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On the Well-posedeness Problem for the Generalized Dissipative Camassa-Holm Equation

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Abstract: In this paper, we study the local well posedeness problem for the generalized dissipative Camassa-Holm Equation. By applying some Sobolev's inequalities and related knowledge of PDE and using Kato's theory, we proof that there is a unique local solution of this problem which continuously depending on the initial value.

Keywords: dissipative Camassa-Holm Equation; Kato's theory; local well posedeness

1 Introduction

In 1993, through Hamilton method R.Camassa and D.Holm [1] presented a new type of completely integrable dispersive shallow water wave equation:

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1}$$

Where u is the fluid velocity in the x direction (equivalently the height of the water's free surface above a flat bottom), κ is a constant related to the critical shallow water wave speed, and subscripts denote partial derivatives. The Camassa-Holm equation (CHE) possesses especial soliton solution-peaked when $\kappa = 0$ and the CHE is bi-hamiltonian. After that, a lot of research and observations about CHE emerge. The conserved quantities and initial value problem of CHE are investigated in [2]. Symmetries of CHE is discussed in [3]. In 1998, A.Constantin and J.Escher [4] investigated the CHE when $\kappa = 0$

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{2}$$

They obtained the existence of global solution and investigated its Blow-up and structure. In [6] Ding Danping and Tian Lixin investigated the dissipative Camassa-Holm equation as follow

$$u_t - u_{xxt} + 3uu_x - \varepsilon(u_{xx} - u_{xxxx}) = 2u_x u_{xx} + uu_{xxx} \tag{3}$$

They obtained that the dissipative Camassa-Holm equation possessed global attractor under condition $u_0 \in H_0^1(R)$, peaked solution and the stationary solution.

In this paper , we are interested in the study of the Cauchy problem for a generalized case of Eq. (3). Thus replaces the term $3uu_x$ of Eq. (3) with $3u^m u_x$. So we write the initial value problem of the generalized dissipative Camassa-Holm equation as

$$\begin{cases} u_t - u_{xxt} + 3u^m u_x - \varepsilon(u_{xx} - u_{xxxx}) = 2u_x u_{xx} + uu_{xxx}, t > 0,\ x \in R \\ u(0, x) = u_0(x) \end{cases} \tag{4}$$

2 Notations

We shall use the following notations without further comment. $\|\cdot\|_X$ for the norm of the Banach space X ;$\mathscr{B}(X, Y)$ denotes the space of all bounded linear operator from X to Y ($\mathscr{B}(X)$ if $X = Y$); $\mathscr{D}(A)$ for the domain of the operator

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A, $\partial = \partial_x = \frac{\partial}{\partial x}$; $\Lambda^s = (1-\partial_x^2)^{s/2}, s \in R$; $H^s = H^s(R)$ with norm$\| f \|_{H^s} = \| f \|_s = (\int_R (1+|\xi|^s)|\hat{f}(\xi)|^2 d\xi)^{1/2}$ and $< \cdot, \cdot >_s$ for its inner product; $[A, B]$ denotes the commutator of the linear operators A and B; and $C(I; X)$ for the space of all continuous functions on an interval I into Banach space X; if I is compact, it is seen as a Banach space with the sup norm.

3 Some useful lemmas

In this paper we apply some well-know lemmas. So we list here without proof (see [7]-[11]).

Lemma 1 *Let $s, t \in R$ such that $-s < t \le s$. then*

$$\| fg \|_t \le c \| f \|_s \| g \|_t \quad if\ s > \frac{1}{2}$$

and

$$\|fg\|_{s+t-m/2} \le c\|f\|_s\|g\|_t \ \ if\ s < \frac{1}{2}$$

where c is a positive constant depending on $s,\ t$.

Lemma 2 (kato's) *Let $f \in H^r,\ r > \frac{3}{2},\ M_f$ is the multiplication operator by f . Then*

$$\Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f]\Lambda^{-\tilde{t}} \in \mathscr{B}(L^2(R)) \ \ if\ |\tilde{t}|,\ |\tilde{s}| \le r-1$$

Moreover

$$\|\Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f]\Lambda^{-\tilde{t}}\omega\|_0 \le c\|f'\|_{r-1}\|\omega\|_0$$

wherec > 0 is a constant.

Lemma 3 *Let $f, g \in H^s$ and $s > \frac{1}{2}$, then*

$$\|fg\| \le c\|f\|_s\|g\|_s$$

That's because H^s is a Banach algebra for $s > \frac{1}{2}$.

Lemma 4 *Let $s > \frac{3}{2}$, then $\|u_x\|_{L^\infty} \le \|u\|_s$*

This lemma derives directly from the Sobolev embedding theorem.

