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Block Hybrid Simpson's Method with Two Offgrid Points for Stiff Systems

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Abstract: A continuous representation based on the Hybrid Simpson's Method (HSM2) with two off-grid points is developed and adapted to cope with the integration of stiff systems of ordinary differential equations (ODEs). This is achieved by combining the HSM2 with three additional methods (obtained from the continuous representation) and applying them as numerical integrators by assembling them into a single block matrix equation which is A-stable and of order $(5, 5, 5, 6)^T$. The performance of the method is demonstrated on some numerical examples to show its accuracy and computational efficiency.

Keywords: Initial value problems; Simpson's method; stability; block methods; relative errors.

1 Introduction

We consider the Stiff IVP

$$y' = f(x, y), \quad y(a) = y_0, \qquad x \in [a, b] \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies a Lipschitz condition (See Henrici [10]) and y, $y_0 \in \mathbb{R}^m$. Stiff problems frequently arise in areas such as quantum mechanics, celestial mechanics, biological sciences and engineering sciences and they represent coupled physical systems having components varying with different time scales. A-stability requirement puts a severe limitation on the choice of suitable methods for stiff systems. The search for higher order A-stable multistep methods is sought majorly in the following directions.

1. Use of higher order derivatives (Enright [5], Hojjati *et al* [22], Sahi *et al* [4], Ezzedine and Hojjati [19] etc).
2. Incorporating additional stages and off-grid points (Gear [7], Gupta [8], Ngwane and Jator [12], Fatokun [2] among others)
3. Incorporating superfuture points (Cash [1], Psihoyios [21])

Lambert [14] noted that the Simpson's Rule, being an optimal method, has no interval of absolute stability and has the positive real axis as its interval of relative stability. This implies that a growing error is unavoidable with the Simpson's Rule (especially when the Jacobian ($\frac{\partial f}{\partial y}$) has eigenvalues with negative real parts). Since for a general problem we do not know in advance the sign of $\frac{\partial f}{\partial y}$, Lambert [14] concluded that the Simpson's Rule cannot be recommended as a general-purpose method for the integration of IVPs. In this paper, we show that the Simpson's Rule can be adapted to cope with the integration of Stiff systems in ODEs by incorporating two off-grid points to obtain the HSM2. This is achieved by combining and implementing the HSM2 with three additional methods to form the Block Hybrid Simpson's Method (BHSM2).

In what then follows, we derive a continuous HSM2 through interpolation and collocation (See Lie and Norsett [13], Onumanyi *et al* [23]) which is used to generate the main dicrete HSM2 and three additional methods for solving (1).

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These methods were shown to enjoy both higher order and good stability properties, but included additional off-grid functions. Gupta [8] noted that the design of algorithms for hybrid methods is more tedious due to the occurrence of off-grid functions which increase the number of predictors needed to implement the methods. In order to avoid this deficiency, we adopt a different approach based on a block-by-block implementation instead of the traditional step by step implementation generally performed in a predictor-corrector mode. Hence, we adopted the approach given in Ngwane and Jator [12], where the continuous HSM2 is used to generate a main method and three additional methods which are combined and used as a BHSM2 to simultaneously produce approximations $\{y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}\}$ to the exact solutions $\{y(x_{n+\frac{1}{2}}), y(x_{n+1}), y(x_{n+\frac{3}{2}}), y(x_{n+2})\}$ where the set of points $x_n = a + nh$, $0(1)N$ belongs to the partition

$$\pi_N : a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} < x_N = b$$

where $h = \dfrac{b-a}{N}$ is the constant stepsize.

In order to apply the block method at the next block to obtain $\{y_{n+\frac{5}{2}}, y_{n+3}, y_{n+\frac{7}{2}}, y_{n+4}\}$, the only necessary starting value is y_{n+2} and the loss of accuracy in y_{n+2} does not affect subsequent points, thus the order of the method is maintained.

Block methods for solving ordinary differential equations have initially been proposed by Milne [9] who advanced their use only as a means of obtaining starting values for predictor-corrector algorithms. Several authors (Rosser [16], Shampine and Watts [15], Fatunla [6] and Ngwane and Jator [12] among others) have modified it to be more efficient as a computational procedure for the integration of IVPs throughout the range of integration rather than just as a starting method for multistep methods.

