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Adaptive Cluster Synchronization of Uncertain Complex Dynamical Networks in Finite Time

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Abstract: This paper studies adaptive cluster synchronization of uncertain complex dynamical networks in finite time. Based on Lyapunov stability theorem and the finite-time control techniques, a useful criteria is established to guarantee the realization of adaptive cluster synchronization in finite time. Finally, a numerical example is given to illustrate the effectiveness and correctness of the derived theoretical results.

Keywords: uncertain complex networks; finite time; adaptive cluster synchronization

1 Introduction

Synchronization as a ubiquitous phenomenon plays a prominent role in nature, social life, engineering and physics. During the last two decades, the investigation on synchronization of complex dynamical networks has dramatically attracted a great deal of attentions. One of the most significant reasons lies in its potential applications in various fields such as communication, physics, biological, network and engineering science [1,2]. As a result, many types of synchronization patterns have been brought forward and deeply studied, including projective synchronization [3], generalized synchronization [4,5], cluster synchronization[6,7], etc.

Cluster synchronization, as a particular synchronization phenomenon, has attracted the interest of many researchers. Up to now, many results about cluster synchronization protocols have been proposed. For instance, Hu and Cao studied the cluster synchronization in directed networks of non-identical systems with noises based on pinning control method [8]. Wu and Lu proposed a novel adaptive strategy for the cluster synchronization of adaptive complex dynamical networks with non-identical nodes [9]. Jiang et al. published their researches on adaptive cluster general projective synchronization (CGPS) of complex dynamic networks in finite time [10]. To sum up, the obtained results take an important step on cluster synchronization in nonlinear science fields. It is noted that, in the real world, the external disturbance, parameter fluctuation and parameter uncertainties are unavoidable, which can make the synchronization more difficult to realize due to measure errors or may destroy the networks stability. Therefore, it is significant and of prime importance to consider the finite-time cluster synchronization problem for uncertain dynamical networks. However, to the best of our knowledge, few results have been reported on this issue.

In this paper, a new cluster synchronization method is presented for uncertain complex dynamic networks in finite time. In order to achieve the cluster synchronization, adaptive controllers are designed. By using the Lyapunov function method, we derived the adaptive cluster synchronization criteria. Finally, a numerical example is provided to show the effectiveness of the proposed method. Your text goes here.

2 Network model and preliminaries

Consider a complex networks consisting of N linearly and diffusively coupled dynamical nodes, which is described as

$$\dot{x}_i(t) = F_i(t, x_i(t), \alpha_i) + \sum_{j=1}^{N} c_{ij}\Gamma x_j(t) + I_i(t), \quad i \in \hat{U} \tag{1}$$

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where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in R_n$ is the state vector of the *i*th node at time$t$,$\hat{U} = \{m_{l-1}+1, \dots, m_l\}$ denotes the index set of all the nodes in the *m*th cluster, $l = 1, 2, \dots, k$, $m_0 = 0$, $m_{l-1} < m_l$. $F_i(t, x_i(t), \alpha_i)$ indicates the dynamics of the *i*th node, and can be expressed as the form: $F_i(t, x_i(t), \alpha_i) = f_i(t, x_i(t)) + g_i(t, x_i(t))\,\alpha_i$, where $f_i(t, x_i(t))$ and $g_i(t, x_i(t))$: $R_n \to R_n$ are nonlinear vector-valued functions, which are distinct for different cluster. Γ is the inner connecting matrix, $I_i(t)$ is a disturbance vector. The matrix $C = (c_{ij})_{N\times N}$ is the zero-row-sum outer coupling matrix, which is defined as follows: if there is a link from node *j* to node *i* ($j \neq i$), then $c_{ij} \neq 0$; otherwise, $c_{ij} = 0$.

Based on the former research, we propose a general complex networks consisting of N dynamical nodes with parameters uncertainties as the networks:

$$\dot{y}_i(t) = F_i(t, y_i(t), \hat{\alpha}_i) + \sum_{j=1}^{N} c_{ij}\Gamma y_j(t) + \hat{I}_i(t) + U_i \qquad i \in \hat{U} \tag{2}$$

where $y_i(t) = (y_{i1}(t), y_{i2}(t), \dots, y_{in}(t))^T \in R^n$ is the state vector of the *i*th node, $f_i(t, y_i(t))$ and $g_i(t, y_i(t)) : R^n \to R^n$ are also nonlinear vector-valued functions, $\hat{\alpha}_i$ is the estimation of the unknown α_i, $\hat{I}_i(t)$ is the estimation of the disturbance vector $I_i(t)$, U_i is an outer controller.

