Mixed Tracking and Projective Synchronization of 6D Hyperchaotic System Using Active Control

Shikha Ayub Khan *
Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India
(Received 6 October 2015, accepted 4 August 2016)

Abstract: This paper examines mixed tracking control and projective synchronization of two identical 6D hyperchaotic systems via active control technique. The designed control functions for the mixed tracking enable each of the system state variables to stabilize at different chosen positions as well as control each state variables of the system to track different desired smooth functions of time. Also, the active control technique is used to design control functions which achieve projective synchronization between the slave state variables and the master state variables. We show that the coupling strength is inversely proportional to the synchronization time. Numerical simulations are carried out to validate the effectiveness of the analytical technique.

Keywords: hyperchaos ; synchronization ; tracking ; active control

1 Introduction

Over the last decades there has been a great interest to harness the very peculiar chaotic behavior in deterministic systems. A chaotic system is a nonlinear deterministic system that displays complex and unpredictable behavior. The sensitive dependence on the initial conditions is a prominent characteristic of chaotic behavior. Over the past two decades, many chaotic systems have been found, such as the Lorenz system [1], the Lur’e system [2], the Duffing-Holmes system [3], the Genesio system [4], the Rössler system [5], Chua’s circuit [6], and so on. Chaos has gradually moved from simply being a scientific curiosity to a promising subject with practical significance and applications in different fields such as communication [7], biological systems [8], economics and other fields.

Chaotic behaviours could be beneficial feature in some cases (e.g., fluid mixing), but can be undesirable in some engineering, biological and other physical applications (e.g., chaos in the brain [9], cardiac chaos [10]); and therefore it is often desired that chaos should be controlled, so as to improve the system performance. Thus, it is of considerable interest and potential utility, to devise control techniques capable of forcing a system to maintain a desired dynamical behaviour even when intrinsically chaotic. The control of chaos and bifurcation is concerned with using some designed control input(s) to modify the characteristics of a parameterized nonlinear system. There might be needed for different components of a chaotic system to follow different trajectories when controlled, therefore, the need for mixed tracking or control. A number of methods such as OGY closed-loop feedback method [11], active control [12], adaptive control [13],[29] and optimal control [14] exist for the control of chaos in systems. Chaos control is considered as a special case of chaos synchronization.

Chaos synchronization, on the other hand, involves the coupling of two chaotic systems so that both systems achieve identical dynamics asymptotically with time. There are two forms of coupling: mutual (bidirectional) coupling and the drive-response (unidirectional) coupling. In mutual coupling, the two systems influence or alter each others dynamics until both systems achieve identical dynamics. In the unidirectional coupling, control functions are designed to force the dynamics of one system referred to as the response system to track the unaltered dynamics of the other system referred to as the drive system. If \( \dot{x} = f(x,y) \) is the drive system and \( \dot{y} = g(x,y) \) is the response system, where \( x \) and \( y \) are phase space or state variables, and \( f \) and \( g \) are the corresponding nonlinear functions, synchronization in a direct sense implies \( |y(t) - \alpha x(t)| \to 0 \) as \( t \to \infty \), where \( \alpha \) is the coupling parameter. When this occurs, the coupled systems are said
to be projective synchronized [15]. Chaos synchronization is directly related to the observer problem in control theory [16]. The problem may be treated as the design of control laws for full chaotic observer (response system) using the known information of the plant (drive system) so as to ensure that the controlled receiver (response system) synchronizes with the transmitter (i.e., the plant or drive system). Hence, the response chaotic system tracks the dynamics of the drive system in the course of time. Projective synchronization based on the drive-response configuration, as employed in this paper, has direct application in secure communication. Complete synchronization, which is a special case of projective synchronization, of two chaotic systems was first demonstrated by Pecora and Carroll in 1990 [15]. Other types of synchronization identified thereafter include sequential, phase, anticipated, measure, generalized, lag projective synchronization [17] (and references therein), and reduced-order synchronization [18] and complete synchronization [30].

