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An Approximate Solution of Inelastic Collision of Two Solitons for Generalized BBM Equation

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Abstract: The paper studies the approximate solution of inelastic collision of two solitons with different velocities for generalized Benjamin-Bona-Mahony equation. The method of this work is developed by F. Merle and Y. Martel. The main result is construction of the approximate solution for the equation in the collision regime.

Keywords: generalized Benjamin-Bona-Mahony equation; solitons; collision; approximate solution.

1 Introduction

This paper is concerned with the following nonlinear partial differential equation:

$$(1 - \frac{1}{2}\partial_x^2)\partial_t u + \partial_x(\partial_x^2 u - u + u^3) = 0, \quad (t, x) \in \mathrm{R} \times \mathrm{R}, \tag{1.1}$$

for the function $u(t, x)$ of time t and a single spatial variable x. Equation (1.1) has a form of generalized BBM equation with the dispersion term. Eq. (1.1) is obtained by using the following change of variable (1.3), which is from the following standard form of the gBBM equation with cubic nonlinearity:

$$(1 - \partial_{\hat{x}}^2)\partial_{\hat{t}} z + \partial_{\hat{x}}(z + z^3) = 0, \quad (\hat{t}, \hat{x}) \in \mathrm{R} \times \mathrm{R}. \tag{1.2}$$

Where the following change of variable are:

$$x = \sqrt{\frac{1}{2}}(\hat{x} - 2\hat{t}), \quad t = \sqrt{\frac{1}{2}}\hat{t}, \quad u(t, x) = z(\hat{t}, \hat{x}). \tag{1.3}$$

If $z(\hat{t}, \hat{x})$ is a solution to Eq. (1.2) then $u(t, x)$ satisfies Eq. (1.1). And Eq. (1.2) was original introduced by Peregrine [1] and Benjamin, Bona and Mahony [2]. What's more, no inverse-scattering theory can be developed for this equation (see [3] and [4]). This situation is in contrast with the generalized KdV equations:

$$\partial_t u + \partial_x(\partial_x^2 u + f(u)) = 0, \tag{1.4}$$

which is completely integrable for $f(u) = u^2$ (KdV equation), $f(u) = u^3$ (mKdV equation) and $f(u) = u^2 - \mu u^3$ (Gardner equation), but not for other functions of $f(u)$.

About the collision of two solitons with different velocities, there were following papers [5-8] etc.

Martel and Merle [5] considered mainly the generalized KdV equations for $f(u) = u^4$. It was investigated that the situation where one soliton Q_{c_1} is supposed to be large with respect to the other one Q_{c_2}; under the assumption $0 < c_2 << 1$, which means that one solitary wave is small in $H^1(\mathrm{R})$ energy space. It was proved that collision of two

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stable solitary waves of gKdV equation is inelastic but almost elastic. As a consequence, the monotonicity properties are strict: the size of large soliton increases and the size of the small soliton decreases through the collision, with explicit lower and upper bounds. Moreover, the first orders of the shifts resulting from the collision could be computed explicitly.

Also, in [6], they extended the same questions to the generalized KdV equation of a general nonlinearity $f(u)$:

$$u_t + (u_{xx} + f(u))_x = 0, \ \ (t, x) \in \mathrm{R}^+ \times \mathrm{R}, \ \ u(0) = u_0 \in H^1(\mathrm{R}),$$

assumed that for $p = 2, \ 3$ or 4

$$f(u) = u^p + f_1(u) \text{ where } f_1 \text{ is } C^{p+4} \text{ and } \lim_{u\to 0} \left|\frac{f_1(u)}{u^p}\right| = 0.$$

And see also [7], with Mizumachi, extending the method [5], [6] to the BBM equation: They considered two solitary waves $\phi_{c_1}(x - c_1 t), \ \phi_{c_2}(x - c_2 t)$ in the case where $1 < c_2 < c_1$ and c_2 is close to 1, so that the function ϕ_{c_2} is small in $H^1(\mathrm{R})$. They still acquired the same results as [5] and [6].

Muñoz [8] studied the inelastic collision of two solitons for gKdV equations with general nonlinearity. It is classified the nonlinearities for which collision are elastic and inelastic. For integrable case, the collision of two solitons is elastic, such as KdV, mKdV and Gardner nonlinearities. But, for non-integrable case, the collision of two solitons is inelastic, such as gKdV, a general case f and BBM etc.

In the present paper, we consider the collision of two solitary waves with different velocities for gBBM equation (1.1). The purpose of this paper is to obtain an approximate solution the equation. The part properties of equation (1.1) are as follows.

Recall the Eq. (1.2) has a two parameter family of solitary wave solutions $\{\phi_c(x - ct - x_0) \mid c > 1, \ \ x_0 \in \mathrm{R}\}$, where ϕ_c satisfies

$$c\phi_c'' - (c-1)\phi_c + \phi_c^3 = 0, \qquad \textit{on R}. \tag{1.5}$$

The unique even solution of Eq. (1.5) is given by

$$\phi_c(x) = (c-1)^{\frac{1}{2}} Q\left(\sqrt{\frac{c-1}{c}}x\right),$$

where $Q(x) = \sqrt{2}\cosh^{-1}(x)$ solves $Q'' + Q^3 = Q$. (1.6)

