ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.30(2020) No.2/3,pp.87-96

Global Subsonic Flow Through a 2-D Infinitely Long Curved Nozzle

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(Received 19 February 2019, accepted 24 May 2019)

Abstract: In this paper, we are concerned with the existence and stability of a subsonic global solution in an infinitely long curved nozzle for the 2-D steady potential flow equation. By introducing some suitably weighted $H\ddot{o}lder$ spaces and establishing a series of a priori estimates on the solution to second order linear elliptic equation in an unbounded nozzle domain with two Neumann boundary conditions with respect to some variables, we show the global existence and stability of a potential flow equation in a 2-D nozzle when the state of subsonic flow at negative infinity is given.

Keywords: Subsonic flow; Potential flow equation; Weighted Hölder space; Global existence

1 Introduction and Main Results

Many authors have already studied the problem of the existence of global subsonic flows in infinitely long nozzles or past obstacles. When two dimensional flow past a profile, the global subsonic potential flow with small Mach number exists outside the profile, see [2]. Finn and Gilbarg [8] proved the uniqueness of subsonic flow past a profile. Regarding the 2-D subsonic potential flows in the infinitely long nozzles, authors in [6] showed that there exists a subsonic flow exists in the nozzle as long as the incoming mass flux is less than the critical value.

To describe the problem mathematically, we use the potential flow equation to describe the motion of the subsonic gas in a 2-D nozzle. Let $\varphi(x)$ be the potential of velocity $u = (u_1, u_2)$, ie., $u_i = \partial_i \varphi$, then it follows from the Bernoulli's law that

$$\frac{1}{2}| \triangle \varphi|^2 + h(\rho) = C_0, \tag{1}$$

where $\nabla = (\partial_1, \partial_2)$, $h(\rho) = \frac{c^2(\rho)}{\gamma-1}$ is the specific enthalpy for the polytropic gas with the state equation $P = A\rho^\gamma$ $(1 < \gamma < 3)$ and the sonic speed $c(\rho) = \sqrt{P'(\rho)}$, $C_0 = \frac{1}{2}q_0^2 + h(\rho_0)$ stands for the Bernoulli's constant, where the far velocity field $(q_0, 0; \rho_0)$ at minus infinity of the nozzle is subsonic, ie., $q_0 < c(\rho_0)$ holds true.

By using (1) and the implicit function theorem, the density function $\rho(x)$ of gas can be expressed as follows

$$\rho = h^{-1}(C_0 - \frac{1}{2}| \bigtriangledown \varphi|^2) \equiv H(\bigtriangledown\varphi). \tag{2}$$

Substituting (2) into the mass conservation equation $\sum_{j=1}^{2} \partial_j(\rho u_j) = 0$ of gas yields

$$((\partial_1\varphi)^2 - c^2)\partial_1^2\varphi + ((\partial_2\varphi)^2 - c^2)\partial_2^2\varphi + 2\partial_1\varphi\partial_2\varphi\partial_{12}^2\varphi = 0, \tag{3}$$

here $c = c(H(\bigtriangledown\varphi))$.

We assume two nozzle walls be $S_1 = \{x : x_2 = \varepsilon f_1(x_1), x_1 \in R\}$ and $S_2 = \{x : x_2 = 1 + \varepsilon f_2(x_1), x_1 \in R\}$, here $f_i(x_1) \in C_0^\infty(-X_0, X_0)$ $(i = 1, 2)$ for some fixed positive constant X_0, and $\varepsilon > 0$ is a suitably small constant. Then the nozzle domain Ω can be expressed as

$$\Omega = \{x : \varepsilon f_1(x_1) \le x_2 \le 1 + \varepsilon f_2(x_1), x_1 \in R\}$$

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IJNS.2020.12.15

For flows passing through a nozzle, when the nozzle walls are impermeable solid walls, the boundary conditions are given by

$$(\partial_1\varphi, \partial_2\varphi)\cdot(\varepsilon\partial_1 f_i, -1) = 0 \quad \text{on} \quad S_i, \quad i = 1,2. \tag{4}$$