The paper is organized as follows: In Section 2, we develop the continuous HSM2 based on a polynomial basis representation in $U(x)$ for the exact solution $y(x)$ which is used to generate the discrete HSM2 and three other discrete by-products which are combined into a block method for solving (1). Section 3 details the error, stability analysis and implementation of the method. Numerical examples are given (in Section 4) on some well known stiff problems to show the performance of the method. Finally, we give some concluding remarks in Section 5.

2 Development of Methods

In this section, we develop the main method HSM2 and three additional methods. The main HSM2 is of the form

$$y_{n+2} - y_n = h\left[\beta_0 f_n + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_{n+\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2}\right] \tag{2}$$

where $\beta_j, j = 0(\frac{1}{2})2$ are unknown constants. We note that y_{n+j} are the numerical solutions to the analytical solutions $y(x_{n+j})$ and $f_{n+j} = f(x_{n+j}, y_{n+j})$, $j = 0(\frac{1}{2})2$

The additional methods are of the form

$$y_{n+\frac{1}{2}} - y_n = h\left[\beta_0 f_n + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_{n+\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2}\right] \tag{3}$$

$$y_{n+1} - y_n = h\left[\beta_0 f_n + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_{n+\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2}\right] \tag{4}$$

$$y_{n+\frac{3}{2}} - y_n = h\left[\beta_0 f_n + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_{n+\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2}\right] \tag{5}$$

To obtain (2)-(5), we seek an approximation to the exact solution $y(x)$ on the interval $[x_n, x_{n+2}]$ by the interpolating function of the form

$$U(x) = \sum_{r=0}^{5} a_r x^r \tag{6}$$

where a_r, $r = 0(1)5$ are coefficients to be uniquely determined.

We impose that the interpolating function (6) coincide with the analytical solution at the point x_0 to obtain

$$U(x_0) = y_0 \tag{7}$$

We also require (6) to satisfy the differential equation (1) at the points x_i, $i = 0(\frac{1}{2})2$ to obtain

$$U(x_i) = f_i, \quad i = 0(\frac{1}{2})2 \tag{8}$$

Equation (7) and (8) lead to a system of six equations which is then solved for the values a_r, $r = 0(1)5$. The continuous HSM2 is developed by substituting the values of a_r, $r = 0(1)5$ into (6). After some algebraic manipulations, our continuous HSM2 is expressed in the form

$$U(x) = y_n(x) + h\left[\beta_0(x)f_n + \beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}} + \beta_1(x)f_{n+1} + \beta_{n+\frac{3}{2}}(x)f_{n+\frac{3}{2}} + \beta_2(x)f_{n+2}\right] \tag{9}$$

Thus, evaluating (9) at $x = \left\{x_{n+\frac{1}{2}}, x_{n+1}, x_{n+\frac{3}{2}}, x_{n+2}\right\}$, (2)-(5) are specified as follows

$$y_{n+2} - y_n = \frac{h}{45}\left[7f_n + 32f_{n+\frac{1}{2}} + 12f_{n+1} + 32f_{n+\frac{3}{2}} + 7f_{n+2}\right] \tag{10}$$

$$y_{n+\frac{1}{2}} - y_n = \frac{h}{1440}\left[251f_n + 646f_{n+\frac{1}{2}} - 264f_{n+1} + 106f_{n+\frac{3}{2}} - 19f_{n+2}\right] \tag{11}$$

$$y_{n+1} - y_n = \frac{h}{180}\left[29f_n + 124f_{n+\frac{1}{2}} + 24f_{n+1} + 4f_{n+\frac{3}{2}} - f_{n+2}\right] \tag{12}$$

$$y_{n+\frac{3}{2}} - y_n = \frac{h}{160}\left[27f_n + 102f_{n+\frac{1}{2}} + 72f_{n+1} + 42f_{n+\frac{3}{2}} - 3f_{n+2}\right] \tag{13}$$

3 Analysis and Implementation

In this section, we discuss the local truncation error and order, convergence, linear stability and implementation of the methods.