2 Construction of an approximate 2-soliton solution

Lemma 1 *(i) Let $c > 1$. By the change of variable (1.3), a solitary wave solution $\phi_c(x - ct)$ to Eq. (1.2) is transformed into $\tilde{Q}_\sigma(y_\sigma)$ which is a solution of Eq. (1.1) where*

$$\tilde{Q}_\sigma(x) = \sqrt{\sigma\theta_\sigma} Q(\sqrt{\sigma}x), \quad Q(x) = \sqrt{2}\cosh^{-1}(x),$$

$$\sigma = \frac{2c-2}{c}, \quad \theta_\sigma = \frac{1}{2-\sigma} = \frac{1}{2}\sum_{j=0}^{\infty}(\frac{1}{2}\sigma)^j, \quad \frac{1}{\theta_\sigma} = 2-\sigma, \quad \mu_\sigma = \frac{2-2\sigma}{2-\sigma} = 1 - \frac{1}{2}\sigma\sum_{j=0}^{\infty}(\frac{1}{2}\sigma)^j, \quad y_\sigma = \hat{x} + \mu_\sigma\hat{t}. \tag{2.1}$$

Especially if $c = c_1$, then $\mu_\sigma = 0, y_\sigma = \hat{x}$ and $\tilde{Q}_\sigma(y_\sigma) = Q(\hat{x})$

$$Q'' + Q^3 = Q, \ \ (Q')^2 + \frac{1}{2}Q^4 = Q^2 \qquad \textit{on } \ R. \tag{2.2}$$

(ii) Moreover, $\tilde{Q}_\sigma$ satisfies the following

$$\tilde{Q}_\sigma'' + \frac{1}{\theta_\sigma}\tilde{Q}_\sigma^3 = \sigma\tilde{Q}_\sigma, \quad (\tilde{Q}_\sigma')^2 + \frac{1}{2\theta_\sigma}\tilde{Q}_\sigma^4 = \sigma\tilde{Q}_\sigma^2. \tag{2.3}$$

For $\sigma > 0$ small,

$$\left\|\tilde{Q}_\sigma\right\|_{L^\infty(\mathrm{R})} \sim \sqrt{\frac{1}{2}\sigma}\,\|Q\|_{L^\infty(\mathrm{R})}, \quad \left\|\tilde{Q}_\sigma\right\|_{L^2(\mathrm{R})} \sim \sqrt{\frac{1}{2}}\sigma^{\frac{1}{4}}\,\|Q\|_{L^2(\mathrm{R})}, \tag{2.4}$$

Proof. The proof of Lemma 1 is similar to the proof of Claim 2.1 (in [7]), so it is omitted. ■

2.1 Decomposition of the approximate solution

First, we construct an approximate solution $z(t,x)$ of

$$(1-\frac{1}{2}\partial_x^2)\partial_t z+\partial_x(\partial_x^2 z-z+z^3)=0, \tag{2.5}$$

which is the sum of the function $Q(y)$, a small soliton $\tilde{Q}_\sigma(y_\sigma)$ and an error term $w(t,x)$. As in [7], we introduce the new coordinates and the approximate solution under the following form

$$y_\sigma=x+\mu_\sigma t, \quad y=x-\alpha(y_\sigma), \quad \alpha(s)=\int_0^s \beta(r)dr,$$

$$\beta(y_\sigma)=\sum_{(k,l)\in\sum_0} a_{k,l}\sigma^l\tilde{Q}_\sigma^k(y_\sigma), \tag{2.6}$$

$$z(t,x)=Q(y)+\tilde{Q}_\sigma(y_\sigma)+w(t,x), \tag{2.7}$$

where

$$w(t,x)=\sum_{(k,l)\in\sum_0}\sigma^l(A_{k,l}(y)\tilde{Q}_\sigma^k(y_\sigma)+B_{k,l}(y)(\tilde{Q}_\sigma^k)'(y_\sigma)), \tag{2.8}$$

$$\sum\nolimits_0=\{(k,l)=(1,0),\ \ (1,1),\ \ (2,0),\ \ (2,1),\ \ (3,0),\ \ (4,0)\}, \tag{2.9}$$

Definition 1 *The operator* L *is defined by*

$$Lf=-f''+f-3Q^2f \tag{2.10}$$

Definition 2 *Let* M *be the set of* C^∞ *functions* f *such that*

$$\forall j\in \mathrm{N},\ \exists K_j,r_j>0,\ \forall x\in \mathrm{R}, \quad \left|f^{(j)}(x)\right|\le K_j(1+|x|)^{r_j}e^{-|x|}. \tag{2.11}$$

Now, we follow the notation introduced in (2.6)-(2.9) and we also set

$$S(z)=(1-\tfrac{1}{2}\partial_x^2)\partial_t z+\partial_x(\partial_x^2 z-z+z^3)=S_{mKdV}(z)+S_{\mathrm{g}BBM}(z),$$
$$S_{mKdV}(z)=\partial_t z+\partial_x(\partial_x^2 z-z+z^3), \quad S_{\mathrm{g}BBM}(z)\ =-\tfrac{1}{2}\partial_t\partial_x^2 z.$$

Then

$$S(z(t,x))=S(Q(y))+S(\tilde{Q}_\sigma(y_\sigma))+\delta S(w(t,x))+S_{int}(t,x), \tag{2.12}$$

where

$$\delta S(w)=\delta S_{mKdV}(w)+S_{\mathrm{g}BBM}(w), \quad \delta S_{mKdV}(w)=\partial_t w-\partial_x Lw, \tag{2.13}$$

$$\begin{aligned} S_{int}(t,x) &= \partial_x(w^3(t,x)+3Q^2(y)\tilde{Q}_\sigma(y_\sigma)+3Q(y)\tilde{Q}_\sigma^2(y_\sigma)+3\tilde{Q}_\sigma^2(y_\sigma)w(t,x)\\ &+ 6Q(y)\tilde{Q}_\sigma(y_\sigma)w(t,x)+3Q(y)w^2(t,x)+3\tilde{Q}_\sigma(y_\sigma)w^2(t,x)).\end{aligned}$$

Since $\tilde{Q}_\sigma(y_\sigma)$is a solution to (2.5), we get $S(\tilde{Q}_\sigma(y_\sigma))=0$.

