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A New Method for Solving Non-Linear Complementarity Problems

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Abstract: The non-linear complementarity problem $NLCP$ is to find a vector z in IR^n satisfying $0 \leqslant z \perp f(z) \geqslant 0$, where f is a given function. This problem can be solved by several methods but the most of these methods require a lot of arithmetic operations, and therefore, it is too difficult, time consuming, or expensive to find an approximate solution of the exact solution. In this paper we give a new method for solving this problem which converges very rapidly relative to most of the existing methods and does not require a lot of arithmetic operations to converge. For this we show that solving the $NLCP$ is equivalent to finding zero of the function F. After we build a sequence of smooth functions $F^{(k)} \in C^\infty$ which is uniformly convergent to the function F and we show that, an approximation of the solution of the $NLCP$ is obtained by solving $F^{(k)}(x) = 0$ for a parameter k large enough. We close our paper with some numerical examples to demonstrate the efficiency of our method. The numerical results obtained in this paper are very favorable and showed that our method works well for the problems tested.

Keywords: Non-linear complementarity problem; Sequence of smooth functions; System of non-linear equations; Approximation of the solution.
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1 Introduction

The non-linear complementarity problem has a number of important applications in operations research, economic equilibrium problems and in the engineering sciences. A variety of economic, biologic and physical phenomena are most naturally modeled by saying that certain pairs of inequality constraints must be complementary, in the sense that at least one must hold with equality. This problem has attracted much attention due its various applications. For a description of many of these applications, we refer the reader to [9–13, 17].

In order to understand the peculiarity of this problem and provide a new effective method to solve it, let IR^n be the n-dimensional Euclidean space and consider the complementarity problem (linear or non-linear), defined by a continuous mapping $f : \mathrm{IR}^n \to \mathrm{IR}^n$, is to find an element $z \in \mathrm{IR}^n$ such that

$$z \geqslant 0, \quad f(z) \geqslant 0 \quad \text{and} \quad z^T f(z) = 0, \tag{1}$$

where $z \geqslant 0$ mean that $z_i \geqslant 0$ for all $i = 1, 2, ..., n$ and the superscript T denotes the transpose of a vector.

We have to note that since $z^T f(z) = \sum_{i=1}^{n} z_i f_i(z)$, this can be equivalently stated as

$$z_i \geqslant 0, \quad f_i(z) \geqslant 0 \quad \text{and} \quad z_i f_i(z) = 0, \text{ for all } i = 1, 2, ..., n. \tag{2}$$

In effect then complementarity states that either z_i or $f_i(z)$ must be zero for each $i = 1, 2, ..., n$.

Note that in the particular case where the function f is linear, that is to say it is written as follows $f(z) = q + Mz$, where $M \in \mathrm{IR}^{n\times n}$ and $q \in \mathrm{IR}^n$ are given, then we say that the problem (1) is a Linear Complementarity Problem noted

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$LCP(q, M)$. This particular problem is an important problem in mathematical programming (see, e.g., EL Foutayeni[14] and Garcia et al.[20] for references). To solve this problem, there are several methods and algorithms, we cite for example, Lemke[23] first presented a solution for this problem. His ideas were later exploited by Scarf[28] in his work on fixed point algorithms. The relationship between the $LCP(q, M)$ and the fixed point problem is well described by Eaves and Scarf[6] and by Eaves and Lemke[4]. Cottle and Dantzig's principal pivot method[2] and recently our modest works [7, 8, 15, 16]. In the first one, we have built an interior point method to solve a linear complementarity problem $LCP(q, M)$; the convergence of this method requires $o(\sqrt{n}L)$ number of iterations where L is the length of a binary coding of the input data of the problem $LCP(q, M)$. This interior point method is globally efficient and has a good iteration complexity but it has the problem of finding a strictly feasible starting point. In the second one, we have given a globally convergent hybrid method which is based on vector divisions and the secant method for solving the $LCP(q, M)$. In the third one, we have given a general characterization of a linear complementarity problem $LCP(q, M)$. Furthermore, through the paper [15], we can provide the solution (if it exists) of a linear complementarity problem in a straightforward manner and according to the data. In the fourth one, we have shown that the linear complementarity problem $LCP(q, M)$ is completely equivalent to finding the fixed point of the map $x = \max(0, (I - M)x - q)$; to find an approximation of the solution to the second problem, we have proposed an algorithm starting from any interval vector $X^{(0)}$ and generating a sequence of the interval vector $(X^{(k)})_{k=1,..}$ which converges to the exact solution of linear complementarity problem.

