). Let $\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_q$ be a q-partition of $\mathbb{N}$, where q-partition means that we partition the natural number $\mathbb{N}$ into q sets; we can regard every q-partition as a point x of the symbolic space $\Omega_q = \{1, 2, \ldots, q\}^{\mathbb{N}}$ by letting

$$x_i = j \quad \text{if and only if} \quad i \in A_j.$$

In this way we construct a one-to-one correspondence between the q-partitions of $\mathbb{N}$ and the points of symbolic space Ω_q. A usually probability measure $\mathbf{P}$ on Ω_q is defined by

$$\mathbf{P}[x_1 x_2 \ldots x_n] = q^{-n},$$

for all $n \geq 1$ and all finite sequence $x_1 x_2 \ldots x_n \in \{1, 2, \ldots, q\}^n$, where

$$[x_1 x_2 \ldots x_n] = \{y \in \Omega_q \colon y_i = x_i \text{ for } 1 \leq i \leq n\}.$$

By the law of large numbers,

$$\lim_{n\to\infty} \frac{\#\{1 \leq i \leq n \colon x_i = j\}}{n} = 1/q, \qquad \text{for } \mathbf{P}\text{-a.s. } x \in \Omega_q \text{ and } j = 1, 2, \ldots, q.$$

Combining this with Szemerédi's theorem, we know that, for $\mathbf{P}$-a.s. $x \in \Omega_q$, the corresponding q-partition satisfies the SWPA. On the other hand, it is clear that, for $\mathbf{P}$-a.s. $x \in \Omega_q$, the corresponding q-partition does not satisfy the SWPB. Roughly speaking, typical partition satisfies the SWPA but does not satisfy the SWPB. A natural question is: how about the size of the sets of non-typical partitions?

It is well known that the space Ω_q can be made a metric space by introducing the metric ρ such that

$$\rho(x, y) = q^{-n} \quad \text{if } x_1 = y_1, \ldots, x_{n-1} = y_{n-1}, x_n \neq y_n$$

and $\rho(x, y) = 0$ if $x = y$. The aim of this paper is to measure the size of the set of partitions without the SWPA and the size of the set of partitions with the SWPB by regarding the two sets as subsets of the symbolic metric space Ω_q. The main tools involved are Hausdorff dimension (see [5]) and the category theory (see [7]). The idea of this study is closely related to fractal geometry. For some recently studies on fractal geometry, we refer to [4, 8, 10, 12–14].

The following two theorems are the main results of this paper.

Theorem 1 *Write*

$$W = \big\{x \in \Omega_q \colon \textit{the } q\textit{-partition corresponding to } x \textit{ does not satisfy the SWPA}\big\}.$$

Then W is of first category in Ω_q and $\dim_H W = \frac{\log(q-1)}{\log q}$.

Theorem 2 *Write*

$$W^* = \big\{x \in \Omega_q \colon \textit{the } q\textit{-partition corresponding to } x \textit{ satisfies the SWPB}\big\}.$$

Then W^* is of first category in Ω_q and $\dim_H W^* = 1$.

This paper is organized as follows: we prove Theorem 1 and 2 in Section 2 and 3, respectively. Section 4 is the conclusion.

2 The proof of Theorem 1

2.1 The category of W

We first show that W is of first category. For this, let W_k denote the set of all point x such that some set of corresponding q-partition does not contains any arithmetic progression of length great than k. Then $W = \bigcup_{k\geq 1} W_k$. Write $V_k = \Omega_q \setminus W_k$; it is sufficient to prove that V_k is a dense open set of Ω_q for each $k \geq 1$.

To see that V_k is dense, for any $x \in \Omega_q$, let $\{x^j\}_{j\geq 1}$ be a sequence of point such that

$$x_i^j = \begin{cases} x_i & \text{if } 1 \leq i \leq j; \\ l & \text{if } i > j \text{ and } i \equiv l \pmod q; \end{cases}$$

where we require $l \in \{1, 2, \ldots, q\}$. It is clear that $x^j \to x$ as $j \to \infty$ and $x^j \in V_k$ for all $j \geq 1$. Thus, V_k is dense.