Formulae (10)-(13) form the BHSM2 in the form

$$A^{(0)}Y_\gamma = A^{(1)}Y_{\gamma-1} + h(B^{(1)}F_{\gamma-1} + B^{(0)}F_\gamma) \tag{14}$$

where

$$\mathbf{A^{(0)}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{A^{(1)}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B^{(0)}} = \begin{bmatrix} \frac{646}{1440} & \frac{-264}{1440} & \frac{160}{1440} & \frac{-19}{1440} \\ \frac{124}{180} & \frac{24}{180} & \frac{4}{180} & \frac{-1}{180} \\ \frac{102}{160} & \frac{72}{160} & \frac{42}{160} & \frac{-3}{160} \\ \frac{32}{45} & \frac{12}{45} & \frac{32}{45} & \frac{7}{45} \end{bmatrix} \qquad \mathbf{B^{(1)}} = \begin{bmatrix} 0 & 0 & 0 & \frac{251}{1440} \\ 0 & 0 & 0 & \frac{29}{180} \\ 0 & 0 & 0 & \frac{27}{160} \\ 0 & 0 & 0 & \frac{7}{45} \end{bmatrix}$$

where $Y_\gamma = [y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}]^T$, $Y_{\gamma-1} = [y_{n-\frac{3}{2}}, y_{n-1}, y_{n-\frac{1}{2}}, y_n]^T$, $F_\gamma = [f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}]^T$, $F_{\gamma-1} = [f_{n-\frac{3}{2}}, f_{n-1}, f_{n-\frac{1}{2}}, f_n]^T$.

3.1 Local Truncation Error and Order

Following Fatunla [6] and Lambert [14], we define the Local Truncation Error (LTE) associated with (14) to be the linear difference operator

$$L[y(x);h] = A^{(0)}Y_\gamma - A^{(1)}Y_{\gamma-1} - h(B^{(1)}F_{\gamma-1} + B^{(0)}F_\gamma). \tag{15}$$

where we have taken $y_{n+j} \equiv y(x_n + jh)$ and $f_{n+j} \equiv y'(x_n + jh)$.

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (15) as a Taylor's series about x_n to obtain the expression

$$L[y(x);h] = C_0y(x) + C_1hy'(x) + \cdots + C_qh^qy^q(x) + \cdots \tag{16}$$

where C_j is a (4×1) matrix.

It is established from our calculations that our block method is of order $(5,5,5,6)^T$ and error constant $(\frac{3}{10240}, \frac{1}{5760}, \frac{3}{10240}, \frac{-1}{15120})^T$.

3.2 Zero Stability

It is worth noting that zero stability is concerned with the stability of the difference system in the limit as h tends to 0. Thus, as $h \to 0$, the method (14) tends to the difference system

$$A^{(0)}Y_\gamma - A^{(1)}Y_{\gamma-1} = 0 \tag{17}$$

The block method (14) is zero stable if the roots R_j, $j = 1, 2, 3, 4$ of the first characteristic polynomial $\rho(R)$ specified by

$$\rho(R) = det\left[\sum_{i=0}^{1} A^{(i)}R^{1-i}\right] = 0 \tag{18}$$

satisfies $|R_j| \leq 1, j = 1, 2, 3, 4$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1. (See Fatunla [6]).

Thus, the BHSM2 is zero stable since

$$\rho(R) = R^3(R-1) \Rightarrow R_1 = R_2 = R_3 = 0, R_4 = 1.$$

The BHSM2 is also consistent since each of its numerical integrator has order $p > 1$. According to Henrici [10], we can assert the convergence of the method.

3.3 Linear Stability Analysis

To analyze the BHSM2 (14), we apply (14) to the Dahlquist test equation $y' = \lambda y$, $\lambda < 0$ to obtain

$$Y_\gamma = M(q)Y_{\gamma-1}, \quad q = \lambda h \tag{19}$$

where

$$M(q) = (A^{(0)} - qB^{(0)})^{-1}(A^{(1)} + qB^{(1)}) \tag{20}$$

where $M(q)$ is the amplification matrix which determines the stability of the method.