In the general case where the function f is non-linear, then we say that the problem (1) is a Non-Linear Complementarity Problem noted $NLCP(f)$, this problem can be solved by methods for finding a zero point in several ways. Converting the $NLCP(f)$ into a zero finding problem. Numerous methods and algorithms exist to solve non-linear complementarity problems such as the homotopy methods of Merrill[26] and several other authors (see for example Eaves[3, 5], Saigal[27]), using a reformulation of the $NLCP(f)$ due to Mangasarian[25] in which the zero finding problem can be made as smooth as desired, Watson[29] applied the homotopy or continuation method of Chow, Mallet-Paret and Yorke[1] to solve the problem. Instead of reformulating the $NLCP(f)$ as a zero finding problem, other authors adjusted simplicial fixed point algorithms to solve the $NLCP(f)$ directly, see e.g. Fisher et al.[18], Garcia[19], Kojima[22] or Lüthi[24]. All of the mentioned methods and algorithms for solving the non-linear complementarity problem require a lot of arithmetic operations, and therefore, it is too difficult, time consuming, or expensive to find an approximate solution of the exact solution of $NCP(f)$.

In this paper, we assume that the problem (1) has a unique solution; our objective is to find this solution. To calculate it, we give a new method for solving the non-linear complementarity problem in general case; this method converges very rapidly relative to most of the existing methods and does not require a lot of arithmetic operations to converge. We assume that the problem has a unique solution and we show that solving $NLCP(f)$ is equivalent to solving the system of non-linear equations $F(x) = 0$, where F is a function from IR^n into itself defined by $F(x) = f(|x| - x) - |x| - x$. We have to note that there is no method, to our knowledge, that gives a solution which converges very rapidly compared to existing methods because of the non-differentiability of the function F. This is why we are building a sequence of smooth functions $F^{(k)} \in C^{\infty}$ which is uniformly convergent to the function F; and we show that, an approximation of the solution of $NLCP(f)$ is obtained by solving $F^{(k)}(x) = 0$ for a parameter k large enough.

The rest of this paper is organized as follows. In section 2 we briefly give some definitions and notations to be used throughout the paper, and we show that the non-linear complementarity problem $NLCP(f)$ is equivalent to solving a system of non-linear equations. In section 3 we give an algorithm for solving this system of non-linear equations. This algorithm is based on the idea of the well known Newton's method. In section 4 we give some numerical examples to illustrate our theoretical results and to show that this method can solve efficiently large-scale non-linear complementarity problems and in the last section, we present our conclusions.

2 Main result

The mathematical formulation of the non-linear complementarity problem is as follows. Let f be a given continuous function from IR^n into itself and let f_i and z_i, $i = 1, 2, ..., n$, denote the components of f and z, respectively. The non-linear complementarity problem, denoted by $NLCP(f)$ for short, which is to find a point z such that

$$z_i \geqslant 0, \quad f_i(z) \geqslant 0 \quad \text{and} \quad z_i f_i(z) = 0, \quad \text{for all } i = 1, 2, ..., n. \tag{3}$$

Here, we assume that f is a uniformly continuous function. In the rest we give a completely equivalent formulation of the non-linear complementarity problem as a system of n non-linear equations in n unknown and thereby make possible the use of the powerful tools of non-linear equations theory. Thereafter, let $F(x) = f(|x| - x) - |x| - x$, where $|x| = (|x_1|, |x_2|, ..., |x_n|)^T \in \mathrm{IR}^n$. Our principal result is the following theorem

Theorem 1 *The vector z solves the non-linear complementarity problem (1) if and only if the vector $x = \frac{1}{2}(f(z) - z)$ solves the equation $F(x) = 0$.*

Proof. Necessity. Let z be a solution of (1) and let

$$x = \frac{1}{2}(f(z) - z) \tag{4}$$

then $|x| = \frac{1}{2}|f(z) - z|$; thus $|x| - x = \frac{1}{2}(|f(z) - z| - (f(z) - z))$.

