ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.19(2015) No.1,pp.47-52

Coefficient Estimates for Pascu-type Subclasses of Bi-univalent Functions Based on Subordination

G. Murugusundaramoorthy[1] *, C. Selvaraj[2], O. S. Babu [3]
[1] School of Advanced Sciences, VIT University, Vellore - 632 014, India
[2]Department of Mathematics, Presidency College (Autonomous), Chennai - 600005, India
[3]Department of Mathematics, Dr. Ambedkar Govt. Arts College, Chennai - 600039, India
(Received 24 March 2013, accepted 28 January 2015)

Abstract: Estimates on the coefficients $|a_2|$ and $|a_3|$ are obtained for normalized analytic function f in the open disk with f and its inverse $g = f^{-1}$ satisfy the condition that

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \quad \text{and} \quad \frac{zg'(z) + \lambda z^2 g''(z)}{(1-\lambda)g(z) + \lambda z g'(z)}$$

$(0 \leq \lambda \leq 1)$ are both subordinate to an analytic function whose range is symmetric with respect to the real axis. Further applications of Salagean operator to this class are obtained and relevant connections are also pointed out.

Keywords: Analytic functions; Univalent functions; Bi-univalent functions; Bi-starlike functions; Bi-convex functions; Subordination and Salagean operator

1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

normailzed by the conditions $f(0) = 0 = f'(0) - 1$ defined in the open unit disk $\triangle = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be bi-univalent in $\triangle$ if both f and f^{-1} are univalent in $\triangle$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\triangle$. Since $f \in \sum$ has the Maclaurian series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots \tag{2}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) which are also univalent in $\triangle$. The subclasses

$$\mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{K}(\alpha)$$

of $\mathcal{S}$, namely starlike and convex functions of order α, $0 \leq \alpha < 1$, are respectively characterized by

$$\Re\left(\frac{z\, f'(z)}{f(z)}\right) > \alpha \quad \text{and} \quad \Re\left(1 + \frac{z\, f''(z)}{f'(z)}\right) > \alpha.$$

Functions in the various subclasses of starlike and convex functions are typically characterized by the quantity $\frac{z\, f'(z)}{f(z)}$ or $1 + \frac{z\, f''(z)}{f'(z)}$ lying in a certain domain starlike with respect to 1 in the right-half plane.

*Corresponding author. *E-mail address*: gmsmoorthy@yahoo.com; pamc9439@yahoo.co.in; osbabu1009@gmail.com

IJNS.2015.02.15/852

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\triangle$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [6] unified various subclasses of starlike and convex functions for which either of the quantity

$$\frac{z\,f'(z)}{f(z)} \qquad \text{(or)} \qquad 1 + \frac{\mathrm{z\,f''(z)}}{\mathrm{f'(z)}}$$

is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the unit disk $\triangle, \varphi(0) = 1, \varphi'(0) > 0$, and φ maps $\triangle$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\frac{z\,f'(z)}{f(z)} \prec \varphi(z)$. Similarly, the class of Ma-Minda convex functions of functions $f \in \mathcal{A}$ satisfying the subordination $1 + \frac{z\,f''(z)}{f'(z)} \prec \varphi(z)$. A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\varphi)$ and $\mathcal{K}_\Sigma(\varphi)$.In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk $\triangle$, satisfying $\varphi(0) = 1, \varphi'(0) > 0$, and $\varphi(\triangle)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{3}$$

Several authors have introduced and investigated subclasses of bi-univalent functions Σ and obtained bounds for the initial coefficients (see [2, 3, 7, 11]. Motivated by the work of Ali et al. [1, 5] ,in this paper, we introduce a new subclass $\mathcal{P}_\Sigma(\lambda, \varphi)$ of bi-univalent functions and obtain the estimates on the coefficients $|a_2|$ and $|a_3|$ by subordination.Further some applications of Salagean [10] operator are discussed .

Definition 1 *A function $f \in \Sigma$ is said to be in the class $\mathcal{P}_\Sigma(\lambda, \varphi)$ if the following subordination hold:*

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \prec \varphi(z) \tag{4}$$

and

$$\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda)g(w) + \lambda w g'(w)} \prec \varphi(w), \tag{5}$$

where $g(w) = f^{-1}(w)$.

We note that $\mathcal{P}_\Sigma(0, \varphi) = \mathcal{S}^*_\Sigma(\varphi)$ and $\mathcal{P}_\Sigma(1, \varphi) = \mathcal{K}_\Sigma(\varphi)$. In order to prove our main results, we require the following Lemma due to [9].