In recent years, active control [19–23] have been widely recognized as powerful design methods to control and synchronize chaos. Active control technique gives the flexibility to construct a control law so that it can be used widely to control and synchronize various nonlinear systems, including chaotic systems.

In this paper tracking control and synchronization of hyperchaos in the new 6D hyperchaotic system is achieved by active control method. This work is motivated by the work done by the authors in [27] and [28]. Numerical simulations are also performed for the purpose of illustration and verification.

2 System Description

The Lorenz system [1] was the first chaotic system to be modeled and one of the most widely studied. The original system was modified into a 4D hyperchaotic system by introducing a linear feedback controller to the second equation of the Lorenz system. In 2009, Hu [24] constructed a 5D hyperchaotic Lorenz system by introducing a linear feedback controller and a nonlinear feedback controller to the Lorenz system. In 2015, Yang constructed a 6D system [25], which can generate hyperchaotic attractors with four positive Lyapunov Exponents which is obtained by coupling a 1D linear system and a 5D hyperchaotic system. As it is well known, a system with more than one positive Lyapunov exponents is called a hyperchaotic system [26]. Using MATLAB we have computed the six Lyapunov exponents of the 6D hyperchaotic system at time $t = 300$, which are $\lambda_1 = 1.0034, \lambda_2 = 0.57515, \lambda_3 = 0.32785, \lambda_4 = 0.020937, \lambda_5 = -0.12087, \lambda_6 = -12.4713$ as shown in the figure figure (1). Since it has four positive Lyapunov exponents, so the system is hyperchaotic. The system
is given as follows

\[
\begin{align*}
\dot{x} &= a(y - x) + u \\
\dot{y} &= cx - y - xz + v \\
\dot{z} &= -bz + xy \\
\dot{u} &= du - xz \\
\dot{v} &= -ky \\
\dot{w} &= hw + ly
\end{align*}
\]  

where \( abhkl \neq 0 \), a, b, c and h are the constant parameters, l is the coupling parameter, d and k are two control parameters, determining the hyperchaotic behaviors of the system. The coupling parameter l, controllers u and v have made the classical Lorenz chaotic system become a 6D hyperchaotic system (1) with four positive Lyapunov exponents. The system (1) has a hyperchaotic attractor when \((a,b,c,d,k,h,l) = (10, 8/3, 28, 2, 8.4, 1, 1)\), as depicted in Figure(2).

### 3 Tracking Control of 6D Hyperchaotic System

#### 3.1 Design of controllers

We aim to design controllers that will enable (1) to be controlled to a predefined rule. To make it more flexible and adaptable, we employ the controls on each component of the 6D hyperchaotic system to different functions. The system (1) with the control parameters is given as

\[
\begin{align*}
\dot{x} &= a(y - x) + u + \mu_1 \\
\dot{y} &= cx - y - xz + v + \mu_2 \\
\dot{z} &= -bz + xy + \mu_3 \\
\dot{u} &= du - xz + \mu_4 \\
\dot{v} &= -ky + \mu_5 \\
\dot{w} &= hw + ly + \mu_6
\end{align*}
\]  

The error function is defined as
S. Ayub Khan: Mixed Tracking and Projective Synchronization of 6D Hyperchaotic System Using Active Control

\[
\begin{align*}
  e_1 &= x - f_1 \\
  e_2 &= y - f_2 \\
  e_3 &= z - f_3 \\
  e_4 &= u - f_4 \\
  e_5 &= v - f_5 \\
  e_6 &= w - f_6 \\
\end{align*}
\]

(3)

where \( f_i \)'s, (i = 1 to 6) are the functions to be determined.