In addition, we suppose that the state of subsonic flow at minus infinity satisfies

$$\lim_{x_1\to-\infty}(\varphi(x) - q_0 x_1) = 0. \tag{5}$$

On the other hand, from the physical point of view (see [2–5, 7, 8] and the references therein), when a subsonic flow in an unbounded domain is called to be stable, it should admit a determined state at infinity. Namely,

$$\lim_{x_1\to+\infty} \bigtriangledown\varphi(x) \quad \text{exists for} \quad x\in\Omega. \tag{6}$$

The main result of the subsonic flow problem in this 2-D infinitely long curved nozzle can be stated as follows.

Theorem 1 *If the 2-D infinitely long curved nozzle Ω is bounded by $S_1 = \{x : x_2 = \varepsilon f_1(x_1), x_1 \in R\}$ and $S_2 = \{x : x_2 = 1 + \varepsilon f_2(x_1), x_1 \in R\}$, here $f_i(x_1) \in C_0^\infty(-X_0, X_0)$ $(i = 1,2)$ for some fixed positive constant X_0, then there exists a small constant $\varepsilon_0 > 0$ such that the problem (3)-(6) has a global smooth solution $\varphi(x)$ as $\varepsilon < \varepsilon_0$, which admits*

(i) $|\bigtriangledown\varphi| < c(H(\bigtriangledown\varphi))$. Namely, the flow is globally subsonic in the whole domain Ω.

(ii) For $x_1 < 0$ and $x \in \Omega$, there exists a suitable constant $\delta_0 > 0$ and a constant $C_0 > 0$ such that

$$|\varphi(x) - q_0x_1| + |\bigtriangledown(\varphi(x) - q_0x_1)| \le C_0\varepsilon e^{-\delta_0|x_1|}.$$

(iii) For $x_1 > 0$ and $x\in\Omega$, there exist a constant $C_0 > 0$ such that

$$|\varphi(x) - q_0x_1| \le C_0\varepsilon(1 + x_1).$$

(iv) $\lim_{x_1\to+\infty, x\in\Omega}\bigtriangledown\varphi(x) = (q_0, 0)$ holds. Furthermore, for $x_1 > 0$ and $x\in\Omega$, there exists a constant $C_0 > 0$ such that

$$|\bigtriangledown_{x_2}\varphi(x)| \le C_0\varepsilon e^{-\delta_0 x_1},$$

here $\delta_0 > 0$ is given in (ii).

This paper is organized as follows. In Section 2, we reformulate the problem (3) with (4)-(6), and then give a more precise description on Theorem 1.1 in some suitably weighted $H\ddot{o}lder$ spaces. In Section 3, we will linearize the nonlinear problem (3) with (4)-(6). We can essentially obtain the Laplacian equation $\triangle u = \tilde{f}(z)$ in the nozzle domain $\tilde{\Omega} = \{(z_1, z_2) : -\infty < z_1 < +\infty, 0 < z_2 < 1\}$ with two Neumann boundary conditions on $z_2 = 0$ and $z_2 = 1$ together with $\lim_{z_1\to-\infty} u(z) = 0$ and the requirement on the existence of $\lim_{z_1\to+\infty}\bigtriangledown_z u(z)$ by such a linearization. By using the Sturm-Liouville theorem and separation variable method, we can derive the formal expression of $u(z)$ in $\tilde{\Omega}$. Subsequently, following from some detailed estimates, we can obtain the existence and regularity of $u(z)$ in $\tilde{\Omega}$. Based on the crucial estimates and properties given in Section 3. Moreover, we can complete the proof of Theorem 2.2 and further obtain the asymptotic behavior of $\bigtriangledown_x\varphi$ at negative and positive infinity in the nozzle domain Ω respectively.

2 The Reformulation on (3)-(6) and More Precise Descriptions on Theorem 1.1

In this section, we first introduce some notations and weighted $H\ddot{o}lder$ norms so that Theorem 1.1 can be given a more precise description.