Proposition 2 *The following holds*

$$\begin{aligned} S(z) &= \sum_{(k,l)\in\Sigma_0}\sigma^l\tilde{Q}_\sigma^k(y_\sigma)\left(-\frac{5}{2}a_{k,l}Q''-3Q^3)'-(LA_{k,l})'+F_{k,l}\right)(y)\\ &+ \sum_{(k,l)\in\Sigma_0}\sigma^l(\tilde{Q}_\sigma^k)'(y_\sigma)\left(\frac{5}{2}A''_{k,l}+3Q^2A_{k,l}-2a_{k,l}Q''-(LB_{k,l})'+G_{k,l}\right)(y)+\varepsilon(t,x),\end{aligned}$$

If the functions $A_{k',l'}$, $B_{k',l'}$ *are bounded then the rest term* $\varepsilon(t,x)$ *satisfies*

$$|\varepsilon(t,x)|\le K\sigma^{\frac{5}{2}}O(\tilde{Q}_\sigma(y_\sigma)). \tag{2.14}$$

Before proofing Proposition 1, we give some properties of the operator L.

Lemma 3 *(Properties of L, Lemma 2.2 in [5]) The operator L defined in $L^2(\mathrm{R})$ by (2.10) is self-adjoint and satisfies the following properties:*
(i) First eigenfunction: $LQ^2 = -3Q^2$;
(ii) Second eigenfunction: $LQ' = 0$; the kernel of L is $\{c_1 Q',\ c_1 \in \mathrm{R}\}$;
(iii) For any function $h \in L^2(\mathrm{R})$ orthogonal to Q' for the L^2 scalar product, there exists a unique function $f \in H^2(\mathrm{R})$ orthogonal to Q' such that $Lf = h$; moreover, if h is even (respectively, odd), then f is even (respectively, odd).
(iv) Suppose that $f \in H^2(\mathrm{R})$ is such that $Lf \in \mathrm{M}$.

Lemma 4 *(Claim B.1 in [7]) Let $h(t,x) = g(y) = g(x-\alpha(y_\sigma))$, where g is a C^3 function. Then*

$$\begin{aligned}
&\partial_t h = -\mu_\sigma \beta(y_\sigma) g'(y),\\
&\partial_x h = (1-\beta(y_\sigma)) g'(y),\\
&\partial_x^2 h = (1-\beta(y_\sigma))^2 g''(y) - \beta(y_\sigma) g'(y),\\
&\partial_x \partial_t h = -\mu_\sigma (1-\beta(y_\sigma))\beta(y_\sigma) g''(y) - \mu_\sigma \beta'(y_\sigma) g'(y),\\
&\partial_x^3 h = (1-\beta(y_\sigma))^3 g'''(y) - 3(1-\beta(y_\sigma))\beta'(y_\sigma) g''(y) - \beta''(y_\sigma) g'(y),\\
&\partial_x^2 \partial_t h = \mu_\sigma \left\{-(1-\beta(y_\sigma))^2 \beta(y_\sigma) g'''(y) + 3\beta(y_\sigma)\beta'(y_\sigma) g''(y) - 2\beta'(y_\sigma) g''(y) - \beta''(y_\sigma) g'(y)\right\}.
\end{aligned}$$

Lemma 5 *Let A and q be C^3-functions. Then*

$$\begin{aligned}
\delta S_{mKdV}(A(y)q(y_\sigma)) &= q(y_\sigma)\left\{-(LA)'(y) + \beta(y_\sigma)\,(-3A'' - 3AQ^2 + (1-\mu_\sigma)A)'(y) - \beta'(y_\sigma)(3A'')(y)\right.\\
&+ \left.\beta^2(y_\sigma)(3A''')(y) + (\beta^2)'(3A''/2)(y) - \beta''(y_\sigma)A'(y) - \beta^3(y_\sigma)A'''(y)\right\}\\
&+ q'(y_\sigma)\left\{3A''(y) + 3A(y)Q^2(y) + (\mu_\sigma - 1)A(y) - \beta(y_\sigma)(6A'')(y)\right.\\
&- \left.\beta'(y_\sigma)(3A')(y) + \beta^2(y_\sigma)(3A'')(y)\right\} + q''(y_\sigma)\left\{3(1-\beta(y_\sigma))A'(y)\right\} + q'''(y_\sigma)A(y).
\end{aligned}$$

Proof. In the proof, we omit the variable y of $A(y)$. Using Lemma 4, we compute

$$\partial_t(A(y)q(y_\sigma)) = -\mu_\sigma \beta(y_\sigma) A' q(y_\sigma) + \mu_\sigma A q'(y_\sigma),$$

and

$$\begin{aligned}
&-\partial_x L(A(y)q(y_\sigma))\\
=\ & \partial_x\left\{(\partial_x^2 A - A + 3AQ^2)q(y_\sigma) + 2(\partial_x A)q'(y_\sigma) + Aq''(y_\sigma)\right\}\\
=\ & \left\{\partial_x(\partial_x^2 A - A + 3AQ^2)\right\} q(y_\sigma) + (\partial_x^2 A - A + 3AQ^2)q'(y_\sigma) + 2(\partial_x^2 A)q'(y_\sigma) + 3(\partial_x A)q''(y_\sigma) + Aq'''(y_\sigma)\\
=\ & q(y_\sigma)\left\{(1-\beta(y_\sigma))^3 A''' - 3(1-\beta(y_\sigma))\beta'(y_\sigma)A'' - \beta''(y_\sigma)A' - (1-\beta(y_\sigma))A' + 3(1-\beta(y_\sigma))(AQ^2)'\right\}\\
+\ & q'(y_\sigma)\left\{3(1-\beta(y_\sigma))^2 A'' - 3\beta'(y_\sigma)A' - A + 3AQ^2\right\} + q''(y_\sigma)\left\{3(1-\beta(y_\sigma))A'\right\} + q'''(y_\sigma)A.
\end{aligned}$$