Differentiating equation (3), we have

\[
\begin{align*}
  \dot{e}_1 &= \dot{x} - \dot{f}_1 \\
  \dot{e}_2 &= \dot{y} - \dot{f}_2 \\
  \dot{e}_3 &= \dot{z} - \dot{f}_3 \\
  \dot{e}_4 &= \dot{u} - \dot{f}_4 \\
  \dot{e}_5 &= \dot{v} - \dot{f}_5 \\
  \dot{e}_6 &= \dot{w} - \dot{f}_6 \\
\end{align*}
\]

(4)

substituting (3) and (4) into (1), we have

\[
\begin{align*}
  \dot{e}_1 &= -a(e_1 + f_1) + a(e_2 + f_2) + (e_4 + f_4) + \mu_1 - \dot{f}_1 \\
  \dot{e}_2 &= c(e_1 + f_1) - (e_2 + f_2) - xz + (e_5 + f_5) + \mu_2 - \dot{f}_2 \\
  \dot{e}_3 &= -b(e_3 + f_3) + xy + \mu_3 - \dot{f}_3 \\
  \dot{e}_4 &= d(e_4 + f_4) - xz + \mu_4 - \dot{f}_4 \\
  \dot{e}_5 &= -k(e_2 + f_2) + \mu_5 - \dot{f}_5 \\
  \dot{e}_6 &= l(e_2 + f_2) + h(e_6 + f_6) + \mu_6 - \dot{f}_6 \\
\end{align*}
\]

(5)

Eliminating terms which cannot be expressed as linear terms in \( e_1, e_2, e_3, e_4, e_5, e_6 \) and solving for \( u(t) \),

\[
\begin{align*}
  \mu_1 &= af_1 - af_2 - f_4 + \dot{f}_1 + \nu_1 \\
  \mu_2 &= cf_1 + f_2 + xz - f_5 + \dot{f}_2 + \nu_2 \\
  \mu_3 &= bf_3 - xy + f_3 + \nu_3 \\
  \mu_4 &= df_4 + xz + f_4 + \nu_4 \\
  \mu_5 &= k f_2 + \dot{f}_5 + \nu_5 \\
  \mu_6 &= h f_6 - l f_2 + \dot{f}_6 + \nu_6 \\
\end{align*}
\]

(6)

the parameters \( \nu_i \)'s (i = 1 to 6) will be obtained later.

Substituting (6) into (5), the differential of the error becomes

\[
\begin{align*}
  \dot{e}_1 &= -ae_1 + ae_2 + e_4 + \nu_1 \\
  \dot{e}_2 &= ce_1 - e_2 + e_5 + \nu_2 \\
  \dot{e}_3 &= -be_3 + \nu_3 \\
  \dot{e}_4 &= de_4 + \nu_4 \\
  \dot{e}_5 &= -ke_2 + \nu_5 \\
  \dot{e}_6 &= he_6 + le_2 + \nu_6 \\
\end{align*}
\]

(7)

IJNS homepage: http://www.nonlinearscience.org.uk/
For simulation, the Mathematica is used to solve the differential equation with the following initial conditions \( (x_0, y_0, z_0, u_0, v_0, w_0) = (1, 0, 1, 10, 1, 1) \). The system parameters are chosen as \( a = 10, b = 8/3, c = 28, d = 2, k = 8.4, h = 1, l = 1 \), so the system behaves hyperchaotically as shown in figure (2). We set \( f_1 = \beta \sin(t), f_2 = \alpha t^2, f_3 = \gamma, f_4 = \delta + \theta \sin(t) \) and \( f_5 = \xi t, f_6 = \beta \cos(t) \) where \( \beta = 8, \alpha = 0.01, \gamma = 2, \delta = 4, \theta = 60, \xi = 0.1 \). The results are presented in figure (3). The effectiveness of the control can be seen as various components of the system converges to the preset functions when the controls are applied at time \( t \geq 20 \). Before the activation of the controls, the system behaves chaotically while the trajectory was changed to the present function on the activation of the control.