In order to derive our results, we introduce definition, assumption, lemmas.

Definition 1 Defining the synchronization error as $e_{\hat{U}}(t) = y_{\hat{U}}(t) - x_{\hat{U}}(t)$, if the complex networks (1) and (2) satisfies: $\lim_{t\to\infty} \|e_{\hat{U}}(t)\| = \lim_{t\to\infty} \|y_{\hat{U}}(t) - x_{\hat{U}}(t)\| = 0$, where $\hat{U}$ represents the index set of all the nodes in the *m*th cluster $m = (1, 2, \dots, k)$, we then call that the networks (1) and (2) can achieve cluster synchronization.

Assumption 1 Suppose that there exist a constant $L > 0$ such that for any time-varying vectors $x(t), y(t) \in R^n$, we have

$$\|F(t, y(t), \alpha) - F(t, x(t), \alpha)\| \leq L \|y(t) - x(t)\| \tag{3}$$

Lemma 1 *Let* $x_i > 0$ $(i = 1, 2, \dots, n)$ *and* $0 < s < q$. *Then*

$$(\sum_{i=1}^{n} {x_i}^q)^{1/q} \leq (\sum_{i=1}^{n} {x_i}^s)^{1/s} \tag{4}$$

Lemma 2 *Assume that function* $V(t)$ *is continuous, positive-definite and satisfies the following differential inequality:* $\dot{V}(t) \leq -qV^\sigma(t)$, $t \geq t_0$, $V(t_0) \geq 0$, *where* $q > 0, 0 < \sigma < 1$ *are constants. For any given time* t_0, $V(t)$ *satisfies the following inequality:* $V^{1-\sigma}(t) \leq V^{1-\sigma}(t_0) - q(1-\sigma)(t - t_0)$, $t_0 \leq t \leq t_1$ *and* $V(t) = 0$, $(\forall t \geq t_1)$ *with* t_1 *given by*

$$t_1 = t_0 + \frac{V^{1-\sigma}(t_0)}{q(1-\sigma)} \tag{5}$$

3 Main result

In this section, in order to realize adaptive cluster synchronization between the drive and response systems in finite time, we give some states before giving the main results. Let the synchronization error be described as $e_i(t) = y_i(t) - x_i(t)$, then according to Eq.(1) and (2), we can get:

$$\begin{aligned}\dot{e}_i(t) &= \dot{y}_i(t) - \dot{x}_i(t) \\ &= F_i(t, y_i(t), \hat{\alpha}_i) - F_i(t, x_i(t), \alpha_i) + \sum_{j=1}^{N} c_{ij}\Gamma\left(y_j(t) - x_j(t)\right) \\ &\quad + \hat{I}_i(t) - I_i(t) + U_i \\ &= g_i(t, y_i(t))\cdot\hat{\alpha}_i - g_i(t, y_i(t))\cdot\alpha_i + F_i(t, y_i(t), \alpha_i) - F_i(t, x_i(t), \alpha_i) \\ &\quad + \sum_{j=1}^{N} c_{ij}\Gamma\left(y_j(t) - x_j(t)\right) + \hat{I}_i(t) - I_i(t) + U_i \qquad i = 1, 2, ..., N.\end{aligned} \tag{6}$$

In order to achieve the main objective of this paper, the nonlinear controllers are designed as follows:

$$U_i = -r_i(t)e_i(t) - \kappa sign(e_i(t))|e_i(t)|^\theta \qquad i \in \hat{U} \tag{7}$$

where $r_i(t) > 0$ $(i = 1, 2, \dots, N)$ are time-varying adaptive control gains that can be suitably chosen by the cluster synchronization system and satisfies the following conditions: $\dot{r}_i(t) = k_i {e^T}_i(t)e_i(t) > 0$.