We obtain the property of A-stability from (19), which requires that for all $q \in \mathbb{C}^-$ and $Re(q) < 0$

$$|\rho(q)| < 1 \tag{21}$$

where $\rho(q)$ is the spectral radius of $M(q)$. Our calculations show that

$$\rho(q) = \frac{240 + 240q + 105q^2 + 25q^3 + 3q^4}{240 - 240q + 105q^2 - 25q^3 + 3q^4} \tag{22}$$

It is obvious from (22) that for $Re(q) < 0$, $|\rho(q)| < 1$. Hence, the block method (14) is A-stable since its region of absolute stability contains the left hand half complex plane $\{q \in \mathbb{C} : Re(q) < 0\}$. Therefore there is no retriction on $q = \lambda h$ which makes (14) a viable candidate for stiff systems.

3.4 Implementation

Formula (14) form the BHSM2 which is implemented to solve (1) without requiring starting values and predictors. For instance, if we let $n = 0$ then $\{y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}\}$ are simultaneously obtained over the subinterval $[x_0, x_2]$, as y_0 is known from the IVP. Similarly if $n = 1$ then $\{y_{n+\frac{5}{2}}, y_{n+3}, y_{n+\frac{7}{2}}, y_{n+4}\}$ are simultaneously obtained over the subinterval $[x_2, x_4]$ as y_2 is known from the previous block (when n=0) and so on; until we reach the final subinterval $[x_{N-1}, x_N]$. We note that the problems discussed in the following section were solved using the feature FindRoot in Mathematica 10.0.

4 Numerical Examples

In this section, we give numerical examples to illustrate the accuracy (small errors) and efficiency (number of steps N) of the BHSM2. We find an approximation in the the partition π_N, where

$$\pi_N : a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} < x_N = b$$

and we give the errors at endpoints (calculated as $Error = Ey = |y(x_N) - y_N|$). All computations were carried out using a written code in Mathematica 10.0 version.

Example 1 *We consider the following IVP which was also solved by Cash [17], Jackson-Kenue [11] and Sahi* et al *[4] on the range* $0 \leq x \leq 1$

$$y_1' = -y_1 + 95y_2, \quad y_1(0) = 1, \quad y_2' = -y_1 - 97y_2, \quad y_2(0) = 1$$

where the analytic solution is given by

$$\textit{Exact}: y_1(x) = \frac{95}{47}e^{-2x} - \frac{48}{47}e^{-96x} \quad y_2(x) = -\frac{1}{47}e^{-2x} + \frac{48}{47}e^{-96x}$$

The errors in the solution are obtained at $x = 1$ using our method for fixed stepsizes as shown in Table 1 below. Similar results were obtained in Cash [17], Jackson-Kenue [11], Sahi *et al* [4]. It is seen that our method is superior to those in [17], [11] and highly competitive with [4].

Table 1: Comparing the Absolute Errors for Example 1

h	Ey_i	Jackson-Kenue	Cash (p = 4)	Cash (p =5)	Sahi *et al*	BHSM2
0.0625	Ey_1	3×10^{-7}	3×10^{-7}	1×10^{-8}	9×10^{-11}	4×10^{-10}
	Ey_2	4×10^{-7}	3×10^{-7}	1×10^{-8}	1×10^{-8}	8×10^{-10}
0.03125	Ey_1	1×10^{-8}	1×10^{-8}	-	4×10^{-12}	7×10^{-12}
	Ey_2	1×10^{-8}	1×10^{-8}	-	4×10^{-12}	7×10^{-14}

Example 2 *As a second test example, we consider the given linear system which has been solved in the range* $0 \leq x \leq 3$

$$y_1' = -21y_1 + 19y_2 - 20y_3,\ y_1(0) = 1,\ y_2' = 19y_1 - 21y_2 + 20y_3,\ y_2(0) = 0,\ y_3' = 40y_1 - 40y_2 + 40y_3,\ y_3(0) = -1.$$