Now, if $z_i > 0$ then $f_i(z) = 0$, therefore $|x_i| - x_i = z_i$;

else i.e. $z_i = 0$ then $f_i(z) \geqslant 0$, therefore $|x_i| - x_i = z_i$.

So in all cases

$$|x| - x = z. \tag{5}$$

Now using (4) and (5) we have $|x| + x = f(z)$, thus, $|x| + x = f(|x| - x)$, and therefore x is a zero of the function $F(x) = f(|x| - x) - |x| - x$.

Sufficiency. Let x be a zero of the function F then we have

$$f(|x| - x) = |x| + x. \tag{6}$$

Let $z := |x| - x$ then we have

(a) $z \geqslant 0$;

(b) $f(z) = f(|x| - x) = |x| + x \geqslant 0$;

(c) $z^T f(z) = (|x| - x)^T (|x| + x) = 0$.

Therefore z is a solution of (1). This completes the proof of the Theorem. ■

We have to note that the previous theorem shows that the problem (1) has a solution if and only if the system of non-linear equations $F(x) = 0$ has a solution. Now we show that

Theorem 2 *Let assume that the non-linear complementarity problem (1) has a unique solution. Then $x = \frac{1}{2}(f(z) - z)$ is a unique solution of the system of non-linear equations $F(x) = 0$.*

Proof. Let assume that the non-linear complementarity problem (1) has a unique solution z and let x_1 and x_2 be two distinct zeros of the function F, then

$$\begin{cases} z_1 := |x_1| - x_1 \text{ is a solution of } (1) \\ z_2 := |x_2| - x_2 \text{ is a solution of } (1) \end{cases}$$

On the one hand, since $z_1 = z_2$ ($= z$, uniqueness of the solution of (1)) then

$$|x_1| - x_1 = |x_2| - x_2. \tag{7}$$

On the other hand, $f(z_1) = f(z_2)$ implies that

$$|x_1| + x_1 = |x_2| + x_2. \tag{8}$$

From (8) and (7) we have that $x_1 = x_2$. This contradicts x_1 and x_2 are two distinct zeros of the function F. This completes the proof of the Theorem. ■

As mentioned above, solving the non-linear complementarity problem $NLCP(f)$ is equivalent to finding the zero of the function F. In the literature, there is no effective and rapid method to solve the equation $F(x) = 0$. Note that to use one of the existing effective and rapid methods, it is necessary that the function F must be at least class C^1, but it is clear that this function is however not the case here. The problem of the function F is that it contains the absolute value $|x|$.

In the following analysis, our goal is to build a sequence of smooth functions $F^{(k)} \in C^\infty$ which uniformly converges to the function F (recall that the sequence of smooth functions $g^{(k)}$ converges uniformly to g on IR^n if $\left\|g^{(k)}(x) - g(x)\right\| \to 0$ as $k \to +\infty$, where $\|x\|$ denotes the Euclidean norm of x), and we show that finding the zero of the function F is e-quivalent to finding the zero of the sequence of smooth functions $F^{(k)}$.

For the rest of this paper, we use the following notations

(a) For $x, y \in \mathrm{IR}^n$, $x \leqslant y$ meaning that $x_i \leqslant y_i$ for all $i = 1, 2, ..., n$;

(b) For $x \in \mathrm{IR}^n$ we denote by $x^2 := (x_1^2, x_2^2, ..., x_n^2)^T \in \mathrm{IR}^n$;

(c) For $x \in \mathrm{IR}^n$ we denote by $\sqrt{x} := (\sqrt{x_1}, \sqrt{x_2}, ..., \sqrt{x_n})^T \in \mathrm{IR}^n$;

(d) For $x \in \mathrm{IR}^n$ and $k \in \mathrm{IN}^*$ we denote by

$$x^2 + \frac{1}{k^2} := (x_1^2 + \frac{1}{k^2}, x_2^2 + \frac{1}{k^2}, ..., x_n^2 + \frac{1}{k^2})^T \in \mathrm{IR}^n.$$

Lemma 3 *The sequence of smooth functions* $x \in \mathrm{IR}^n \mapsto \sqrt{x^2 + \frac{1}{k^2}}$ *converges uniformly to the absolute value function* $x \in \mathrm{IR}^n \mapsto |x|$ *on* IR^n *when* $k \to +\infty$.