Now let $x \in V_k$, we will show that x is an interior point of V_k. Let

$$\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_q$$

be the corresponding q-partition of $\mathbb{N}$. By the definition of V_k, each A_i $(1 \leq i \leq q)$ contains an arithmetic progression of length $k+1$. Therefore, there is an $N \in \mathbb{N}$ such that each $A_i \cap \{1, 2, \ldots, N\}$ also contains an arithmetic progression of length $k+1$. Write

$$[x_1 x_2 \ldots x_N] = \{y \in \Omega_q \colon y_i = x_i \text{ for } 1 \leq i \leq N\}.$$

Then $[x_1 x_2 \ldots x_N] \subset V_k$ is an open set and $x \in [x_1 x_2 \ldots x_N]$, and so V_k is open.

2.2 The Hausdorff dimension of W

We first show that $\dim_H W \geq \frac{\log(q-1)}{\log q}$. Write

$$W' = \{x \in \Omega_q \colon x_1 = 1 \text{ and } x_i \neq 1 \text{ for all } i > 1\}.$$

It suffices to show that $\dim_H W' = \frac{\log(q-1)}{\log q}$ since $W' \subset W$. For $w = w_1 w_2 \ldots w_n \in \{1, 2, \ldots, q\}^n$, we call the set

$$[w] = \{x \in \Omega_q \colon x_i = w_i \text{ for } 1 \leq i \leq n\}$$

a cylinder of order n. By the definition of W', we know that there are exactly $(q-1)^{n-1}$ many cylinders of order n which intersection W'. By Kolmogorov's consistency theorem, there exists a unique Borel probability measure μ on W' such that, for every $n \geq 1$ and every cylinder $[w]$ of order n,

$$\mu(W' \cap [w]) = \begin{cases} (q-1)^{-(n-1)} & \text{if } W' \cap [w] \neq \emptyset; \\ 0 & \text{if } W' \cap [w] = \emptyset. \end{cases}$$

For every $x \in \Omega_q$, let $C_n(x)$ denote the cylinder of order n containing x. Then for all $x \in W'$, we have

$$\lim_{n\to\infty} \frac{\log \mu(C_n(x))}{\log |C_n(x)|} = \lim_{n\to\infty} \frac{(n-1)\log(q-1)}{(n+1)\log q} = \frac{\log(q-1)}{\log q},$$

where $|C_n(x)|$ denotes the diameter of $C_n(x)$. Using Billingsley's theorem, we conclude that $\dim_H W' = \frac{\log(q-1)}{\log q}$. And so

$$\dim_H W \geq \frac{\log(q-1)}{\log q}. \tag{2}$$

It remains to obtain the upper bound of $\dim_H W$. For this, for every $1 \leq j \leq q$, let U_j denote the set of all points x in Ω_q such that

$$\lim_{n\to\infty} \frac{\#\{1 \leq i \leq n \colon x_i = j\}}{n} = 0.$$

It follows from Szemerédi's theorem that $W \subset U_1 \cup U_2 \cup \cdots \cup U_q$. By symmetry, it suffices to show that $\dim_H U_1 \leq \frac{\log(q-1)}{\log q}$. To this end, pick two numbers $\theta, \eta \in (0,1)$ such that

$$\theta + (q-1)\eta = 1. \tag{3}$$

By Kolmogorov's consistency theorem, there exists a unique Borel probability measure ν on Ω_q such that, for every $n \geq 1$ and every cylinder $[w]$ of order n,

$$\nu([w]) = \theta^{n_1}\eta^{n_2},$$

where $n_1 = \#\{1 \leq i \leq n \colon w_i = 1\}$ and $n_2 = n - n_1$. Then for all $x \in U_1$, we have