The analytical solution of the system is given by

$$\textit{Exact}: y_1(x) = \frac{1}{2}(e^{-2x} + e^{-40x}(\cos(40x) + \sin(40x))), \quad y_2(x) = \frac{1}{2}(e^{-2x} - e^{-40x}(\cos(40x) + \sin(40x))),$$

$$y_3(x) = -e^{-40x}(\cos(40x) - \sin(40x)).$$

Table 2: Maximum Errors for Example 2

h	BHSM2	ETRs	ETR_2s	TOMs
2×10^{-2}	5.374×10^{-5}	3.778×10^{-3}	3.513×10^{-3}	1.552×10^{-3}
1×10^{-2}	2.012×10^{-6}	1.007×10^{-4}	8.615×10^{-5}	9.775×10^{-6}
5×10^{-3}	4.179×10^{-8}	1.091×10^{-6}	7.231×10^{-7}	1.197×10^{-7}
2.5×10^{-3}	8.920×10^{-10}	1.793×10^{-8}	8.864×10^{-9}	1.853×10^{-9}

The problem has also been solved by Brugnano and Trigiante [18] using the Extended Trapezoidal Rules (ETRs), Extended Trapezoidal Rules of Second kind (ETR_2s) and Top Order Methods (TOMs). These methods are of order 6 and thus comparable with the BHSM2 in this paper. The results for the maximum errors ($MaxError = maximum|y(x) - y|$) are given in Table 2 and compared with the results given by our methods. It is seen from Table 2 that our methods perform better than those in [18]. This high accuracy is due to the smaller error constant of each of the numerical integrators in BHSM2 as compared to the methods in Brugnano and Trigiante [18]. Thus, for this example, our method is superior in terms of accuracy.

Example 3 *We also consider the following nonlinear IVP which was also solved by Wu and Xia [24], Ngwane and Jator [12] and Sahi* et al *[4] using different step sizes.*

$$y_1' = -1002y_1 + 1000y_2^2, \quad y_1(0) = 1, \;\; y_2' = y_1 - y_2(1 + y_2), \quad y_2(0) = 1.$$

where the exact solution is given by

$$Exact : y_1(x) = e^{-2x} \quad y_2(x) = e^{-x}$$

The methods given in Wu and Xia [24], Ngwane and Jator [12], and Sahi *et al* [4] were shown to perform excellently on stiff problems. It is obvious from the computational results in Table 3-5 that while our method performs better both in terms of accuracy and number of steps than the method in Wu and Xia [24], it has accuracy of same error magnitude with the methods in [12] and [4]. However, our method uses fewer number of steps in achieving this high accuracy. Thus, we may conclude that for this example, our method is superior to the method in [24] and highly competitive (in terms of accuracy) but superior (in terms of computational efficiency) to those in [12] and [4].

Table 3: Comparison of Exact Errors for Example 3

			Wu and Xia [24]			BHSM2	
x	h	N	$\|Ey_1\|$	$\|Ey_2\|$	N	$\|Ey_1\|$	$\|Ey_2\|$
1	0.002	500	2.5606×10^{-7}	8.0150×10^{-8}	250	4.4409×10^{-16}	6.1062×10^{-16}
10	0.001	10000	5.5468×10^{-16}	6.0936×10^{-12}	5000	5.1940×10^{-20}	4.7434×10^{-19}

Table 4: Comparison of Exact Errors for Example 3

			Ngwane and Jator [12]			BHSM2	
x	h	N	$\|Ey_1\|$	$\|Ey_2\|$	N	$\|Ey_1\|$	$\|Ey_2\|$
1	0.1	10	5.6763×10^{-13}	6.5675×10^{-13}	5	3.3588×10^{-9}	2.3048×10^{-11}
10	0.01	1000	7.0972×10^{-22}	7.8198×10^{-18}	500	4.7728×10^{-22}	5.6175×10^{-18}

Example 4 *Lastly, we consider the nonlinear system which has been solved by Jeltch [20] and Sahi* et al*[4] in the range* $0 \leq x \leq 48$

$$y_1' = -0.013y_1 - 1000y_1y_2 - 2500y_1y_3, \;\; y_1(0) = 0,$$
$$y_2' = -0.013y_2 - 1000y_1y_2, \;\; y_2(0) = 1, \;\; y_3' = -2500y_1y_3, \;\; y_3(0) = 1.$$
$$Exact : y_i(2) = \left\{-3.616933169289 \times 10^{-6}, 0.9815029948230, 1.018493388244\right\}$$