Proof. Since $x^2 + \frac{1}{k^2} \geqslant x^2$ for all $x \in \mathrm{IR}^n$ and all $k \in \mathrm{IN}^*$, then we have

$$\sqrt{x^2 + \frac{1}{k^2}} \geqslant \sqrt{x^2} = |x|. \tag{9}$$

Moreover the inequality $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$ is valid for all positive real numbers a and b, we deduce that

$$\sqrt{x^2 + \frac{1}{k^2}} \leqslant \sqrt{x^2} + \sqrt{\frac{1}{k^2}} = |x| + \frac{1}{k}. \tag{10}$$

Thus, for any $x \in \mathrm{IR}^n$ we have

$$0 \leqslant \sqrt{x^2 + \frac{1}{k^2}} - |x| \leqslant \frac{1}{k}, \tag{11}$$

this implying that

$$\sup_{x \in \mathrm{IR}^n} \left\| \sqrt{x^2 + \frac{1}{k^2}} - |x| \right\| \leqslant \frac{1}{k}. \tag{12}$$

As the numerical sequence $(u_k)_k$ general term $u_k = \frac{1}{k}$ converges to 0, we deduce that the sequence of smooth functions $x \in \mathrm{IR}^n \mapsto \sqrt{x^2 + \frac{1}{k^2}}$ converges uniformly to $|x|$ on IR^n. This completes the proof of the Lemma. ■

We now consider the sequence of smooth functions $F^{(k)} : \mathrm{IR}^n \to \mathrm{IR}^n$ defined by

$$F^{(k)}(x) := f\left(\sqrt{x^2 + \frac{1}{k^2}} - x\right) - \sqrt{x^2 + \frac{1}{k^2}} - x,$$

and we show that

Theorem 4 *The sequence of smooth functions* $(F^{(k)})_{k \geqslant 1}$ *converges uniformly to* F *on* IR^n *when* $k \to +\infty$.

Proof. Using the previous Lemma we get

$$\sqrt{x^2 + \frac{1}{k^2}} \text{ converges uniformly to } |x| \text{ as } k \to +\infty;$$

and using the fact that f is a uniformly continuous function then we have

$$f\left(\sqrt{x^2 + \frac{1}{k^2}} - x\right) \text{ converges uniformly to } f(|x| - x) \text{ as } k \to +\infty.$$

Thus, the sequence of smooth functions $F^{(k)}$ converges uniformly to F when $k \to +\infty$. This completes the proof of the Theorem. ■

Now we show that if $x^*_{(k)}$ is the zero of the sequence of smooth functions $F^{(k)}$, then $x^*_{(k)}$ is an approximation to the solution of the system of non-linear equations $F(x) = 0$ for k large enough.

Corollary 5 *If* $x^*_{(k)}$ *is the solution of the equation* $F^{(k)}(x) = 0$, *then* $x^*_{(k)}$ *is an approximation to the solution of the equation* $F(x) = 0$ *for* k *is large enough.*

Proof. To show that, we use the previous theorem which we can interpret as

$\forall \epsilon > 0, \exists\, k^* > 0$ such that for all $k > k^*$ we have

$$||F(x^*_{(k)})|| = ||F(x^*_{(k)}) - F^{(k)}(x^*_{(k)})|| \leqslant \epsilon$$

then we have for any $\epsilon > 0$, $x^*_{(k)}$ is the approximation to the solution of the equation $F(x) = 0$. This completes the proof of the Corollary. ■

Now we can use any effective and rapid method to solve the system of non-linear equations $F^{(k)}(x) = 0$ for k is large enough. In the next section, we will look at one method in particular, Newton's method (a method other than Newton may be used). We are going to write our own algorithm for solving the system of non-linear equations using Newton's method. Note that Matlab has its own algorithms for solving systems of non-linear equations (e.g. fsolve, which is loosely based on Newton's method).