4 Kato's theory

Consider the Cauchy problem associated to a quasilinear evolution equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u)u = f(u) \in X,\ t \ge 0 \\ u(0) = u_0 \in Y \end{cases} \tag{5}$$

where $A(u)$ is a linear operator depending on the unknown , and the initial value. To study the Cauchy problem (local in the time) associated to (5) we will make the following assumptions:

(X) X and Y are reflexive Banach spaces where $X \subset Y$ with the inclusion continuous and dense, and there is an isomorphism S from Y onto X such that $\|\phi\|_Y = \|S\phi\|_X$ for all $\phi \in Y$.

(A_1) Let W be an open ball centered in 0 and contained in Y. The linear operator$A(u) \in G(X, 1, \beta)$ where β is a real number, i.e., $-A(u)$ generates a c_0-semigroup such that

$$\|e^{-sA(u)}\|_{\mathscr{B}(X)}$$

Note that if X is a Hilbert space, then $A(u) \in G(X, 1, \beta)$ if and only if

$(a)\langle A\phi, \phi\rangle_X \ge -\beta\|\phi\|_X^2,\ \forall\phi \in \mathscr{D}(A)$

$(b)(A+\lambda)$ is onto for some (all) $\lambda > \beta$ Under these conditions $A(u)$ is said to be quasi-m-accretive.

(A_2) The map

$$w \in W \to B(w) = [S, A(w)]S^{-1} \in \mathscr{B}(X)$$

is uniformly bounded and Lipchitz continuous, that is, there exist constants $\lambda_1, \mu_1 > 0$, such that

$$\begin{aligned} &\|B(w)\|_{\mathscr{B}(X)} \le \lambda_1 \\ &\|B(w) - B(y)\|_{\mathscr{B}(X)} \le \mu_1 \|w - y\|_y \end{aligned} \tag{6}$$

for all $w, y \in W$

(A_3) $Y \subseteq D(A(w))$ for each $w \in W$ (so that $A|_Y \in \mathscr{B}(X,Y)$ by the Closed Graph theorem). Moreover, the map $w \in W \to A(w) \in \mathscr{B}(Y,X)$ satisfies the following Lipschitz condition:

$$\|A(w) - A(y)\|_{\mathscr{B}(Y,X)} \le \mu_2 \|w - y\|_X \tag{7}$$

for all $w, y \in W$, where μ_2 is a non-negative constant.

(f_1) The function $f : W \to Y$ is bounded, i.e. there is a constant $\lambda_2 > 0$ such that

$$\|f(w)\|_Y \le \lambda_2 \tag{8}$$

for all $w \in W$, and the function $w \in W \to f(w)$ is Lipschitz in $X(Y)$, i.e.

$$\|f(w) - f(y)\|_X \le \mu_3 \|w - y\|_X, \ \ \forall w, y \in W \tag{9}$$

$$\|f(w) - f(y)\|_Y \le \mu_4 \|w - y\|_Y, \ \ \forall w, y \in W \tag{10}$$

where μ_3, μ_4 is non-negative constant.

In practice, as we will see, the value of $\beta, \lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3, \mu_4$ are functions of r, the radius of ball W.

Theorem 5 (Kato) *Assume conditions $(X), (A_1) - -(A_3), (f_1)$. Given $u_0 \in Y$; there is $T > 0$ and unique solution $u \in C(I, Y) \bigcap C^1(I, X)$ to (5) with $u(0) = u_0$, where $I = [0, T]$. Moreover, the map $u_0 \in Y \to u \in C([0, T], Y)$ is continuous in the following sense.*

Assume that $s-\lim\limits_{n\to\infty} A_n(w) \overset{\mathscr{B}(X,Y)}{=} A_\infty(w)$, $s-\lim\limits_{n\to\infty} B_n(w) \overset{\mathscr{B}(X)}{=} B_\infty(w)$, $\lim\limits_{n\to\infty} f_n(w) \overset{Y}{=} f_\infty(w)$, $\lim\limits_{n\to\infty} u_{0,n} = u_{0,\infty}$. where $s - \lim$denotes the strong limit. Consider the sequence of Cauchy problems

$$\begin{cases} \partial_t u_n + A_n(u_n) u_n = f_n(u_n) \\ u_n(0) = u_{0,n}, \quad n \in Z \bigcup \{\infty\} \end{cases} \tag{11}$$

Suppose also that $(X), (A_1) - -(A_3), (f_1)$ hold for all equations in (10) with the same X, Y, S, W, and that the constants $\beta, \lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3, \mu_4$can be chosen independently of n. Let T_n be the time of existence of the solutions u_n. Then all u_n's with $n < \infty$ sufficiently large, can be extended (if necessary) to $[0, T_\infty]$ and

$$\lim_{n\to\infty} \sup_{[0,T_\infty]} \|u_n(t) - u_\infty(t)\|_Y = 0$$

5 Local theory

In this section we will apply Kato's theory to establish local well posedness for the Cauchy problem associated to the generalized dissipative Camassa-Holm equation.