Combining the above, we obtain Lemma 5. ■

Lemma 6 *(Claim B.3 in [7]) Let A and q be C^3-functions. Then*

$$\begin{aligned}
S_{gBBM}(A(y)q(y_\sigma)) &= \frac{1}{2}\mu_\sigma q(y_\sigma)\left\{\beta(y_\sigma)A'''(y) + \beta'(y_\sigma)(2A''(y))\right\}\\
&+ \frac{1}{2}\mu_\sigma q(y_\sigma)\left\{\beta^2(y_\sigma)(-2A''')(y) + (\beta^2)'(y_\sigma)(-3A''/2)(y) + \beta''(y_\sigma)A'(y) + \beta^3(y_\sigma)A'''(y)\right\}\\
&+ \frac{1}{2}\mu_\sigma q'(y_\sigma)\left\{-A''(y) + \beta(y_\sigma)(4A'')(y) + \beta'(y_\sigma)(3A')(y) + \beta^2(y_\sigma)(-3A'')(y)\right\}\\
&+ \frac{1}{2}\mu_\sigma q''(y_\sigma)\left\{-2A'(y) + \beta(y_\sigma)(3A')(y)\right\} + \frac{1}{2}\mu_\sigma q'''(y_\sigma)(-A)(y).
\end{aligned}$$

Lemma 7 *Let*

$$\beta = a_{1,0}\tilde{Q}_\sigma + a_{1,1}\sigma\tilde{Q}_\sigma + a_{2,0}\tilde{Q}_\sigma^2 + a_{2,1}\sigma\tilde{Q}_\sigma^2 + a_{3,0}\tilde{Q}_\sigma^3 + a_{4,0}\tilde{Q}_\sigma^4.$$

Then,

$$\beta' = a_{1,0}(\tilde{Q}_\sigma)' + a_{1,1}\sigma(\tilde{Q}_\sigma)' + a_{2,0}(\tilde{Q}_\sigma^2)' + a_{2,1}\sigma(\tilde{Q}_\sigma^2)' + a_{3,0}(\tilde{Q}_\sigma^3)' + a_{4,0}(\tilde{Q}_\sigma^4)',$$

$$
\begin{aligned}
\beta'' &= \sigma\tilde{Q}_\sigma a_{1,0} + \tilde{Q}_\sigma^3(-2a_{1,0}) + \sigma\tilde{Q}_\sigma^2(4a_{2,0}) + \tilde{Q}_\sigma^4(-6a_{2,0}) + \sigma\tilde{Q}_\sigma^3(a_{1,0} - 2a_{1,1} + 9a_{3,0}) \\
&+ \sigma^2\tilde{Q}_\sigma a_{1,1} + \tilde{Q}_\sigma^5(-12a_{3,0}) + \sigma^{\frac{5}{2}}O(\tilde{Q}_\sigma),
\end{aligned}
$$

$$
\begin{aligned}
\beta^2 &= a_{1,0}^2\tilde{Q}_\sigma^2 + 2a_{1,0}a_{2,0}\tilde{Q}_\sigma^3 + 2a_{1,0}a_{1,1}\sigma\tilde{Q}_\sigma^2 + (2a_{1,0}a_{3,0} + a_{2,0}^2)\tilde{Q}_\sigma^4 + (2a_{2,0}a_{3,0} + 2a_{1,0}a_{4,0})\tilde{Q}_\sigma^5 \\
&+ (2a_{1,0}a_{2,1} + 2a_{1,1}a_{2,0})\sigma\tilde{Q}_\sigma^3 + \sigma^{\frac{5}{2}}O(\tilde{Q}_\sigma),
\end{aligned}
$$

$$
(\beta^2)' = a_{1,0}^2(\tilde{Q}_\sigma^2)' + 2a_{1,0}a_{2,0}(\tilde{Q}_\sigma^3)' + 2a_{1,0}a_{1,1}\sigma(\tilde{Q}_\sigma^2)' + (2a_{1,0}a_{3,0} + a_{2,0}^2)(\tilde{Q}_\sigma^4)' + \sigma^{\frac{5}{2}}O(\tilde{Q}_\sigma).
$$

Proof. The proof follows by elementary calculations from (2.3). ■

In the next four lemmas, we expand the various terms in (2.12).

Lemma 8

$$
\begin{aligned}
S(Q) &= \sum_{(k,l)\in\Sigma_0} \sigma^l \left(\tilde{Q}_\sigma^k(y_\sigma)a_{k,l} \left\{ \frac{5}{2}Q'' - 3Q^3 \right\}'(y) - 2(\tilde{Q}_\sigma^k)'(y_\sigma)a_{k,l}Q''(y) \right) \\
&+ \sum_{(k,l)\in\Sigma_0} \sigma^l \left(\tilde{Q}_\sigma^k(y_\sigma)F_{k,l}^I(y) + (\tilde{Q}_\sigma^k)'(y_\sigma)G_{k,l}^I(y) \right) + \sigma^{\frac{5}{2}}O(\tilde{Q}_\sigma(y_\sigma)) \qquad (2.15)
\end{aligned}
$$

Proof. By Lemma 5 with $A(y) = Q(y)$ and $q = 1$,

$$
\begin{aligned}
S_{mKdV}(Q(y)) &= \delta S_{mKdV}(Q) - \partial_x(2Q^3) = (Q'' - Q + Q^3)' + \beta(y_\sigma)(-3Q'' - 3Q^3 + (1-\mu_\sigma)Q)' \\
&- \beta'(y_\sigma)(3Q'') + \beta^2(y_\sigma)(3Q''') + (\beta^2)'(3Q''/2) - \beta''(y_\sigma)Q' - \beta^3(y_\sigma)Q'''.
\end{aligned}
$$