3 Algorithm

The following algorithm tries to solve the non-linear complementarity problem by solving the equivalent non-linear system of equations

$$F^{(k)}(x) = f\left(\sqrt{x^2+\frac{1}{k^2}}-x\right)-\sqrt{x^2+\frac{1}{k^2}}-x=0.$$

Generally, a non-linear system of equations for $x=(x_1,x_2,...,x_n)^T$ has the following form

$$F_i^{(k)}(x_1,x_2,...,x_n)=0, \quad i=1,2,...,n$$

If we start with a guessed value x^i, we can find a better guess by extrapolation. Near our guessed value, the function can be expanded

$$\begin{bmatrix} F_1^{(k)}(x^i+dx) \\ F_2^{(k)}(x^i+dx) \\ ... \\ F_n^{(k)}(x^i+dx) \end{bmatrix} = \begin{bmatrix} F_1^{(k)}(x^i) \\ F_2^{(k)}(x^i) \\ ... \\ F_n^{(k)}(x^i) \end{bmatrix} + \begin{bmatrix} \frac{\partial F_1^{(k)}}{\partial x_1}dx_1+\frac{\partial F_1^{(k)}}{\partial x_2}dx_2+...+\frac{\partial F_1^{(k)}}{\partial x_n}dx_n \\ \frac{\partial F_2^{(k)}}{\partial x_1}dx_1+\frac{\partial F_2^{(k)}}{\partial x_2}dx_2+...+\frac{\partial F_2^{(k)}}{\partial x_n}dx_n \\ ... \\ \frac{\partial F_n^{(k)}}{\partial x_1}dx_1+\frac{\partial F_n^{(k)}}{\partial x_2}dx_2+...+\frac{\partial F_n^{(k)}}{\partial x_n}dx_n \end{bmatrix}$$

Note that these derivatives are evaluated at the $x=x^i$.

$$\begin{bmatrix} F_1^{(k)}(x^i+dx) \\ F_2^{(k)}(x^i+dx) \\ ... \\ F_n^{(k)}(x^i+dx) \end{bmatrix} = \begin{bmatrix} F_1^{(k)}(x^i) \\ F_2^{(k)}(x^i) \\ ... \\ F_n^{(k)}(x^i) \end{bmatrix} + \begin{bmatrix} \frac{\partial F_1^{(k)}}{\partial x_1} & \frac{\partial F_1^{(k)}}{\partial x_2} & ... & \frac{\partial F_1^{(k)}}{\partial x_n} \\ \frac{\partial F_2^{(k)}}{\partial x_1} & \frac{\partial F_2^{(k)}}{\partial x_2} & ... & \frac{\partial F_2^{(k)}}{\partial x_n} \\ ... & ... & ... & ... \\ \frac{\partial F_n^{(k)}}{\partial x_1} & \frac{\partial F_n^{(k)}}{\partial x_2} & ... & \frac{\partial F_n^{(k)}}{\partial x_n} \end{bmatrix}_{(x^i)} \begin{bmatrix} dx_1 \\ dx_2 \\ ... \\ dx_n \end{bmatrix}$$

In other word

$$F^{(k)}(x^i+dx)=F^{(k)}(x^i)+Jdx, \tag{13}$$

where J is the Jacobian matrix, it is given by

$$J=\begin{bmatrix} \tilde{X}_1D_1f_1(X)-\bar{X}_1 & \tilde{X}_2D_2f_1(X) & ... & \tilde{X}_nD_nf_1(X) \\ \tilde{X}_1D_1f_2(X) & \tilde{X}_2D_2f_2(X)-\bar{X}_2 & ... & \tilde{X}_nD_nf_2(X) \\ .... & .... & ... & .... \\ \tilde{X}_1D_1f_n(X) & \tilde{X}_2D_2f_n(X) & ... & \tilde{X}_nD_nf_n(X)-\bar{X}_n \end{bmatrix} \tag{14}$$

where

$X=(X_1,X_2,...,X_n)^T$ such as $X_i=\sqrt{\left(x_i^2+\frac{1}{k^2}\right)}-x_i$ for all $i=1,2,...,n$;

$\tilde{X}=(\tilde{X}_1,\tilde{X}_2,...,\tilde{X}_n)^T$ such as $\tilde{X}_i=\frac{1}{\sqrt{\left(x_i^2+\frac{1}{k^2}\right)}}x_i-1$ for all $i=1,2,...,n$;

$\bar{X}=(\bar{X}_1,\bar{X}_2,...,\bar{X}_n)^T$ such as $\bar{X}_i=-\frac{1}{\sqrt{\left(x_i^2+\frac{1}{k^2}\right)}}x_i-1$ for all $i=1,2,...,n$; and where we have denoted by $D_if_j(X)$ the partial derivative of f_j with respect to x_i evaluated at the point X.