The equation (4) can be rewritten in the following way

$$\begin{cases} u_t + uu_x - \varepsilon u_{xx} = \partial_x (1 - \partial_x^2)^{-1} (\frac{1}{2} u^2 - \dfrac{3}{m+1} u^{m+1} - \frac{1}{2} (u_x)^2) \\ u(0, x) = u_0(x) \end{cases} \tag{12}$$

Theorem 6 *Let $u_0 \in H^s, s > \frac{5}{2}$ then there exist $T > 0$ depending on $\|u_0\|_s$, and unique solution u to (4) (or (12)) such that $u \in C([0, T], H^s) \bigcap C^1([0, T], H^{s-1})$ Moreover, the map $u_0 \in H^s \to u \in C([0, T], H^s)$ is continuous in the sense described in theorem 5(Kato)*

To prove this result we will apply Theorem 1, with $X = H^{1/2}$, $Y = H^s$, $S = \Lambda^{s-1/2}$, $\Lambda = (1-\partial_x^2)^{1/2}$, $A(u) = (u-\varepsilon\partial_x)\partial_x$, $f(u) = \partial_x(1-\partial_x^2)^{-1}(\frac{1}{2}u^2 - \frac{3}{m+1}u^{m+1} - \frac{1}{2}(u_x)^2)$ and $W = \{\varphi \in H^s | \|\varphi\|_s \le R\} = \overline{B}(0,R)$
We begin with the following lemma.

Lemma 7 *The operator* $A(u) = (u-\varepsilon\partial_x)\partial_x \in G(H^{1/2}, 1, \beta)$ *with* $u \in H^s, s > \frac{5}{2}$.

Proof. First, we claim that

$$\langle (u-\varepsilon\partial_x)\partial_x\varphi, \varphi\rangle_{1/2} \ge -\beta\|\varphi\|_{1/2}^2 \tag{13}$$

We write the left hand side of (13) as follows:

$$\begin{aligned}
\langle (u-\varepsilon\partial_x)\partial_x\varphi, \varphi\rangle_{1/2} &= \langle \Lambda^{1/2}(u-\varepsilon\partial_x)\partial_x\varphi, \Lambda^{1/2}\varphi\rangle_0 \\
&= \langle [\Lambda^{1/2}, u]\partial_x\varphi + u\partial_x\Lambda^{1/2}\varphi - \varepsilon\partial_x^2\varphi, \Lambda^{1/2}\varphi\rangle_0 \\
&= \Big\langle \Big[\Lambda^{1/2}, u\Big]\partial_x\varphi, \Lambda^{1/2}\varphi\Big\rangle_0 - \Big\langle \partial_x u, (\Lambda^{1/2}\varphi)^2\Big\rangle_0 - \varepsilon\Big\langle \Lambda^{1/2}\partial_x^2\varphi, \Lambda^{1/2}\varphi\Big\rangle_0
\end{aligned}$$

We will show that each of the terms on the RHD of above can be estimated by $c\|u\|_s\|\varphi\|_{1/2}^2$, where c is a positive constant. We have

$$\begin{aligned}
\Big|\Big\langle \Big[\Lambda^{1/2}, u\Big]\partial_x\varphi, \Lambda^{1/2}\varphi\Big\rangle_0\Big| &\le \Big\|\Big[\Lambda^{1/2}, u\Big]\Lambda^{1/2}\Lambda^{-1/2}\partial_x\varphi\Big\|_0\Big\|\Lambda^{1/2}\varphi\Big\|_0 \\
&\le \Big\|u\Big\|_s\Big\|\varphi\Big\|_{1/2}^2
\end{aligned} \tag{14}$$

Where we applied Lemma 2 with $\tilde{s} = 0, \tilde{t} = -\frac{1}{2}$. According to lemma 4 the second term is bounded by

$$\Big|\Big\langle \partial_x u, (\Lambda^{1/2}\varphi)^2\Big\rangle_0\Big| \le \Big\|u_x\Big\|_{L^\infty}\Big\|\Lambda^{1/2}\varphi\Big\|_0^2 \le \Big\|u\Big\|_s\Big\|\varphi\Big\|_{1/2}^2 \tag{15}$$

and the last term of the RHS can be easily estimated as follow

$$\Big|\varepsilon\Big\langle \Lambda^{1/2}\partial_x^2\varphi, \Lambda^{1/2}\varphi\Big\rangle_0\Big| \le \varepsilon\Big\|\varphi\Big\|_{1/2}^2 \tag{16}$$