Let $\Omega \in R^2$ be an open set including the origin $O = (0,0)$, if $u \in C^{m,\alpha}(\Omega)$ with $0 \le \alpha < 1$, then we define the following weighted $H\ddot{o}lder$ norms for $x, y \in \Omega$, some positive constant $\delta > 0$ and $m \in N\cup\{0\}$:

$$\begin{aligned}
[u]^{(\delta)}_{m,0;\Omega} &\equiv \sum_{|\beta|=m}\sup_{x\in\Omega} e^{\delta|x_1|}|D^\beta u(x)|;\\
[u]^{(\delta)}_{m,\alpha;\Omega} &\equiv \sum_{|\beta|=m}\sup_{x,y\in\Omega} e^{\delta d_{x,y}}\frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}, \quad \text{where} \quad d_{x,y} = min\{|x_1|, |y_1|\};\\
|u|^{(\delta)}_{m,\alpha;\Omega} &\equiv \sum_{0\le k\le m}[u]^{(\delta)}_{k,0;\Omega} + [u]^{(\delta)}_{m,\alpha;\Omega};
\end{aligned}$$

$$\begin{aligned}\|u\|_{m,\alpha;\Omega}^{(\delta)} \equiv & \sup_{x\in\Omega;x_1<0} e^{\delta|x_1|}|u(x)| + \sup_{x\in\Omega;x_1>0}(1+x_1)^{-1}|u(x)| \\ & + \sup_{x\in\Omega;x_1<0} e^{\delta|x_1|}|\partial_{x_1}u(x)| + \sup_{x\in\Omega;x_1>0}|\partial_{x_1}u(x)| \\ & + \sup_{x\in\Omega} e^{\delta|x_1|}|\partial_{x_2}u(x)| + \sum_{2\le k\le m}[u]_{k,0;\Omega}^{(\delta)} + [u]_{m,\alpha;\Omega}^{(\delta)},\end{aligned}$$

and the corresponding function spaces are defined as

$$\begin{aligned} H_{m,\alpha}^{(\delta)} &= \{u(x)\in C^{m,\alpha}(\Omega) : |u|_{m,\alpha}^{(\delta)} < +\infty\}, \\ \mathbf{H}_{m,\alpha}^{(\delta)} &= \{u(x)\in C^{m,\alpha}(\Omega) : \|u\|_{m,\alpha}^{(\delta)} < +\infty\}.\end{aligned}$$

By use of the weighted $H\ddot{o}lder$ norms introduced above, Theorem 1.1 can be stated more precisely as follows.

Theorem 2 *Under the assumptions of Theorem 1.1, in the domain* $\Omega = \{x : -\infty < x_1 < +\infty, \varepsilon f_1(x_1) < x_2 < 1+\varepsilon f_2(x)\}$*, problem* $(3)-(6)$ *has a unique solution* $\varphi(x)\in C^{4,\alpha}(\Omega)$ *(any fixed constant* $0<\alpha<1$*), which satisfies*

(i) $\|\varphi(x) - q_0x_1\|_{4,\alpha;\Omega}^{(\delta_0)} \le \tilde{C}\varepsilon$*, here* $\delta_0 > 0$ *is some suitable constant.*

(ii) $\lim_{x\in\Omega;x_1\to+\infty} \triangledown\varphi(x) = (q_0, 0)$.