### 3.2 Numerical Simulation

Using the active control method, a constant matrix \( A \) is chosen which will control the error dynamics (7) such that the feedback matrix is

\[
\begin{bmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4 \\
\nu_5 \\
\nu_6 \\
\end{bmatrix} = A
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\epsilon_5 \\
\epsilon_6 \\
\end{bmatrix}
\]

Thus, the matrix \( A \) is chosen to be of the form

\[
A = \begin{bmatrix}
\lambda_1 + a & -a & 0 & -1 & 0 & 0 \\
-c & \lambda_2 & 0 & 0 & -1 & 0 \\
0 & 0 & \lambda_3 + b & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 - d & 0 & 0 \\
0 & k & 0 & 0 & \lambda_5 & 0 \\
0 & -l & 0 & 0 & 0 & \lambda_6 - h \\
\end{bmatrix}
\]

so that the eigenvalues \( \lambda_i \)'s, \( i = 1 \) to 6) are negative.

\[48\]
4 Projective synchronization of 6D system

We consider the identical hyperchaotic 6D systems with subscript (1) and (2) described as the drive and the response systems respectively,

\[
\begin{align*}
    x_1 &= a(y_1 - x_1) + u_1 \\
    y_1 &= cx_1 - y_1 - x_1z_1 + v_1 \\
    z_1 &= -bz_1 + x_1y_1 \\
    u_1 &= du_1 - x_1z_1 \\
    v_1 &= -ky_1 \\
    w_1 &= hw_1 + ly_1
\end{align*}
\]

\[
\begin{align*}
    x_2 &= a(y_2 - x_2) + u_2 + \mu_1 \\
    y_2 &= cx_2 - y_2 - x_2z_2 + v_2 + \mu_2 \\
    z_2 &= -bz_2 + x_2y_2 + \mu_3 \\
    u_2 &= du_2 - x_2z_2 + \mu_4 \\
    v_2 &= -ky_2 + \mu_5 \\
    w_2 &= hw_2 + ly_2 + \mu_6
\end{align*}
\]

where \(x_i, y_i, z_i, u_i, v_i, w_i\), \((i = 1, 2)\) are the state vectors and \(a, b, c, d, h, k, l\) are the parameters of the system and \(\mu = [\mu_1 \ \mu_2 \ \mu_3 \ \mu_4 \ \mu_5 \ \mu_6]^t\) is the nonlinear controller to be designed.

The projective synchronization error is defined as

\[
\begin{align*}
    e_1 &= x_2 - \alpha x_1 \\
    e_2 &= y_2 - \alpha y_1 \\
    e_3 &= z_2 - \alpha z_1 \\
    e_4 &= u_2 - \alpha u_1 \\
    e_5 &= v_2 - \alpha v_1 \\
    e_6 &= w_2 - \alpha w_1
\end{align*}
\]

where \(\alpha\) is the scaling factor.

4.1 Design of Control Function

The error dynamics is obtained as

\[
\begin{align*}
    \dot{e}_1 &= -ae_1 + ae_2 + e_4 + \mu_1 \\
    \dot{e}_2 &= ce_1 - e_2 + e_5 - x_2z_2 + \alpha x_1z_1 + \mu_2 \\
    \dot{e}_3 &= -be_3 + x_2y_2 - \alpha x_1y_1 + \mu_3 \\
    \dot{e}_4 &= de_4 - x_2z_2 + \alpha x_1z_1 + \mu_4 \\
    \dot{e}_5 &= -ke_2 + \mu_5 \\
    \dot{e}_6 &= he_6 + le_2 + \mu_6
\end{align*}
\]

To achieve asymptotic stability of system (12), we eliminate terms which cannot be expressed as linear terms in \(e_1, e_2, e_3, e_4, e_5, e_6\) as follows :

\[
\begin{align*}
    \mu_1 &= \nu_1 \\
    \mu_2 &= x_2z_2 - \alpha x_1z_1 + \nu_2 \\
    \mu_3 &= -x_2y_2 + \alpha x_1y_1 + \nu_3 \\
    \mu_4 &= x_2z_2 - \alpha x_1z_1 + \nu_4 \\
    \mu_5 &= \nu_5 \\
    \mu_6 &= \nu_6
\end{align*}
\]