Table 5: Comparison of Exact Errors for Example 3

		Sahi *et al* [4]			BHSM2		
x	h	N	$\|Ey_1\|$	$\|Ey_2\|$	N	$\|Ey_1\|$	$\|Ey_2\|$
1	0.04	250	1.3112×10^{-13}	1.7186×10^{-13}	13	2.2513×10^{-12}	3.9024×10^{-14}
10	0.02	500	1.3235×10^{-22}	1.4162×10^{-18}	250	2.6474×10^{-21}	4.1471×10^{-18}

$$y_i(48) = \{-1.945338956808 \times 10^{-6}, 0.6110474831446, 1.3889505715163\}$$

(See Jeltch [20])

We give the relative errors at $x = 2$ and at $x = 48$. Our method produce higher accuracy than the method in Sahi *et al* [4] as shown by the results in Table 6. The exact solution at $x = 2$ and $x = 48$ were taken from Jeltsch [20]. Although, Jeltsch [20] also gave the relative errors for this problem, we chose not to compare with the method because the results were presented graphically with a different emphasis.

Table 6: Comparison of Relative Errors computed as Rel error = $(|y - y(x)|)/(|1 + y(x)|)$ for Example 4

		Sahi *et al* [4]			BHSM2		
h	x	$\|Ey_1\|$	$\|Ey_2\|$	$\|Ey_3\|$	$\|Ey_1\|$	$\|Ey_2\|$	$\|Ey_3\|$
$\frac{1}{8}$	2	2.633×10^{-6}	1.501×10^{-4}	1.460×10^{-4}	2.720×10^{-6}	3.824×10^{-7}	9.725×10^{-7}
	48	1.113×10^{-9}	4.770×10^{-4}	3.217×10^{-4}	3.775×10^{-9}	1.122×10^{-9}	8.238×10^{-10}
$\frac{1}{16}$	2	9.850×10^{-7}	2.492×10^{-5}	2.398×10^{-5}	1.094×10^{-6}	1.539×10^{-7}	3.910×10^{-7}
	48	1.918×10^{-10}	3.054×10^{-5}	2.060×10^{-5}	1.755×10^{-15}	2.794×10^{-10}	1.882×10^{-10}
$\frac{1}{32}$	2	1.927×10^{-8}	2.119×10^{-6}	2.070×10^{-6}	2.867×10^{-8}	4.112×10^{-9}	1.017×10^{-8}
	48	1.205×10^{-11}	1.918×10^{-6}	1.294×10^{-6}	4.844×10^{-16}	7.722×10^{-11}	5.186×10^{-11}
$\frac{1}{64}$	2	1.370×10^{-12}	1.327×10^{-7}	1.303×10^{-7}	1.343×10^{-14}	2.201×10^{-11}	2.168×10^{-11}
	48	7.517×10^{-13}	1.197×10^{-7}	8.072×10^{-8}	1.241×10^{-16}	1.984×10^{-11}	1.319×10^{-11}

5 Conclusion

A continuous representation of the HSM2 with two off-grid points is proposed and used to obtain discrete methods which are combined and applied as a BHSM2 for solving Stiff IVPs. It is interesting to note that the Simpson's Rule which is a bad method for IVPs has been adapted via the BHSM2 to be competitive with standard methods for integrating (1). The BHSM2 is A-stable, hence it is an excellent candidiate for stiff problems. In particular, this method has only two off-grid points and has the advantage of being self-starting, requiring no predictors (hence devoid of complicated routines) and simultaneously generating solutions at two grid and two off-grid points over subintervals that do not overlap. We have demonstrated the accuracy of the method on some well known stiff problems (See Table 1, 2, 3, 4, 5 and 6). The numerical results show that our methods is efficient and highly competitive with the existing methods cited in this paper. In the future, we would like to implement our method in a variable stepsize mode with an estimation of the global errors.

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