Lemma 1 *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions h analytic in $\triangle$ for which $\Re\{h(z)\} > 0$, where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ for $z \in \triangle$.*

2 Coefficient estimates for the function class $\mathcal{P}_\Sigma(\lambda, \varphi)$

Theorem 2 *Let f given by* (1) *be in the class $\mathcal{P}_\Sigma(\lambda, \varphi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|(1 + 2\lambda - \lambda^2)B_1^2 + (1+\lambda)^2(B_1 - B_2)|}} \tag{6}$$

and

$$|a_3| \leq B_1 \left(\frac{B_1}{(1+\lambda)^2} + \frac{1}{2(1+2\lambda)} \right). \tag{7}$$

Proof: Let $f \in \mathcal{P}_\Sigma(\lambda, \varphi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \triangle \longrightarrow \triangle$, with $u(0) = 0 = v(0)$, satisfying

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} = \varphi(u(z)) \tag{8}$$

and

$$\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda)g(w) + \lambda w g'(w)} = \varphi(v(w)). \tag{9}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots$$

or, equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right] \tag{10}$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \cdots\right]. \tag{11}$$

Then $p(z)$ and $q(z)$ are analytic in $\triangle$ with $p(0) = 1 = q(0)$. Since $u, v : \triangle \to \triangle$, the functions $p(z)$ and $q(z)$ have a positive real part in $\triangle$, and $|p_i| \leq 2$ and $|q_i| \leq 2$. Using (10) and (11) in (8) and (9) respectively, we have

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} = \varphi\left(\frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right]\right) \tag{12}$$

and

$$\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda)g(w) + \lambda w g'(w)} = \varphi\left(\frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right) w^2 + \cdots\right]\right). \tag{13}$$

In light of (1) - (3), from (12) and (13), it is evident that

$$1 + (1+\lambda)a_2 z + [2(1+2\lambda)a_3 - (1+\lambda)^2 a_2^2]z^2 + \cdots$$
$$= 1 + \frac{1}{2}B_1 p_1 z + [\frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2]z^2 + \cdots$$

and

$$1 - (1+\lambda)a_2 w - [2(1+2\lambda)a_3 + (\lambda^2 - 6\lambda - 3)a_2^2]w^2 + \cdots$$
$$= 1 + \frac{1}{2}B_1 q_1 w + [\frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2]w^2 + \cdots$$

which yields the following relations.

$$(1+\lambda)a_2 = \frac{1}{2}B_1 p_1 \tag{14}$$

$$-(1+\lambda)^2 a_2^2 + 2(1+2\lambda)a_3 = \frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2 \tag{15}$$

$$-(1+\lambda)a_2 = \frac{1}{2}B_1 q_1 \tag{16}$$

and

$$-(\lambda^2 - 6\lambda - 3)a_2^2 - 2(1+2\lambda)a_3 = \frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2. \tag{17}$$

From (14) and (16), it follows that

$$p_1 = -q_1 \tag{18}$$

and

$$8(1+\lambda)^2 a_2^2 = B_1^2(p_1^2 + q_1^2). \tag{19}$$

From (15), (17) and (19), we obtain

$$a_2^2 = \frac{B_1^3(p_2 + q_2)}{4[(1+2\lambda-\lambda^2)B_1^2 + (1+\lambda)^2(B_1 - B_2)]}.$$

Applying Lemma 1, for the coefficients p_2 and q_2, we immediately got the desired estimate on $|a_2|$ as asserted in (6).

By subtracting (17) from (15) and using (18) and (19), we get

$$a_3 = \frac{B_1^2(p_1^2 + q_1^2)}{8(1+\lambda)^2} + \frac{B_1(p_2 - q_2)}{8(1+2\lambda)}.$$

Applying Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, we get the desired estimate on $|a_3|$ as asserted in (7)

Remark 3 *For $\lambda = 0$ and $\lambda = 1$, the inequality* (6) *reduces to the estimate of $|a_2|$ ([1], Corollary 2.1) and ([1], Corollary 2.2) respectively.*

Remark 4 *For the class of strongly starlike functions, the function ϕ is given by*

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \le 1),$$

which gives $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$. Hence the inequalities (6) *and* (7) *becomes ([7], Theorem 2.1)*

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha(1-\lambda)(1+3\lambda) + (1+\lambda)^2}} \qquad \textit{and} \qquad |a_3| \le \frac{4\alpha^2}{(1+\lambda)^2} + \frac{\alpha}{1+2\lambda}$$

In particular, when $\lambda = 0$, for the class $\mathcal{S}^*_\Sigma(\alpha)$, we get [3]

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha+1}} \qquad \text{and} \qquad |a_3| \le 4\alpha^2 + \alpha.$$

When $\lambda = 1$, for the class $\mathcal{K}_\Sigma(\alpha)$, we get [3]

$$|a_2| \le \alpha \qquad \text{and} \qquad |a_3| \le \alpha^2 + \frac{\alpha}{3}.$$