Theorem 3 *Suppose that Assumption 1 holds, The complex dynamical networks will realize adaptive cluster synchronization if the following condition holds:*

$$\gamma = -\lambda_{\max}\left((L_i - r^*)I_{n\times n}\otimes I_{N\times N} + C\otimes\Gamma\right) > 0 \tag{8}$$

when employing the outer controller (6) and updating laws:

$$\dot{\hat{I}}_i = -\xi_i e_i^{\mathrm{T}}(t) \tag{9}$$

$$\dot{\hat{\alpha}}_i = -\mu_i g_i(t, y_i(t)) {e_i}^T(t) \tag{10}$$

Proof. In order to verify the conclusion of Theorem 3, it is necessary to construct a Lyapunov function as

$$V(t) = \frac{1}{2}\sum_{i=1}^{\mathrm{N}} e_i^{\mathrm{T}}(t)e_i(t) + \frac{1}{2}\sum_{i=1}^{\mathrm{N}}\frac{1}{\mu_i}(\alpha_i - \hat{\alpha}_i)^2 + \frac{1}{2}\sum_{i=1}^{N}\frac{1}{\xi_i}(\hat{I}_i(t) - I_i(t))^2 + \sum_{i=1}^{N}\frac{(r_i - r^*)^2}{2k_i}$$

where r^* is a positive constant.

Then, the derivative of $V(t)$ with respect to time *t* along the trajectory of the error system (6) yields:

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{\mathrm{N}} e_i^{\mathrm{T}}(t)\dot{e}_i(t) - \sum_{i=1}^{\mathrm{N}}\tfrac{1}{\mu_i}\dot{\hat{\alpha}}_i(\alpha_i - \hat{\alpha}_i) + \sum_{i=1}^{N}\tfrac{1}{\xi_i}\dot{\hat{I}}_i(\hat{I}_i(t) - I_i(t)) + \sum_{i=1}^{N}\tfrac{r_i - r^*}{k_i}\dot{r}_i \\
&= \sum_{i=1}^{\mathrm{N}} e_i^{\mathrm{T}}(t)[g_i(t, y_i(t))\cdot\hat{\alpha}_i - g_i(t, y_i(t))\cdot\alpha_i + F_i(t, y_i(t), \alpha_i) - F_i(t, x_i(t), \alpha_i) \\
&\quad + \sum_{j=1}^{N} c_{ij}\Gamma\left(y_j(t) - x_j(t)\right) + \hat{I}_i(t) - I_i(t) + U_i] - \sum_{i=1}^{\mathrm{N}}\tfrac{1}{\mu_i}\dot{\hat{\alpha}}_i(\alpha_i - \hat{\alpha}_i) \\
&\quad + \sum_{i=1}^{N}\tfrac{1}{\xi_i}\dot{\hat{I}}_i(\hat{I}_i(t) - I_i(t)) + \sum_{i=1}^{N}\tfrac{r_i - r^*}{k_i}\dot{r}_i
\end{aligned} \tag{11}$$

Furthermore, one term of $\dot{V}(t)$ is estimated as follows:

$$\sum_{i=1}^{N}\sum_{j=1}^{N} e_{ij}^{T}\kappa sign(e_{ij}(t))|e_{ij}(t)|^{\theta} = \kappa\sum_{i=1}^{N}\sum_{j=1}^{N}\left|{e_i}^T(t)\right||e_i(t)|^{\theta} = \kappa\sum_{i=1}^{N}\sum_{j=1}^{N}|e_i(t)|^{\theta+1}$$

By Lemma 1, we have the following inequation:

$$(\sum_{i=1}^{N}\sum_{j=1}^{N} e_{ij}^2(t))^{1/2} \le (\sum_{i=1}^{N} e_{ij}^{1+\theta}(t))^{1/1+\theta}$$

Hence,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}|e_i(t)|^{\theta+1} \ge \left(\sum_{i=1}^{N}\sum_{j=1}^{N}|e_{ij}(t)|^2\right)^{(\theta+1)/2} = \left(\sum_{i=1}^{N}\sum_{j=1}^{N} e_{ij}^{T}(t)e_{ij}(t)\right)^{(\theta+1)/2}$$