Using $Q'' = Q - Q^3$ and (2.1), we obtain

$$
\begin{aligned}
S_{mKdV}(Q(y)) &= \beta(y_\sigma)(-3Q'' - 3Q^3)' - \beta'(y_\sigma)(3Q'') + \beta^2(y_\sigma)(3Q''') + (\beta^2)'(y_\sigma)(3Q''/2) + \sigma\beta(y_\sigma)\frac{1}{2}Q' \\
&- \beta''(y_\sigma)Q' - \beta^3(y_\sigma)Q''' + \frac{1}{4}\sigma^2\beta(y_\sigma)Q' + \sigma^{\frac{5}{2}}O(\tilde{Q}_\sigma(y_\sigma)).
\end{aligned}
$$

Next, using Lemma 6 and (2.1), we get

$$
\begin{aligned}
&S_{gBBM}(Q(y)) \\
&= \frac{1}{2}\mu_\sigma \{\beta(y_\sigma)Q''' + \beta'(y_\sigma)(2Q'') + \beta^2(y_\sigma)(-2Q''') + (\beta^2)'(y_\sigma)(-3Q''/2) + \beta''(y_\sigma)Q' + \beta^3(y_\sigma)(Q''')\} \\
&= \beta(y_\sigma)(\frac{1}{2}Q''') + \beta'(y_\sigma)(Q'') + \beta^2(y_\sigma)(-Q''') + (\beta^2)'(y_\sigma)(-3Q''/4) + \sigma\beta(y_\sigma)\left\{\frac{1}{4}Q'''\right\} \\
&+ \sigma\beta'(y_\sigma)\left\{\frac{1}{2}Q''\right\} + \beta''(y_\sigma)(\frac{1}{2}Q') + \beta^3(y_\sigma)(\frac{1}{2}Q''') + \sigma\beta^2(y_\sigma)(-\frac{1}{2}Q''') \\
&+ \sigma(\beta^2)'(y_\sigma)(-3Q''/8) + \sigma\beta''(y_\sigma)\frac{1}{4}Q' + \sigma^2\beta(y_\sigma)\frac{1}{8}Q''' + \sigma^2\beta'(y_\sigma)\frac{1}{4}(Q'') + \sigma^{\frac{5}{2}}Q_\sigma(y_\sigma)).
\end{aligned}
$$

Combining the above, we obtain

$$
\begin{aligned}
S(Q) &= \beta(y_\sigma)\left\{-\frac{5}{2}Q'' - 3Q^3\right\}' - 2\beta'(y_\sigma)Q'' + 2\beta^2(y_\sigma)Q''' + (\beta^2)'(y_\sigma)(3Q''/4) + \beta''(y_\sigma)\frac{1}{2}Q' \\
&+ \sigma\beta(y_\sigma)\frac{1}{2}\left\{\frac{1}{2}Q'' - Q\right\}' + \sigma\beta'(y_\sigma)\left\{\frac{1}{2}Q''\right\} - \beta^3(y_\sigma)\frac{1}{2}Q''' + \sigma\beta^2(y_\sigma)(-\frac{1}{2}Q''') + \sigma(\beta^2)'(y_\sigma)(-3Q''/8) \\
&+ \sigma\beta''(y_\sigma)\frac{1}{4}Q' + \sigma^2\beta(y_\sigma)\frac{1}{4}(\frac{1}{2}Q'' - Q)' + \sigma^2\beta'(y_\sigma)(\frac{1}{4}Q'') + \sigma^{\frac{5}{2}}O(\tilde{Q}_\sigma(y_\sigma)).
\end{aligned}
$$

Hence using Lemma 7, we can obtain

$$\begin{aligned} S(Q) &= \tilde{Q}_\sigma(y_\sigma)a_{1,0}\left\{-\frac{5}{2}Q''-3Q^3\right\}'-2\tilde{Q}'_\sigma(y_\sigma)a_{1,0}Q''+\tilde{Q}^2_\sigma(y_\sigma)\left(a_{2,0}\left\{-\frac{5}{2}Q''-3Q^3\right\}'+2a_{2,0}^2Q'''\right) \\ &+ (\tilde{Q}^2_\sigma)'(y_\sigma)\left(-2a_{2,0}Q''+\frac{3}{4}a_{1,0}^2Q''\right)+\sigma\tilde{Q}_\sigma(y_\sigma)\left(a_{1,1}\left\{-\frac{5}{2}Q''-3Q^3\right\}'+\frac{1}{4}a_{1,0}Q'''\right) \\ &+ \sigma\tilde{Q}'_\sigma(y_\sigma)\left(-2a_{1,1}Q''-\frac{1}{2}a_{1,0}Q''\right)+\sum_{3\le k+l\le 4}\sigma^l\left(\tilde{Q}^k_\sigma(y_\sigma)a_{k,l}\left\{-\frac{5}{2}Q''-3Q^3\right\}'(y)+(\tilde{Q}^k_\sigma)'(y_\sigma)a_{k,l}(-2Q''(y))\right) \\ &+ \sum_{3\le k+l\le 4}\sigma^l\left(\tilde{Q}^k_\sigma(y_\sigma)F^I_{k,l}+(\tilde{Q}^k_\sigma)'(y_\sigma)G^I_{k,l}\right)+\sigma^{\frac{5}{2}}O(\tilde{Q}_\sigma(y_\sigma)), \end{aligned}$$