$$\lim_{n\to\infty} \frac{\log \mu(C_n(x))}{\log |C_n(x)|} = -\frac{\log\theta}{\log q} \cdot \lim_{n\to\infty} \frac{n_1}{n+1} - \frac{\log\eta}{\log q} \cdot \lim_{n\to\infty} \frac{n_2}{n+1} = -\frac{\log\eta}{\log q},$$

since $\lim_{n\to\infty} n_1/n = 0$ for all $x \in U_1$. Using Billingsley's theorem, we conclude that $\dim_H U_1 \leq -\frac{\log\eta}{\log q}$ for all $\theta, \eta \in (0,1)$ satisfying (1). It follows that $\dim_H U_1 \leq \frac{\log(q-1)}{\log q}$. Thus,

$$\dim_H W \leq \frac{\log(q-1)}{\log q}. \tag{4}$$

Combining (2) and (4), we have $\dim_H W = \frac{\log(q-1)}{\log q}$.

3 Proof of Theorem 2

3.1 The category of W^*

For $j = 1, 2, \ldots, q$, $a \in \mathbb{N}$ and $d \in \mathbb{N}$, let

$$U_{j,a,d} = \{x \in \Omega_q \colon x_{a+kd} = j \text{ for } k \geq 0\}. \tag{5}$$

In other word, for all $x \in U_{j,a,d}$, let $\mathbb{N} = A_1 \cup A_2 \cup \ldots A_q$ be the corresponding q-partition, then the set A_j contains the infinite arithmetic progression: a, $a+d$, $a+2d$, $\ldots$. It is clear that

$$W^* = \bigcup_{j=1}^{q} \bigcup_{a\in\mathbb{N}} \bigcup_{d\in\mathbb{N}} U_{j,a,d}.$$

By the definition, every set $U_{j,a,d}$ is closed. And these sets are all nowhere dense, because in each cylinder, say $[w_1w_2\ldots w_n]$, there is a point $x \notin U_{j,a,d}$. For example, choose $x = x_1x_2\ldots$ with

$$x_i = \begin{cases} w_i & \text{if } 1 \leq i \leq n; \\ k & \text{if } i > n; \end{cases}$$

where $k = 1$ when $j \neq 1$ and $k = 2$ when $j = 1$. Thus, W^* is of first category.

3.2 The Hausdorff dimension of W^*

Let $U_{j,a,d}$ be as in (5). We first show that $\dim_H U_{j,a,d} = (d-1)/d$ for all j and a. By Kolmogorov's consistency theorem, there exists a unique Borel probability measure λ on $U_{j,a,d}$ such that, for every $n \geq 1$ and every cylinder $[w]$ of order n,

$$\lambda[w] = \begin{cases} q^{n-\tau(n)} & \text{if } U_{j,a,d} \cap [w] \neq \emptyset; \\ 0 & \text{if } U_{j,a,d} \cap [w] = \emptyset; \end{cases}$$

where $\tau(n) = \#\{k = 0, 1, 2, \ldots \colon a + kd \leq n\}$. Then for all $x \in U_{j,a,d}$, let $C_n(x)$ be the cylinder of order n containing x, we have

$$\lim_{n\to\infty} \frac{\log \lambda(C_n(x))}{\log |C_n(x)|} = \lim_{n\to\infty} \frac{(n-\tau(n))\log q}{(n+1)\log q} = \frac{d-1}{d}.$$

Using Billingsley's theorem, we conclude that $\dim_H U_{j,a,d} = (d-1)/d$. Therefore, $\dim_H W^* = 1$ since $W^* = \bigcup_{j=1}^{q} \bigcup_{a\in\mathbb{N}} \bigcup_{d\in\mathbb{N}} U_{j,a,d}$.

4 Conclusion

Notice that Ω_q is of second category and $\dim_H \Omega_q = 1$. Thus, Theorem 1 states that the set of partitions without the SWPA is small in the sense of category and Hausdorff dimension; while Theorem 2 states that the set of partitions with the SWPB is small in the sense of category but has full Hausdorff dimension.

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