IJNS homepage: http://www.nonlinearScience.org.uk/
Substituting (14) into (13)

\[
\begin{align*}
\dot{e}_1 &= -ae_1 + ce_1 + e_4 + \nu_1 \\
\dot{e}_2 &= ce_1 - e_2 + e_5 + \nu_2 \\
\dot{e}_3 &= -be_3 + \nu_3 \\
\dot{e}_4 &= de_4 + \nu_4 \\
\dot{e}_5 &= -ke_2 + \nu_5 \\
\dot{e}_6 &= he_6 + le_2 + \nu_6
\end{align*}
\]  

(15)

Using the active control method, a constant matrix \( A \) is chosen such that the error dynamics (13) is controlled. For that the feedback matrix is

\[
\begin{bmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4 \\
\nu_5 \\
\nu_6
\end{bmatrix} = A
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5 \\
e_6
\end{bmatrix}
\]

(16)

with

\[
A = \begin{bmatrix}
\lambda_1 + a & -a & 0 & 0 & 0 & 0 \\
-c & \lambda_2 & 0 & 0 & -1 & 0 \\
0 & 0 & \lambda_3 + b & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 - d & 0 & 0 \\
0 & k & 0 & 0 & \lambda_5 & 0 \\
0 & -l & 0 & 0 & 0 & \lambda_6 - h
\end{bmatrix}
\]

(17)

In (16) the six eigenvalues \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \) are chosen to be negative in order to achieve a stable projective synchronization between two identical 6D hyperchaotic system.

### 4.2 Numerical Simulation

Numerical solutions were carried out in Mathematica to solve systems (10) and (11) with the following initial conditions \((x_1, y_1, z_1, u_1, v_1, w_1) = (1, 0, 1, 10, 1, 1)\) and \((x_2, y_2, z_2, u_2, v_2, w_2) = (4, 4, 3, 4, 5, -2)\). The system parameters are chosen as \(a = 10, b = 8/3, c = 28, d = 2, h = 8.4, k = 1, l = 1\), so the system behaves hyperchaotically as shown in figure (2). The error dynamics of the system when the controls are activated at time \( t \geq 10 \) as shown in figure (4). Then the synchronization errors between the two system is seen to converge to zero. Figure (5) shows the dynamics of the state variables \((x, y)\) of the system when compared after activation of control at time \( t = 0 \) and value of \( \alpha = 2.0 \). The trajectory of the master system is seen to be twice that of the slave as expected. Also complete synchronization and anti-synchronization are the special cases of the projective synchronization when \( \alpha = 1 \) and \( \alpha = -1 \) respectively. A quantity called synchronization time which gives a value of speed of synchronization when the error between the two synchronization approaches zero was also computed. Figure (6) depicts the time it takes for synchronization to occur as the coupling strength is increased. From the graph, an exponential decrease is seen. This synchronization time-coupling strength graph can be used as a measure of the speed of synchronization. In effect, for the system under consideration the synchronization time is seen to decrease with increasing coupling strength.

### 5 Conclusion

This work demonstrates that chaos synchronization of 6D hyperchaotic systems using active control method is achieved. We have used the same method to enable tracking of a desired trajectories which are achieved in a systematic way. Numerical simulations are used to verify the effectiveness of the proposed active control technique. Computational and analytical results are in excellent agreement.
Figure 4: Error dynamics of the state variables when the control functions are activated for $t \geq 10$

Figure 5: Dynamics of the state variables when the control functions are activated at $t = 0$ and $\alpha = 2.0$

IJNS homepage: http://www.nonlinearscience.org.uk/
Figure 6: Dependence of the synchronization time on the coupling parameter $\lambda$

References