So combining (14) (15) (16) we get (13)

To complete the proof , we need to show that $(A(u)+\lambda)$ is onto $H^{1/2}$ for some $\lambda > \beta$. The fact that $A(u)$ is a closed operator, together with inequality (13) shows that $(A(u)+\lambda)$ has closed range for $\lambda > \beta$. Thus, it suffices to prove that $(A(u)+\lambda)$ has dense range for $\lambda > \beta$. Let $\varphi \in H^{1/2}$be such that $\langle (A(u)+\lambda)\phi, \varphi\rangle_{1/2} = 0$ for all $\phi \in \mathscr{D}(A(u)) = \{\phi \in H^{1/2} | (u-\varepsilon\partial_x)\partial_x\phi \in H^{1/2}\}$. Then $\varphi \in \mathscr{D}(((u-\varepsilon\partial_x)\partial_x)^*) = \{g \in H^{1/2} | (u-\varepsilon\partial_x)\Lambda g \in H^{1/2}\} \subseteq \mathscr{D}(A(u))$ and satisfies the equation $-\Lambda^{-1}\partial_x((u-\varepsilon\partial_x)\Lambda\varphi) + \lambda\varphi = 0$. Applying Λ to this equation, multiplying by $\Lambda^{1/2}\varphi$ integrating by parts and invoking (13), we obtain

$$0 = \Big\langle \Lambda^{1/2}((u-\varepsilon\partial_x)\partial_x\varphi) + \lambda\Lambda^{1/2}\varphi, \Lambda^{1/2}\varphi\Big\rangle_0 \ge (\lambda-\beta)\Big\|\varphi\Big\|_{1/2}^2,\ \lambda > \beta$$

Since $(\lambda-\beta) > 0$, we conclude that $\varphi = 0$. ■

Lemma 8 *(i)*$B(u) = [\Lambda^{s-1/2}, (u-\varepsilon\partial_x)\partial_x]\Lambda^{1/2-s} \in \mathscr{B}(H^{1/2})$ *for* $u \in H^s, s > \frac{5}{2}$
(ii)$\|(B(u)-B(v))w\|_{1/2} \le \|w\|_{1/2}\|u-v\|_s,\ s > \frac{5}{2}$

Proof. (i)Note that

$$\begin{aligned}
[\Lambda^{s-1/2}, (u-\varepsilon\partial_x)\partial_x]\Lambda^{1/2-s} &= [\Lambda^{s-1/2}, u\partial_x]\Lambda^{1/2-s} - [\Lambda^{s-1/2}, \varepsilon\partial_x^2]\Lambda^{1/2-s} \\
&= [\Lambda^{s-1/2}, u]\Lambda^{1/2-s}\partial_x - 0
\end{aligned}$$

Therefore

$$\begin{aligned}
\|B(u)w\|_{1/2} &= \|\Lambda^{1/2}[\Lambda^{s-1/2}, u]\Lambda^{1-s}\Lambda^{-1/2}\partial_x w\|_0 \\
&\le \|u\|_s\|\Lambda^{-1/2}\partial_x w\|_0 \\
&\le \|u\|_s\|w\|_{1/2}
\end{aligned} \tag{17}$$

where we used Lemma 2 once again (with $\tilde{s} = -\frac{1}{2}, \tilde{t} = s-1$).
(ii) Note that, replacing u by $u-v$ in inequality (17) we get

$$\|(B(u)-B(v))w\|_{1/2} \le \|w\|_{1/2}\|u-v\|_s$$

We complete the proof of lemma 8 ■

Lemma 9 *(i)*$H^s \subseteq \mathscr{D}((u-\varepsilon\partial_x)\partial_x) = \{f \subseteq H^{1/2} | (u-\varepsilon\partial_x)\partial_x f \in H^{1/2}\},\ s > \frac{5}{2}$
(ii)$A(u) \in \mathscr{B}(H^s, H^{1/2}),\ s > \frac{5}{2}$
(iii)$\|A(u)-A(v)\|_{\mathscr{B}(H^s,H^{1/2})} \le \mu\|u-v\|_{1/2}$