According to Eqs.(3), (5), (7) and (11), we obtain:

$$\begin{aligned}
\dot{V}(t) &\le \sum_{i=1}^{\mathrm{N}} e_i^{\mathrm{T}}(t)[(L_i - r^*)I_{n\times n}\otimes I_{N\times N} + C\otimes\Gamma]e_i(t) - \kappa\left(\sum_{i=1}^{N}\sum_{j=1}^{N} e_{ij}^{\mathrm{T}}(t)e_{ij}(t)\right)^{(\theta+1)/2} \\
&\le -\gamma\sum_{i=1}^{\mathrm{N}} e_i^{\mathrm{T}}(t)e_i(t) - \kappa\left(\sum_{i=1}^{N}\sum_{j=1}^{N} e_{ij}^{\mathrm{T}}(t)e_{ij}(t)\right)^{(\theta+1)/2} \\
&\le -3\kappa(V(t))^{(\theta+1)/2}
\end{aligned} \tag{12}$$

From Lemma 2 and the conditions in Theorem 3, we can obtain that $\lim_{t\to\infty} e_i(t) \to 0$, which implies that the error system (6) is exponentially stable. Namely the drive system (1) can achieve the general cluster synchronization with response system (2) in finite time. This completes the proof of Theorem 3.

Remark 1. In Theorem 3, we define the derivative of disturbance $\hat{I}$ so as to its application. In the practical complex networks, we often find the disturbance has few effect on the system, so we can design feedback controller to eliminate the effect.

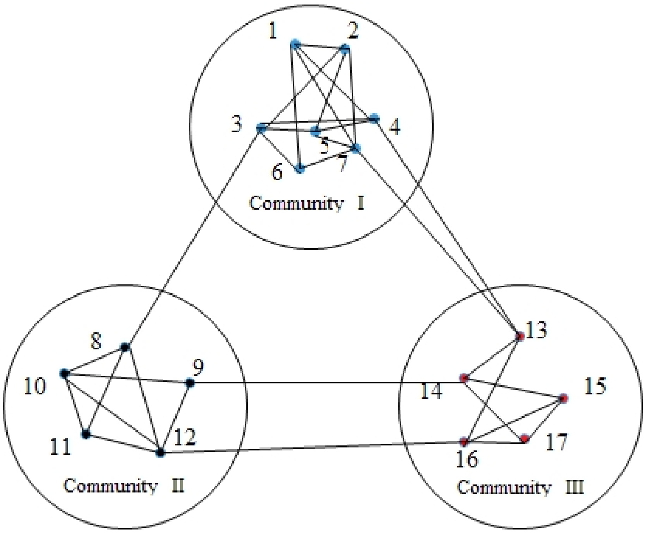

Fig. 1: A community networks with three communities consisting of 17 nodes.

4 Numerical simulation

In this section, illustrative examples are given to show the effectiveness of the proposed theory for general cluster synchronization. we show that the networks with three communities consisting of 17 nodes in Fig.1.

Consider the following Lorenz system as the 17 nodes dynamics:

$$F(t,x(t),\alpha)=\begin{pmatrix} a(x_2-x_1) \\ cx_1-x_1x_3-x_2 \\ -bx_3+x_1x_2 \end{pmatrix}=f(t,x(t))+g(t,x(t))\cdot\alpha$$

where $a=10, b=8/3, c=28$ and $\alpha=(a,b,c)^T$

$$f(t,x(t))=\begin{pmatrix} 0 \\ -x_1x_3-x_2 \\ x_1x_2 \end{pmatrix},\ g(t,x(t))=\begin{pmatrix} x_2-x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_3 \end{pmatrix}$$

The coupling matrix *C* is characterized as follows:

$$C_1=\begin{pmatrix} -9 & 4 & 0 & 3 & 0 & 2 & 0 \\ 1 & -4 & 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & -6 & 1 & 1 & 3 & 0 \\ 2 & 0 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -7 & 0 & 2 \\ 1 & 0 & 3 & 0 & 0 & -5 & 1 \\ 3 & 2 & 0 & 0 & 2 & 1 & -8 \end{pmatrix}$$

$$C_2=\begin{pmatrix} -7 & 4 & 0 & 3 & 0 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 2 & 1 & -4 \end{pmatrix},\ C_3=\begin{pmatrix} -8 & 0 & 2 & 4 & 2 \\ 0 & -6 & 1 & 3 & 2 \\ 2 & 1 & -7 & 4 & 0 \\ 1 & 0 & 1 & -3 & 1 \\ 0 & 3 & 2 & 0 & -5 \end{pmatrix}$$