Thus the Jacobian matrix J given by (14) can be written in the following form

$$J=J_f(X)\times Diag(\tilde{X})-Diag(\bar{X}), \tag{15}$$

where

$$J_f(X)=\begin{bmatrix} D_1f_1(X) & D_2f_1(X) & ... & D_nf_1(X) \\ D_1f_2(X) & D_2f_2(X) & ... & D_nf_2(X) \\ .... & .... & ... & .... \\ D_1f_n(X) & D_2f_n(X) & ... & D_nf_n(X) \end{bmatrix}$$

$$Diag(\tilde{X})=\begin{bmatrix} \tilde{X}_1 & 0 & ... & 0 \\ 0 & \tilde{X}_2 & ... & 0 \\ ... & .... & ... & ... \\ 0 & 0 & ... & \tilde{X}_n \end{bmatrix}$$

$$Diag(\bar{X}) = \begin{bmatrix} \bar{X}_1 & 0 & ... & 0 \\ 0 & \bar{X}_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \bar{X}_n \end{bmatrix}$$

Now we want our new vector $x^{i+1} = x^i + dx$ to be a better approximation to the solution, so we set $F^{(k)}(x^{i+1}) = 0$ in equation (13). Then

$$-F^{(k)}(x^i) = Jdx.$$

The procedure for Newton's method is

- Calculate the function values at the guessed value of $x^i = (x_1, x_2, ..., x_n)^T$;
- Calculate the Jacobian matrix using the current guess for the solution;
- Solve the linear system $-F^{(k)}(x^i) = Jdx$ for the values of dx;
- Update the guessed value $x^{i+1} = x^i + dx$;

This procedure should be repeated, using the updated value of x^i as the guess, until the values of $F^{(k)}(x)$ are sufficiently close to zero. $F^{(k)}(x)$ is a vector of residual errors. For sufficiently close to zero we could use

$$\left\|F^{(k)}(x)\right\| = \left(\sum_{i=1}^{n}(F_i^{(k)}(x))^2\right)^{1/2} < Tolerance.$$

Thus, x is an approximation to the solution of $F(x) = 0$; and therefore $z = |x| - x$ is an approximation to the solution of (1). In the next section we give some numerical examples to illustrate our theoretical results and to show that this method can solve efficiently large-scale non-linear complementarity problems.

4 Numerical tests

In this section, we provide numerical examples to demonstrate the efficiency of our method. To do so, we conducted the numerical experiments on some test problems.

In the following, we will implement our algorithm in Matlab 7.2 and run it on a personal computer with a 1.66 GHZ CPU processor and 1 Go memory. We stop the iterations if the condition $||F^{(k)}(x)|| \leqslant 10^{-6}$ is satisfied for k large enough.

We begin first by noting that this method can be used to solve linear complementarity problems. To test this method for solving the LCP, we take the following examples

Example 1. *Let us consider the following linear complementarity problem $LCP(q, M)$, find vector z satisfying $Mz + q \geqslant 0$, $z \geqslant 0$ and $z^T(Mz + q) = 0$,*

where $M = (m_{ij})_{1\leq i,j\leq n}$ such as $m_{ii} = 4$, $m_{i,i+1} = m_{i+1,i} = -1$ for all $i = 1, ..., n$ and zero in the rest and $q = (q_i)_{1\leq i\leq n}$ such as $q_i = -1$.

This example is used by Geiger and Kanzow[21]. The test results of this example are summarized in Table 1.

Example 2. *Let us consider the following linear complementarity problem $LCP(q, M)$ where $M = (m_{ij})_{1\leq i,j\leq n}$ such as $m_{ij} = i\delta_{ij}/n$* where δ is the Kronecker's delta ($\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$) and $q = (q_i)_{1\leq i\leq n}$ *such as* $q_i = -1$.

This example is used by Geiger and Kanzow[21]. The test results of this example are summarized in Table 2.

We use our method and several known methods (Lemke, Chen et al., Yu and Qin, EL Foutayeni and Khaladi) to solve these examples and compare it with the known exact solution z^* and compare the number of iterations and the execution time for each methods.