In order to show Theorem 2.1, we intend to introduce the following transformation so that the domain Ω can be changed into a standard nozzle domain $\tilde{\Omega} \equiv \{z = (z_1, z_2) : -\infty < z_1 < +\infty, 0 < z_2 < 1\}$:

$$z_1 = x_1, \quad z_2 = \frac{x_2 - \varepsilon f_1(x_1)}{1+\varepsilon f_2(x_1) - \varepsilon f_1(x_1)}. \tag{7}$$

In this case, for the notational convenience, we still denote the solution by $\varphi(z)$ instead of $\varphi(x)$ under the transformation (7). Following from a direct computation, the problem $(3)-(6)$ can be changed into

$$\begin{cases} \sum_{i,j=1}^2 A_{ij}(z,\triangledown_z\varphi)\partial_{z_iz_j}^2 + B(z,\triangledown_z\varphi)\partial_{z_2}\varphi = 0 \quad \text{in} \quad \tilde{\Omega}, \\ b_{11}(z)\partial_{z_1}\varphi + \partial_{z_2}\varphi = 0 \quad \text{on} \quad z_2 = 0, \\ b_{21}(z)\partial_{z_1}\varphi + \partial_{z_2}\varphi = 0 \quad \text{on} \quad z_2 = 1, \\ \lim_{z_1\to-\infty}(\varphi(z) - q_0z_1) = 0, \\ \lim_{z\in\tilde{\Omega};z_1\to+\infty} \triangledown_z\varphi(z) \quad \text{exists}, \end{cases} \tag{8}$$

where

$$\begin{cases} A_{11}(z,\triangledown_z\varphi) = c^2(H(\triangledown_x\varphi)) - (\partial_{x_1}\varphi)^2, \\ A_{22}(z,\triangledown_z\varphi) = \sum_{i=1}^2(c^2(H(\triangledown_x\varphi)) - (\partial_{x_i}\varphi)^2)(\frac{\partial z_2}{\partial x_i})^2 - 2\partial_{x_1}\varphi\partial_{x_2}\varphi\frac{\partial^2 z_2}{\partial x_1\partial x_2}, \\ A_{12}(z,\triangledown_z\varphi) = A_{21}(z,\triangledown_z\varphi) = (c^2(H(\triangledown_x\varphi)) - (\partial_{x_1}\varphi)^2)\frac{\partial z_2}{\partial x_1} - \partial_{x_1}\varphi\partial_{x_2}\varphi\frac{\partial z_2}{\partial x_2}, \\ B(z,\triangledown_z\varphi) = \sum_{i=1}^2(c^2(H(\triangledown_x\varphi)) - (\partial_{x_i}\varphi)^2)\frac{\partial^2 z_2}{\partial x_i^2} - 2\partial_{x_1}\varphi\partial_{x_2}\varphi\frac{\partial^2 z_2}{\partial x_1\partial x_2}, \\ b_{ij} = \frac{\varepsilon\partial_{x_j}f_i}{\varepsilon\partial_{x_1}f_i\frac{\partial z_2}{\partial x_1} - \frac{\partial z_2}{\partial x_2}}, \quad i=1,2; \quad j=1 \end{cases}$$

with

$$\partial_{x_1}\varphi = \partial_{z_1}\varphi + \partial_{z_2}\varphi\frac{\partial z_2}{\partial x_1}, \quad \partial_{x_2}\varphi = \partial_{z_2}\varphi\frac{\partial z_2}{\partial x_2}.$$

By the transformation(7), together with the properties of $f_i(x_1)$ $(i=1,2)$ and the definition of the norm $\|.\|_{m,\alpha}^{(\delta)}$, in order to show Theorem 2.1, we only need to establish Theorem 2.2.

Theorem 3 *Under the assumptions of Theorem 1.1, problem (8) has a unique solution* $\varphi(z)\in C^{4,\alpha}(\tilde{\Omega})$ *which satisfies*

(i) $\|\varphi(z) - q_0z_1\|_{4,\alpha;\tilde{\Omega}}^{(\delta_0)} \le \tilde{C}\varepsilon$.

(ii) $\lim_{z\in\tilde{\Omega};z_1\to+\infty} \triangledown_z\varphi(z) = (q_0, 0)$.

In next sections, we will focus on the proof of Theorem 2.2.