Here we assume that $I_i(t)=0$, the initial values are chosen as $x_i(0)=(0.8+0.1i, 0.8+0.1i, 0.8+0.1i)^T$, $y_i(0)=(3.0+0.6i, 3.0+0.6i, 3.0+0.6i)^T$. In numerical simulation, we fix the inner coupling matrix $\Gamma=diag(1,1,1)$, let $\kappa_1=\kappa_2=\kappa_3=4$, $L=1, \theta=0.4, \mu_1=\mu_2=\mu_3=1$ and the initial values of estimated parameters are:$\hat{a}=0.7,\ \hat{b}=2.8,\ \hat{c}=1.6$.

The synchronization errors of Communities I – III are shown in Fig. 2. 3 and 4. From the simulations, we could see that the uncertain complex networks (1) surely achieve cluster synchronization in finite time.

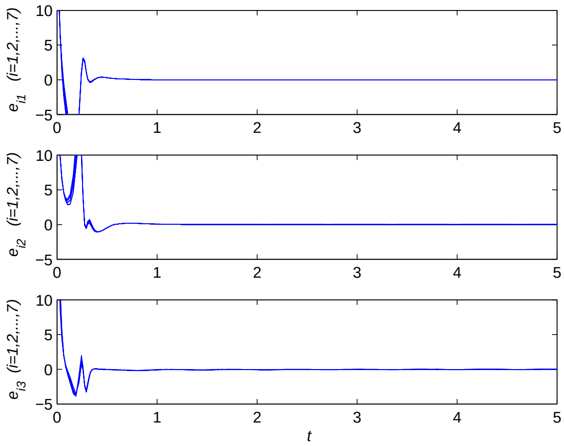

Fig. 2: Synchronization errors in community I

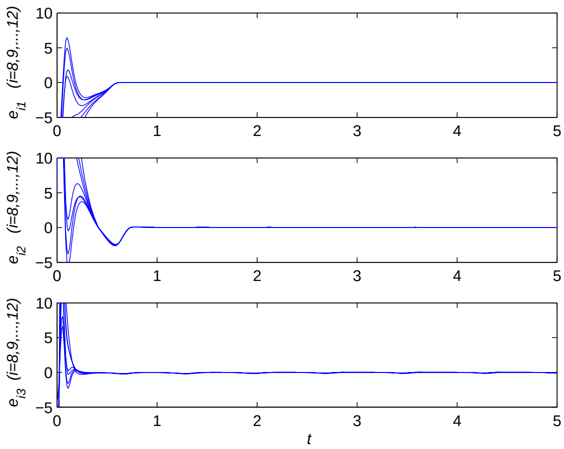

Fig. 3: Synchronization errors in community II

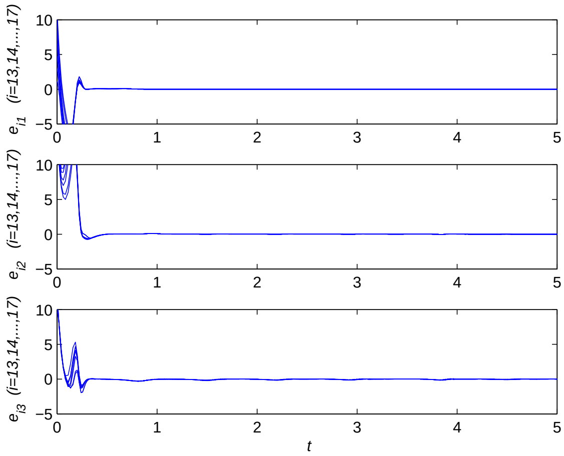

Fig. 4: Synchronization errors in community III

5 Conclusion

In this paper, a detailed analysis is presented for the finite-time cluster synchronization of uncertain complex dynamical networks by means of adaptive controllers in finite time. A cluster synchronization criterion for the uncertain complex networks is obtained via utilizing the Lyapunov method. Finally, numerical simulations have been presented to demonstrate the effectiveness of the theoretical results.

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