When looking for an approximation with six significant digits, we obtain that (see Table 1 and Table 2) using our method who does not require a lot of iterations and not a lot of CPU time by against, the other methods require a lot of arithmetic operations, and therefore, it is too difficult, time consuming, and expensive to find an approximate solution of the exact solution as shown in Tables 1 and 2, where Iter denotes the iteration number when the algorithm terminates; and Time denotes the total cost time (in second) for solving the $LCP(q, M)$.

The table 1 shows that the numerical results of the methods of Lemke, Chen et al., Yu et al. and our method to solve the first example. We have to note that this example is taken from [21]. In this example we take $k = 100$ and $x^{(0)} = (2, 1, ..., 1)^T$.

Table 1: Numerical results of the first example

Method	n	Approximation to the solution of LCP(q,M)	Iter	Time(s)
Lemke	4	(0.363636;0.454545;0.454545;0.363636)	5	0.009567
	8	(0.366013;0.464052;0.490196;0.496732;	5	0.057053
		0.496732;0.490196;0.464052;0.366013)		
Chen, Harker	4	(0.363636;0.454545;0.454545;0.363636)	5	0.016000
Kanzow and	8	(0.366013;0.464052;0.490196;0.496732;	5	0.031000
Smale		0.496732;0.490196;0.464052;0.366013)		
Yu	4	(0.363636;0.454545;0.454545;0.363636)	5	0.031000
and	8	(0.366013;0.464052;0.490196;0.496732;	5	0.016000
Qin		0.496732;0.490196;0.464052;0.366013)		
EL Foutayeni	4	(0.363636;0.454545;0.454545;0.363636)	2	0.004935
and	8	(0.366013;0.464052;0.490196;0.496732;	2	0.008255
Khaladi		0.496732;0.490196;0.464052;0.366013)		

Now let's compare these methods with ours, using the second example. This example is taken from [21]. In this example we take $k = 100$ and $x^{(0)} = (2, 1, ..., 1)^T$. The following table (Table 2) clearly shows the effectiveness of our method over existing methods.

Table 2: Numerical results of the second example

Method	n	Approximation to the solution of LCP(q,M)	Iter	Time(s)
Lemke	4	(4.000000;2.000000;1.333333;1.000000)	5	0.029403
	8	(8.000000;4.000000;2.666667;2.000000;	6	0.049984
		1.600000;1.333333;1.142857;1.000000)		
Chen, Harker	4	(4.000000;2.000000;1.333333;1.000000)	5	0.016000
Kanzow and	8	(8.000000;4.000000;2.666667;2.000000;	7	0.031000
Smale		1.600000;1.333333;1.142860;1.000000)		
Yu	4	(4.000000;2.000000;1.333333;1.000000)	5	0.016000
and	8	(8.000000;4.000000;2.666667;2.000000;	8	0.047000
Qin		1.600000;1.333333;1.142860;1.000000)		
EL Foutayeni	4	(4.000000;2.000000;1.333333;1.000000)	2	0.006999
and	8	(8.000000;4.000000;2.666667;2.000000;	2	0.009289
Khaladi		1.600000;1.333333;1.142857;1.000000)		

Now we test two well-known non-linear complementarity problems by our method. For each test problem, we also compare the numerical performance of the proposed method with various values of k and various initial states x^0. The test instances are described below.

Example 3. *Consider the NLCP, where $f : IR^4 \to IR^4$ is given by*

$$f(z) := \begin{cases} f_1(z) = -z_2 + z_3 + z_4 \\ f_2(z) = z_1 - (4.5z_3 + 2.7z_4)/(z_2 + 1) \\ f_3(z) = 5 - z_1 - (0.5z_3 + 0.3z_4)/(z_3 + 1) \\ f_4(z) = 3 - z_1 \end{cases}$$

Example 4. *Consider the NLCP, where $f : IR^4 \to IR^4$ is given by*

$$f(z) := \begin{cases} f_1(z) = 3z_1^2 + 2z_1z_2 + 2z_2^2 + z_3 + 3z_4 + 6 \\ f_2(z) = 2z_1^2 + z_1 + z_2^2 + 10z_3 + 2z_4 - 2 \\ f_3(z) = 3z_1^2 + z_1z_2 + 2z_2^2 + 2z_3 + 9z_4 - 9 \\ f_4(z) = z_1^2 + 3z_2^2 + 2z_3 + 3z_4 - 3 \end{cases}$$

The numerical results reported in Tables 3 and 4 show that the method proposed in this paper is quite well for solving the complementarity problems.