On the other hand if we take

$$\phi(z) = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)^2 z^2 + \cdots \quad (0 \le \beta < 1),$$

then $\beta_1 = \beta_2 = 2(1-\beta)$ in this case inequalities (6) and (7) becomes ([7], Theorem 3.1)

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{1+2\lambda-\lambda^2}} \qquad \text{and} \qquad |a_3| \le \frac{4(1-\beta)^2}{(1+\lambda)^2} + \frac{(1-\beta)}{(1+2\lambda)}.$$

In particular, when $\lambda = 0$, for the class $\mathcal{S}^*_\Sigma(\beta)$, we get [3]

$$|a_2| \le \sqrt{2(1-\beta)} \qquad \text{and} \qquad |a_3| \le 4(1-\beta)^2 + (1-\beta).$$

When $\lambda = 1$, for the class $\mathcal{K}_\Sigma(\beta)$, we get [3]

$$|a_2| \le \sqrt{(1-\beta)} \qquad \text{and} \qquad |a_3| \le \frac{(4-3\beta)(1-\beta)}{3}.$$

3 Applications of Salagean derivative operator

In 1983, Salagean [10] introduced the differential operator

$$D^k : \mathcal{A} \to \mathcal{A}$$

defined by

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z), \quad D^k f(z) = D(D^{k-1}f(z)) = z(D^{k-1}f(z))', \quad k \in \mathbb{N} = \{1, 2, 3, \ldots\}.$$

We note that

$$D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

In this section, we introduce a subclass $\mathcal{P}^k_\Sigma(\lambda, \phi)$ of $\mathcal{P}_\Sigma(\lambda, \phi)$ and find estimate on the coefficients $|a_2|$ and $|a_3|$ for the functions in this new subclass $\mathcal{P}^k_\Sigma(\lambda, \phi)$.

Definition 2 *A function $f \in \Sigma$ is said to be in the class $\mathcal{P}_\Sigma^k(\lambda, \phi)$ if the following subordination hold:*

$$\frac{(1-\lambda)D^{k+1}f(z) + \lambda D^{k+2}f(z)}{(1-\lambda)D^k f(z) + \lambda D^{k+1}f(z)} \prec \phi(z) \tag{20}$$

and

$$\frac{(1-\lambda)D^{k+1}g(w) + \lambda D^{k+2}g(w)}{(1-\lambda)D^k g(w) + \lambda D^{k+1}g(w)} \prec \phi(w), \quad g(w) := f^{-1}(w). \tag{21}$$

Remark 5 *Taking $\lambda = 0$ in the class $\mathcal{P}_\Sigma^k(\lambda, \phi)$, we have $\mathcal{P}_\Sigma^k(0, \phi)$ and $f \in \Sigma$ is said to be in the class $\mathcal{P}_\Sigma^k(0, \phi)$ if the following subordination hold;*

$$\frac{D^{k+1}f(z)}{D^k f(z)} \prec \phi(z)$$

and

$$\frac{D^{k+1}g(w)}{D^k g(w)} \prec \phi(w), \quad g(w) := f^{-1}(w).$$

Remark 6 *Taking $\lambda = 1$ in the class $\mathcal{P}_\Sigma^k(\lambda, \phi)$, we have $\mathcal{P}_\Sigma^k(1, \phi)$ and $f \in \Sigma$ is said to be in the class $\mathcal{P}_\Sigma^k(1, \phi)$ if the following subordination hold:*

$$\frac{D^{k+2}f(z)}{D^{k+1}f(z)} \prec \phi(z)$$

and

$$\frac{D^{k+2}g(w)}{D^{k+1}g(w)} \prec \phi(w), \quad g(w) := f^{-1}(w).$$

We note that $\mathcal{P}_\Sigma^0(\lambda, \phi) = \mathcal{P}_\Sigma(\lambda, \phi), \mathcal{P}_\Sigma^0(0, \phi) = \mathcal{S}_\Sigma^*(\phi), \mathcal{P}_\Sigma^0(1, \phi) = \mathcal{K}_\Sigma(\phi)$.