■

Lemma 9

$$\begin{aligned} \delta S_{mKdV}(w) &= \sum_{(k,l)\in\Sigma_0}\sigma^l\left(\tilde{Q}^k_\sigma(y_\sigma)(-LA_{k,l})'(y)+(\tilde{Q}^k_\sigma)'(y_\sigma)((-LB_{k,l})'+3A''_{k,l}+3Q^2A_{k,l})(y)\right) \\ &+ \sum_{(k,l)\in\Sigma_0}\sigma^l\left(\tilde{Q}^k_\sigma(y_\sigma)F^{II}_{k,l}(y)+(\tilde{Q}^k_\sigma)'(y_\sigma)G^{II}_{k,l}(y)\right)+\sigma^{\frac{5}{2}}O(\tilde{Q}_\sigma(y_\sigma)), \end{aligned}$$

Proof.

$$\delta S_{mKdV}(w)=\sum_{(k,l)\in\Sigma_0}\sigma^l\left(\delta S_{mKdV}(A_{k,l}(y)\tilde{Q}_\sigma(y_\sigma))+\delta S_{mKdV}(B_{k,l}(y)(\tilde{Q}_\sigma)'(y_\sigma))\right).$$

First, we compute $\delta S_{mKdV}(A_{1,0}(y)\tilde{Q}_\sigma(y_\sigma))$. By Lemmas 5 and 7, we have

$$\begin{aligned} &\delta S_{mKdV}(A_{1,0}(y)\tilde{Q}_\sigma(y_\sigma)) \\ =\ &\tilde{Q}_\sigma(y_\sigma)\left\{-(LA_{1,0})'+a_{1,0}\tilde{Q}_\sigma(y_\sigma)\,(-3A''_{1,0}-3A_{1,0}Q^2)'-a_{1,0}\tilde{Q}'_\sigma(y_\sigma)(3A''_{1,0})+a_{1,0}^2\tilde{Q}^2_\sigma(y_\sigma)(3A'''_{1,0})\right\} \\ +\ &\tilde{Q}'_\sigma(y_\sigma)\left\{3A''_{1,0}+3A_{1,0}Q^2-a_{1,0}\tilde{Q}_\sigma(y_\sigma)(6A''_{1,0})\right\}+\tilde{Q}''_\sigma(y_\sigma)(3A'_{1,0})+\sigma^{\frac{3}{2}}O(\tilde{Q}_\sigma(y_\sigma)). \end{aligned}$$

Note in particular that we have used $(\tilde{Q}'_\sigma)^2(y_\sigma)=\sigma^{3/2}O(\tilde{Q}_\sigma(y_\sigma))$ from (2.3). Next, by (2.1) and (2.3), we have

$$\tilde{Q}''_\sigma(y_\sigma)(3A'_{1,0})=\left(\sigma\tilde{Q}_\sigma(y_\sigma)-2\tilde{Q}^3_\sigma(y_\sigma)\right)(3A'_{1,0})+\sigma^{\frac{3}{2}}O(\tilde{Q}_\sigma(y_\sigma)).$$

Thus,

$$\begin{aligned} \delta S_{mKdV}(A_{1,0}(y)\tilde{Q}_\sigma(y_\sigma)) &= \tilde{Q}_\sigma(y_\sigma)(-LA_{1,0})'+\tilde{Q}'_\sigma(y_\sigma)\left(3A''_{1,0}+3A_{1,0}Q^2\right)+\tilde{Q}^2_\sigma(y_\sigma)\left(a_{1,0}(-3A''_{1,0}-3A_{1,0}Q^2)'\right) \\ &+(\tilde{Q}^2_\sigma)'(y_\sigma)\left(-\frac{9}{2}a_{1,0}A''_{1,0}\right)+\tilde{Q}^3_\sigma(y_\sigma)(3a_{1,0}^2A'''_{1,0}-6A'_{1,0})+\sigma\tilde{Q}_\sigma(y_\sigma)(3A'_{1,0})+\sigma^{\frac{5}{2}}O(\tilde{Q}_\sigma(y_\sigma)). \end{aligned}$$

Now, we compute $\delta S_{mKdV}(B_{1,0}(y)\tilde{Q}'_\sigma(y_\sigma))$ in a similar way:

$$\begin{aligned} \delta S_{mKdV}(B_{1,0}(y)\tilde{Q}'_\sigma(y_\sigma)) &= \tilde{Q}'_\sigma(y_\sigma)\left\{-(LB_{1,0})'+a_{1,0}\tilde{Q}_\sigma(y_\sigma)(-3B''_{1,0}-3B_{1,0}Q^2)'\right\} \\ &+ \tilde{Q}''_\sigma(y_\sigma)(3B''_{1,0}+3B_{1,0}Q^2)+\sigma^{\frac{3}{2}}O(\tilde{Q}_\sigma(y_\sigma)) \\ &= \tilde{Q}'_\sigma(y_\sigma)(-LB_{1,0})'+(\tilde{Q}^2_\sigma)'(y_\sigma)\left(a_{1,0}\frac{(-3B''_{1,0}-3B_{1,0}Q^2)'}{2}\right) \\ &+ \sigma\tilde{Q}_\sigma(y_\sigma)(3B''_{1,0}+3B_{1,0}Q^2)+\tilde{Q}^3_\sigma(y_\sigma)\left(-6B''_{1,0}-6B_{1,0}Q^2\right)+\sigma^{\frac{3}{2}}O(\tilde{Q}_\sigma(y_\sigma)). \end{aligned}$$

Similarly, for all (k,l) with $2\le k+l\le 4$, combining the above, we obtain Lemma 9. ■

Lemma 10

$$\begin{aligned} S_{gBBM}(w) &= \sum_{(k,l)\in\Sigma_0} \sigma^l (\tilde{Q}_\sigma^k)'(y_\sigma)(-\frac{1}{2}A''_{k,l})(y) + \sum_{(k,l)\in\Sigma_0} \sigma^l \left(\tilde{Q}_\sigma^k(y_\sigma)F_{k,l}^{III}(y)) + (\tilde{Q}_\sigma^k)(y_\sigma)G_{k,l}^{III}(y))\right) \\ &+ \sigma^{\frac{3}{2}} O(\tilde{Q}_\sigma(y_\sigma)). \end{aligned}$$