3 Solvability and a priori Estimates for the Linearized Problem of (8)

In terms of the smallness of perturbed nozzle walls and by use of direct computations, the linearized problem of (8) can be essentially expressed as:

$$\begin{cases} \bar{L}(v)\dot{u} \equiv \sum\limits_{i,j=1}^{2} a_{ij}(\bigtriangledown_z v)\partial^2_{z_iz_j}\dot{u}, \equiv \sum\limits_{i=1}^{2}(c^2(H(\bigtriangledown_z v)) - (\partial_{z_i}v)^2)\partial^2_{z_i}\dot{u} - 2\partial_{z_1}v\partial_{z_2}v\partial^2_{z_1z_2}\dot{u} = \dot{f} \quad \text{in} \quad \tilde{\Omega}, \\ \partial_{z_2}\dot{u} = \dot{g_1} \quad \text{on} \quad z_2 = 0, \\ \partial_{z_2}\dot{u} = \dot{g_2} \quad \text{on} \quad z_2 = 1, \\ \lim_{z_1\to-\infty, z\in\tilde{\Omega}} \dot{u}(z) = 0, \\ \lim_{z_1\to+\infty, z\in\tilde{\Omega}} \bigtriangledown_z \dot{u}(z) \quad \text{exists}, \end{cases} \tag{9}$$

Where $v \in H^{(\delta_0)}_{4,\alpha}(\tilde{\Omega})$ with $\|v - q_0z_1\|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} < \varepsilon$ and $\delta_0(> 0)$ is a suitably fixed constant.

It is easy to verify that the coefficients of the problem (9) satisfy the following uniformly elliptic condition in $\tilde{\Omega}$:

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(z, \triangle_z v)\xi_i\xi_j \leq \Lambda|\xi|^2, \tag{10}$$

for all $\xi = (\xi_1, \xi_2) \in R^2$ and $z \in \tilde{\Omega}$, here λ and Λ are two appropriate constants.

Next, we study the solvability of problem (9) as well as the regularity and a priori estimates of solution $\dot{u}(z)$ to (9). To this end, we first study the Laplacian equation in R^2 with the following boundary conditions:

$$\begin{cases} L_0u \equiv \triangle u = \tilde{f} \quad \text{in} \quad \tilde{\Omega}, \\ \partial_{z_2}u = \tilde{g_1} \quad \text{on} \quad z_2 = 0, \\ \partial_{z_2}u = \tilde{g_2} \quad \text{on} \quad z_2 = 1, \\ \lim_{z_1\to-\infty} u(z) = 0, \\ \lim_{z_1\to+\infty} \bigtriangledown_z u(z) \quad \text{exists}. \end{cases} \tag{11}$$

where $\tilde{f}(z) \in H^{(\delta_0)}_{2,\alpha}(\tilde{\Omega})$ and $\tilde{g}_i(z_1) \in H^{(\delta_0)}_{3,\alpha}(\tilde{\Omega})$ (i=1,2).

We now give a lemma on the function $\tilde{f}(z)$ for the later uses.

Lemma 4 *For $\tilde{f} \in H^{(\delta_0)}_{2,\alpha}(\tilde{\Omega})$, if we set*

$$f_m(z_1) = 2\int_0^1 \tilde{f}(z)cos(m\pi z_2)dz_2,$$

for $m \in N$, then

$$|f_m(z_1)| \leq \frac{C}{m^2}|\tilde{f}|^{(\delta_0)}_{2,\alpha;\tilde{\Omega}}e^{-\delta_0|z_1|},$$

Proof. Integrating by parts, we arrive at

$$\begin{aligned} f_m(z_1) &= 2(\frac{1}{m\pi}\tilde{f}sin(m\pi z_2)|^{z_2=1}_{z_2=0} - \frac{1}{m\pi}\int_0^1 \partial_{z_2}\tilde{f}sin(m\pi z_2)dz_2) \\ &= 2(\frac{1}{m^2\pi^2}\partial_{z_2}\tilde{f}cos(m\pi z_2)|^{z_2=1}_{z_2=0} - \frac{1}{m^2\pi^2}\int_0^1 \partial^2_{z_2}\tilde{f}cos(m\pi z_2)dz_2). \end{aligned}$$