Theorem 7 *Let f given by* (1) *be in the class $\mathcal{P}_\Sigma^k(\lambda, \phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|[2(1+2\lambda)3^k - (1+\lambda)^2 2^{2k}]B_1^2 + (1+\lambda)^2 2^{2k}(B_1 - B_2)|}} \tag{22}$$

and

$$|a_3| \leq B_1\left(\frac{B_1}{(1+\lambda)^2.2^{2k}} + \frac{1}{2(1+2\lambda)3^k}\right). \tag{23}$$

Proof: Proceeding as in the proof of Theorem 2 we can arrive the following relations

$$(1+\lambda)2^k a_2 = \frac{1}{2}B_1 p_1 \tag{24}$$

$$-(1+\lambda)^2 2^{2k} a_2^2 + 2(1+2\lambda)3^k a_3 = \frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2 \tag{25}$$

$$-(1+\lambda)2^k a_2 = \frac{1}{2}B_1 q_1 \tag{26}$$

and

$$(4(1+2\lambda)3^k - (1+\lambda)^2 2^{2k})a_2^2 - 2(1+2\lambda)3^k a_3 = \frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2 \tag{27}$$

From (24) and (26) it follows that

$$p_1 = -q_1 \tag{28}$$

and

$$4(1+\lambda)^2 2^{2k+1} a_2^2 = B_1^2(p_1^2 + q_1^2). \tag{29}$$

From (25), (27) and (29), we obtain

$$a_2^2 = \frac{B_1^3(p_2+q_2)}{4[\{2(1+2\lambda)3^k - (1+\lambda)^2 2^k\}B_1^2 + (1+\lambda)^2 2^{2k}(B_1 - B_2)]}.$$

Applying Lemma 1 for the coefficients p_2 and q_2, we immediately get the desired estimate on $|a_2|$ as asserted in (22).

By subtracting (27) from (25) and using (28) and (29), we get

$$a_3 = \frac{B_1^2(p_1^2 + q_1^2)}{8(1+\lambda)^2.2^{2k}} + \frac{B_1(p_2 - q_2)}{8(1+2\lambda).3^k}.$$

Applying Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, we get the desired estimate on $|a_3|$ as asserted in (23).

Remark 8 *For $k = 0$, Theorem 7 reduces to Theorem 2.*

Remark 9 *As in Remark 4, by taking $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$, for the class of strongly starlike functions, inequalities* (22) *and* (23) *becomes*

$$|a_2| \leq \frac{2\alpha}{\sqrt{|\{4(1+2\lambda)3^k - 3(1+\lambda)^2 2^{2k}\}\alpha + (1+\lambda)^2 2^{2k}|}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(1+\lambda)^2.2^{2k}} + \frac{\alpha}{(1+2\lambda)3^k}.$$

On the other-hand, by taking $B_1 = B_2 = 2(1-\beta)$, the inequalities (22) and (23) becomes

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{|2(1+2\lambda)3^k - (1+\lambda)^2 2^{2k}|}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(1+\lambda)^2.2^{2k}} + \frac{(1-\beta)}{(1+2\lambda)3^k}.$$

We note that, for $k = 0$, these estimates coincides with that in Remark 4. Consequently, when $\lambda = 0$ and $\lambda = 1$ the estimates for the classes $\mathcal{S}^*_\Sigma(\alpha), \mathcal{S}^*_\Sigma(\beta)$ and $\mathcal{K}_\Sigma(\alpha), \mathcal{K}_\Sigma(\beta)$ are arrived respectively.

References

[1] R.M. Ali, S.K. Leo, V. Ravichandran, S. Supramaniam. Coefficient estimates for bi-univalent Ma-Minda star-like and convex functions. *Appl. Math. Lett.*, 25 (2012): 344 -351.

[2] D.A. Brannan, J. Clunie, W.E. Kirwan. Coefficient estimates for a class of star-like functions. *Canad. J. Math.*, 22 (1970): 476 - 485.

[3] D.A. Brannan, T.S. Taha. On some classes of bi-univalent functions. *Studia Univ. Babes-Bolyai Math.*, 31(2) (1986): 70 - 77.

[4] P.L. Duren, Univalent Functions, in: Grundlehren der Mathematischen Wissenchaften. Vol. 259, Springer, New York 1983.

[5] M. Lewin. On a coefficient problem for bi-univalent functions. *Proc. Amer. Math. Soc.*, 18 (1967): 63 - 68.

[6] W.C. Ma, D. Minda. A unified treatment of some special classes of functions.in: Proceedings of the Conference on Complex Analysis, Tianjin, 1992, 157 - 169, Conf. Proc. Lecture Notes Anal. 1. Int. Press, Cambridge, MA, 1994.

[7] G. Murugusundaramoorthy, C. Selvaraj, O.S. Babu. On Pascu-Type subclasses of bi-univalent functions and applications of Hohlov operator (communicated).

[8] E. Netanyahu. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Arch. Ration. Mech. Anal.*, 32 (1969): 100 -112.

[9] Ch. Pommerenke. Univalent functions, Vandenhoeck and Rupercht, Göttingen, 1975.

[10] G.S. Salagean. Subclasses of univalent functions. Complex Analysis - Fifth Romanian Finish Seminar, Bucharest, 1 (1983): 362 - 372.

[11] H.M. Srivastava, A.K. Mishra, P. Gochhayat. Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.*, 23(10) (2010): 1188 - 1192.