Proof. Let $w \in H^s, s > \frac{5}{2}$. Then

$$\|(u-\varepsilon\partial_x)\partial_x w\|_{1/2} \le \|u\|_{1/2}\|\partial_x w\|_{s-1} + \varepsilon\|\partial_x^2 w\|_{s-2} \le (\|u\|_{1/2}+\varepsilon)\|w\|_s$$

where, we have used Lemma 1 This proves (i) and (ii). Part (iii) follows at once from this inequality, replacing u by $u-v$. ■

Lemma 10 *Let* $f(u) = \partial_x(1-\partial_x^2)^{-1}(\frac{1}{2}u^2 - \frac{3}{m+1}u^{m+1} - \frac{1}{2}(u_x)^2)$. *Then*
(i)$\|f(u)\|_s \le \mu,\ s > \frac{5}{2}$
(ii)$\|f(u)-f(v)\|_{1/2} \le c\|u-v\|_{1/2}$
(iii)$\|f(u)-f(v)\|_s \le c\|u-v\|_s,\ s > \frac{5}{2}$

Where μ and c are positive constants.
Proof. We begin our proof with (ii). Note that

$$\begin{aligned}
\|f(u)-f(v)\|_{1/2} &\le \|\Lambda^{-1/2}(u^2-v^2)\|_0 + \|\Lambda^{-1/2}\partial_x(u-v)\partial_x(u+v)\|_0 \\
&+ \frac{3}{m+1}\|\Lambda^{-1/2}(u^{(m+1)}-v^{(m+1)})\|_0 \\
&\le \|(u+v)(u-v)\|_0 + \|\partial_x(u+v)\partial_x(u-v)\|_{-1/2} + \frac{3}{m+1}\|u^{m+1}-v^{m+1}\|_0
\end{aligned}$$

The RHS of this inequality can be estimated as follows

$$\|(u-v)(u+v)\|_0 \le \|u+v\|_{L^\infty}\|u-v\|_0 \le \|u+v\|_s\|u-v\|_{1/2} \tag{18}$$
$$\|\partial_x(u+v)\partial_x(u-v)\|_{-1/2} \le \|\partial_x(u-v)\|_{-1/2}\|\partial_x(u+v)\|_{s-1} \le \|u+v\|_s\|u-v\|_{1/2} \tag{19}$$
$$\begin{aligned}
\|u^{m+1}-v^{m+1}\|_0 &= \|(u-v)(u^m+u^{m-1}v+\cdots+v^m)\|_0 \\
&\le \|u-v\|_0\|u^m+u^{m-1}v+\cdots+v^m\|_s
\end{aligned}$$

In these terms , we applied lemma 1 and lemma 4. but for the third term we have to make further estimate. Note that $s > \frac{5}{2}$, so we can use lemma 3. Therefore we obtain

$$\|u^m+u^{m-1}v+\cdots+v^m\|_s \le \|u\|_s^m\|v\|_s + \cdots + \|v\|_s^m \triangleq K < \infty$$

Then

$$\|u^{m+1}-v^{m+1}\|_0 \le K\|u-v\|_{1/2} \tag{20}$$

Combining (18)-(20) we completed the proof of (ii)
For (iii) we can directly apply lemma 3 and get the following conclusion

$$\begin{aligned}
\|f(u)-f(v)\|_s &\le \|\partial_x(1-\partial_x^2)^{-1}(\frac{1}{2}(u^2-v^2) - \frac{1}{2}(u_x^2-v_x^2))\|_s \\
&+ \|\partial_x(1-\partial_x^2)^{-1}(\frac{3}{m+1}(u^{m+1}-v^{m+1})\|_s \\
&\le \|\frac{1}{2}(u_x^2-v_x^2)\|_{s-1} + \|\frac{1}{2}(u^2-v^2)\|_{s-1} + \|\frac{3}{m+1}(u^{m+1}-v^{m+1})\|_{s-1}
\end{aligned}$$

Where

$$\|u_x^2 - v_x^2\|_{s-1} \leq \|(u+v)(u-v)\|_s \leq \|u+v\|_s \|u-v\|_s$$
$$\|u^2 - v^2\|_{s-1} \leq \|(u-v)(u+v)\|_s \leq \|u+v\|_s \|u-v\| s$$
$$\|u-v\|_{s-1} \leq \|u-v\|_s$$
$$\|u^{m+1} - v^{m+1}\|_{s-1} \leq K\|u-v\|_s$$

Therefore, there is constant $c > 0$ satisfies (iii). We can easily find that (i) is an immediate consequence of (iii), since we choose $v = 0$. So far, we have verified conditions of theorem 5(Kato). that is to say the proof of theorem 10 complete. ■

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