Proof. By definition,

$$S_{gBBM}(w) = \sum_{(k,l)\in\Sigma_0} \sigma^l \left(S_{gBBM}(A_{k,l}(y)\tilde{Q}_\sigma^k(y_\sigma)) + S_{gBBM}(B_{k,l}(y)(\tilde{Q}_\sigma^k)'(y_\sigma))\right).$$

First, we compute $S_{gBBM}(A_{1,0}(y)\tilde{Q}_\sigma(y_\sigma))$, as in the proof Lemma 9, it follows from Lemma 6 and then (2.1), (2.3) that

$$\begin{aligned} S_{gBBM}(A_{1,0}(y)\tilde{Q}_\sigma(y_\sigma)) &= \tfrac{1}{2}\mu_\sigma \tilde{Q}_\sigma(y_\sigma)\left\{\beta(y_\sigma)A'''_{1,0} + \beta'(y_\sigma)(2A''_{1,0}) + \beta^2(y_\sigma)(-2A'''_{1,0})\right\} \\ &+ \frac{1}{2}\mu_\sigma \tilde{Q}'_\sigma(y_\sigma)\left\{-A''_{1,0} + \beta(y_\sigma)(4A''_{1,0})\right\} + \frac{1}{2}\mu_\sigma \tilde{Q}''_\sigma(y_\sigma)(-2A'_{1,0}) + \sigma^{\frac{3}{2}} O(\tilde{Q}_\sigma(y_\sigma)) \\ &= \frac{1}{2}\tilde{Q}_\sigma(y_\sigma)\left\{a_{1,0}\tilde{Q}_\sigma(y_\sigma)A'''_{1,0} + a_{1,0}\tilde{Q}'_\sigma(y_\sigma)(2A''_{1,0}) + a_{1,0}^2\tilde{Q}_\sigma^2(y_\sigma)(-2A'''_{1,0})\right\} \\ &+ \frac{1}{2}\tilde{Q}'_\sigma(y_\sigma)\left\{-A''_{1,0} + a_{1,0}\tilde{Q}_\sigma(y_\sigma)(4A''_{1,0})\right\} + \frac{1}{2}\tilde{Q}''_\sigma(y_\sigma)(-2A'_{1,0}) + \sigma^{\frac{3}{2}} O(\tilde{Q}_\sigma(y_\sigma)) \\ &= \tilde{Q}'_\sigma(y_\sigma)(-\frac{1}{2}A''_{1,0}) + \sigma\tilde{Q}_\sigma(y_\sigma)(-A'_{1,0}) + \tilde{Q}_\sigma^2(y_\sigma)(\frac{1}{2}a_{1,0}A'''_{1,0}) + (\tilde{Q}_\sigma^2)'(y_\sigma)(\frac{3}{2}a_{1,0}A''_{1,0}) \\ &+ \tilde{Q}_\sigma^3(y_\sigma)(-a_{1,0}^2A'''_{1,0} + 2A'_{1,0}) + \sigma^{\frac{3}{2}} O(\tilde{Q}_\sigma(y_\sigma)). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} S_{gBBM}(B_{1,0}(y)\tilde{Q}'_\sigma(y_\sigma)) &= \frac{1}{2}\tilde{Q}'_\sigma(y_\sigma)a_{1,0}\tilde{Q}_\sigma(y_\sigma)B'''_{1,0} + \frac{1}{2}\tilde{Q}''_\sigma(y_\sigma)(-B''_{1,0}) \\ &= \sigma\tilde{Q}_\sigma(y_\sigma)(-\frac{1}{2}B''_{1,0}) + (\tilde{Q}_\sigma^2)'(y_\sigma)\left(\frac{a_{1,0}}{4}B'''_{1,0}\right) + \tilde{Q}_\sigma^3(y_\sigma)\left(B''_{1,0}\right) + \sigma^{\frac{3}{2}} O(\tilde{Q}_\sigma(y_\sigma)). \end{aligned}$$

Finally, we check that for (k,l) such that $2 \le k+l \le 4$,

$$\begin{aligned} &S_{gBBM}(\sigma^l \tilde{Q}_\sigma^k(y_\sigma)A_{k,l}(y)) = \sigma(\tilde{Q}_\sigma^k)'(y_\sigma)(-\tfrac{1}{2}A''_{k,l}) + \sigma^{\frac{3}{2}} O(\tilde{Q}_\sigma(y_\sigma)), \\ &S_{gBBM}(\sigma^l (\tilde{Q}_\sigma^k)'(y_\sigma)B_{k,l}(y)) = \sigma^{\frac{3}{2}} O(\tilde{Q}_\sigma(y_\sigma)). \end{aligned}$$

■

Lemma 11

$$S_{int}(w) = \sum_{(k,l)\in\Sigma_0} \sigma^l \left(\tilde{Q}_\sigma^k(y_\sigma)F_{k,l}^{int}(y)) + (\tilde{Q}_\sigma^k)(y_\sigma)G_{k,l}^{int}(y))\right) + \sigma^{\frac{3}{2}} O(\tilde{Q}_\sigma(y_\sigma)),$$

Proof. We have

$$\begin{aligned} \partial_x(w^3) &= \partial_x(A_{1,0}^3(y)\tilde{Q}_\sigma^3(y_\sigma)) + \sigma^{\frac{3}{2}} O(\tilde{Q}_\sigma(y_\sigma)) = (1-\beta(y_\sigma))\left\{(A_{1,0}^3)'\tilde{Q}_\sigma^3(y_\sigma)\right\} + \sigma^{\frac{3}{2}} O(\tilde{Q}_\sigma(y_\sigma)) \\ &= \tilde{Q}_\sigma^3(y_\sigma)(A_{1,0}^3)' + \sigma^{\frac{3}{2}} O(\tilde{Q}_\sigma(y_\sigma)) \end{aligned}$$