We compare the results obtained by our method with that obtained by the CHKS and Yu-Qin methods. The results are summarized in Tables 3 and 4, where Iter denotes the iteration number when the algorithm terminates; and Time denotes the total cost time (in second) for solving the NCP problem. We also stop the execution when 500 iterations were completed without achieving convergence and denoted by fail. From Tables 3 and 4, we can see that our method can comparable with the CHKS and Yu-Qin methods from the iteration number and the CPU times.

Table 3: Numerical results for the third example

Method	Approximation to the solution of NLCP(f)	Iter	Time(s)
CHKS	Fail		
Yu-Qin	(3;2.3948E-16;2.8833E-16;-1.0612E-17)	8	0.016000
EL Foutayeni-Khaladi	(3;2.3948E-16;2.8833E-16;-1.0612E-17)	7	0.006999

The table 3 shows that the numerical results of different methods to solve the non-linear complementarity problem $NLCP(f)$ of the third example, the exact solution is $z^* = (3, 0, 0, 0)^T$. Note that this example is taken from [30]. In this example we take $k = 100$ and $x^{(0)} = (2, 1, 1, 1)^T$.

Table 4: Numerical results for the fourth example

Method	Approximation to the solution of NLCP(f)	Iter	Time(s)
CHKS	(1.3100E-15;1.3994E-10;1.4931E-10;1)	6	0.016000
Yu-Qin	(1.3100E-15;1.3994E-10;1.4931E-10;1)	12	0.046000
EL Foutayeni-Khaladi	(1.3100E-15;1.3994E-10;1.4931E-10;1)	4	0.005984

The table 4 shows that the numerical results of different methods to solve the non-linear complementarity problem $NLCP(f)$ of the fourth example, the exact solution is $z^* = (0, 0, 0, 1)^T$. Note that this example is taken from [30]. In this example we take $k = 100$ and $x^{(0)} = (2, 1, 1, 1)^T$.

Now we show that the influence of the parameter k on the value of the solution z of the non-linear complementarity problem. For this we consider the following example

Example 5. *Consider the $NLCP(f)$, where $f : IR^5 \to IR^5$ is given by*

$$f(z) = 2\exp\left(\sum_{i=1}^{5}(z_i - i + 2)^2\right)\begin{pmatrix} z_1 + 1 \\ z_2 \\ z_3 - 1 \\ z_4 - 2 \\ z_5 - 3 \end{pmatrix}.$$

Note that the solution to this problem is $z^* = (0, 0, 1, 2, 3)^T$. We have to note that this example is taken from [29].

Table 5 shows that at a certain rank of k, any solution of the equation $F^{(k)}(x) = 0$ is an approximate solution of the exact solution z^* of non-linear complementarity problem $NLCP(f)$.

5 Conclusion

In this paper we have given a new method for solving the non-linear complementarity problem. This method converges very rapidly relative to most of the existing methods and does not require a lot of arithmetic operations to converge. The results discussed in this paper are very favorable. Even stronger results have been obtained for monotone complementarity problems. The numerical results showed that our method works well for the problems tested. The numerical tests confirm the efficiency of our method. With regard to the nice theoretical results of our algorithm, the computational results reported are very encouraging. We expect our algorithm can also solve large-scale problems well. As one of the remarks we would like to point out that we can use this new method for solving linear complementarity problems.

Table 5: Numerical results for the fifth example

k=1	Iter=27 Time(s)=0.053104 z=(0.1128;0.3543;1.1128;2.0609;3.0413)
k=2	Iter=24 Time(s)=0.062295 z=(0.0689;0.2713;1.0689;2.0362;3.0243)
k=3	Iter=23 Time(s)=0.046551 z=(0.0050;0.0266;0.7754;1.8790;2.9183)
k=10	Iter=22 Time(s)=0.051145 z=(0.0017;0.0156;0.9296;1.9640;2.9759)
k=50	Iter=22 Time(s)=0.0469141 z=(0.0004;0.0072;0.9857;1.9928;2.9952)
k=100	Iter=21 Time(s)=0.0401580 z=(0.0000;0.0000;0.9899;1.9998;2.9952)
k=10^6	Iter=21 Time(s)=0.0467373 z=(0.0000;0.0000;1.0000;2.0000;3.0000)

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