Next, by similar arguments,

$$\begin{aligned} \partial_x(3Q^2\tilde{Q}_\sigma(y_\sigma) + 3Q\tilde{Q}_\sigma^2(y_\sigma)) &= 3(1-\beta(y_\sigma))\left\{\tilde{Q}_\sigma(y_\sigma)(Q^2)' + \tilde{Q}_\sigma^2(y_\sigma)Q'\right\} + 3Q^2(\tilde{Q}_\sigma)'(y_\sigma) + 3Q(\tilde{Q}_\sigma^2)'(y_\sigma) \\ &= \tilde{Q}_\sigma(y_\sigma)(3Q^2)' + (\tilde{Q}_\sigma)'(y_\sigma)(3Q^2) + \tilde{Q}_\sigma^2(y_\sigma)(3Q' - 3a_{1,0}(Q^2)') \\ &+ (\tilde{Q}_\sigma^2)'(y_\sigma)(3Q) + \tilde{Q}_\sigma^3(y_\sigma)(3a_{1,0}Q'), \end{aligned}$$

$$\partial_x(3\tilde{Q}_\sigma^2(y_\sigma)w) = \tilde{Q}_\sigma^3(y_\sigma)(3A'_{1,0}) + \sigma^{\frac{3}{2}} O(\tilde{Q}_\sigma(y_\sigma)),$$

$$\begin{aligned}\partial_x(6Q\tilde{Q}_\sigma(y_\sigma)w) &= \partial_x(6A_{1,0}Q\tilde{Q}_\sigma^2(y_\sigma)+3B_{1,0}Q(\tilde{Q}_\sigma^2)'(y_\sigma)) = \tilde{Q}_\sigma^2(y_\sigma)(6A_{1,0}Q)' \\ &+ (\tilde{Q}_\sigma^2)'(y_\sigma)\left(6A_{1,0}Q+(3B_{1,0}Q)'\right) - \tilde{Q}_\sigma^3(y_\sigma)(6a_{1,0}A_{1,0}Q)' + \sigma^{\frac{3}{2}}O(\tilde{Q}_\sigma(y_\sigma)),\end{aligned}$$

Finally,

$$\begin{aligned}\partial_x(3Qw^2) &= \partial_x\left(3A_{1,0}Q\tilde{Q}_\sigma^2(y_\sigma)+3A_{1,0}B_{1,0}Q(\tilde{Q}_\sigma^2)'(y_\sigma)\right) = \tilde{Q}_\sigma^2(y_\sigma)(3A_{1,0}Q)' \\ &+ (\tilde{Q}_\sigma^2)'(y_\sigma)\left((3QA_{1,0}B_{1,0})'+3A_{1,0}Q\right) + \tilde{Q}_\sigma^3(y_\sigma)(-3a_{1,0}(A_{1,0}Q)') + \sigma^{\frac{3}{2}}O(\tilde{Q}_\sigma(y_\sigma),\end{aligned}$$

$$\partial_x(3\tilde{Q}_\sigma(y_\sigma)w^2) = \tilde{Q}_\sigma^3(y_\sigma)(3A'_{1,0}) + \sigma^{\frac{3}{2}}O(\tilde{Q}_\sigma(y_\sigma)).$$

■

Putting together Lemmas 8-11, we obtain Proposition 1, in particular, the explicit expressions of $F_{k,l}$ and $G_{k,l}$ for $1 \leq k+l \leq 2$.

Proposition 1 means that if the system

$$(\Omega_{k,l}) \qquad \begin{cases} (LA_{k,l})' = -\frac{5}{2}a_{k,l}(Q''-3Q^3)' + F_{k,l} \\ (LB_{k,l})' = \frac{5}{2}A''_{k,l} + 3Q^2A_{k,l} - 2a_{k,l}Q'' + G_{k,l} \end{cases} \tag{2.16}$$

is solved for every $(k,l) \in \Sigma_0$, then $S(z) = \varepsilon(t,x)$ is small. Indeed, the $\Omega_{k,l}$ is solved for every $(k,l) \in \Sigma_0$. The proof of this part is similar to [5,7], so it is omitted. Therefore, we complete the proof of Proposition 1.

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References

[1] D. H. Peregrine. Long waves on a beach, *J. Fluid Mechanics*, 27 (1967):815-827.

[2] T. B. Benjamin, J. L. Bona and J. J. Mahony. Model equations for long waves in nonlinear dispersion systems, *Philos. Trans. Roy. Soc. London Ser. A*, 272(1972): 47-78.

[3] J. B. Mcleod and P. J. Olver. The connection between completely integrable partial differential equations and ordinary differential equation of Painleve type, *SIAM J. Math. Analysis*, 14(1983): 56-75.

[4] P. J. Olver. Euler operators and conservation laws of the BBM equation, *Math. Proc. Camb. Phil. Soc*, 85(1979): 143-160.

[5] Y. Martel and F. Merle. Description of two soliton collision for the quartic gKdV equation, *Ann. Math*, 174(2011): 757-857.

[6] Y. Martel and F. Merle. Stability of two soliton collision for nonintegrable gKdV equations, *Commun. Math. Phys*, 286(2009): 39-79.

[7] Y. Martel, F. Merle and T. Mizumachi. Description of the inelastic collision of two solitary waves for the BBM equation, *Arch. Ration. Mech. Anal*, 196(2010): 517-574.

[8] C. Muñoz. On the inelastic two-soliton collision for gKdV equations with general nonlinearity, *Inter. Math. Res. Not*, 9(2010): 1624-1719.