Because $\tilde{f} \in H^{(\delta_0)}_{2,\alpha}(\tilde{\Omega})$, we have $|D^\beta\tilde{f}(z)| \leq |\tilde{f}|^{(\delta_0)}_{2,\alpha;\tilde{\Omega}}e^{-\delta_0|z_1|}$ for $|\beta| \leq 2$. From this, we can derive that

$$|f_m(z_1)| \leq \frac{C}{m^2}|\tilde{f}|^{(\delta_0)}_{2,\alpha;\tilde{\Omega}}e^{-\delta_0|z_1|}.$$

Lemma 5 *If $\tilde{f} \in H^{(\delta_0)}_{2,\alpha}(\tilde{\Omega})$ and $\tilde{g}_i \in H^{(\delta_0)}_{3,\alpha}(\tilde{\Omega})$ (i=1.2) with $0 < \delta_0 < \pi$, then the equation(11) has a solution $u \in C(\bar{\tilde{\Omega}})$, which satisfies the estimate*

$$\|u\|^{(\delta_0)}_{0,0;\tilde{\Omega}} \le C(|\tilde{f}|^{(\delta_0)}_{2,\alpha;\tilde{\Omega}} + \sum_{i=1}^{2} |\tilde{g}_i|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}}). \tag{12}$$

Proof. We intend to use the method of separation variables to study the solvability and regularities of solution u to (11). To this end, we firstly focus on its corresponding homogeneous problem.

Let us consider the nontrivial solutions of the problem

$$\begin{cases} \triangle u = 0 & \text{in} \quad \Omega, \\ \partial_{z_2} u = 0 & \text{on} \quad z_2 = 0, \\ \partial_{z_2} u = 0 & \text{on} \quad z_2 = 1. \end{cases} \tag{13}$$

Set $u(z) = X(z_1)Y(z_2)$, then from (13) it follows that

$$\begin{cases} Y''(z_2) + \lambda Y(z_2) = 0, \\ Y'(0) = 0, \quad Y'(1) = 0, \end{cases} \tag{14}$$

and

$$X''(z_1) - \lambda X(z_1) = 0, \tag{15}$$

here $\lambda \in R$.

By a simple computation, we can show that the eigenvalues of (14) are $\lambda_m = (m\pi)^2$ $(m = 0, 1, 2, ...)$, and the corresponding eigenfunctions are $cos(m\pi z_2)$.

Set $h(z) = \frac{1}{2}(\tilde{g}_2(z_1) - \tilde{g}_1(z_1))z_2^2 + \tilde{g}_1(z_1)z_2$ and $v(z) = u(z) - h(z)$, then it follows from (11) that $v(z)$ satisfies

$$\begin{cases} \triangle v = \tilde{f} - \triangle h \equiv f \quad \text{in} \quad \tilde{\Omega}, \\ \partial_{z_2} v = 0 \quad \text{on} \quad z_2 = 0, \\ \partial_{z_2} v = 0 \quad \text{on} \quad z_2 = 1, \\ \lim_{z_1 \to -\infty} v(z) = 0, \\ \lim_{z_1 \to +\infty} \bigtriangledown_z v(z) \quad \text{exists}. \end{cases} \tag{16}$$

Let

$$v(z) = X_0(z_1) + \sum_{m=1}^{\infty} X_m(z_1) cos(m\pi z_2) \tag{17}$$

and

$$f(z) = f_0(z_1) + \sum_{m=1}^{\infty} f_m(z_1) cos(m\pi z_2),$$

where

$$f_0(z_1) = \int_0^1 f(z) dz_2,$$
$$f_m(z_1) = 2\int_0^1 f(z) cos(m\pi z_2) dz_2.$$

Next, we determine the terms $X_0(z_1)$ and $X_m(z_1)$ in (17). Following from (16) and (17), we can formally obtain

$$\begin{aligned} &{X_0}''(z_1) = f_0(z_1), \\ &\lim_{z_1 \to -\infty} X_0(z_1) = 0, \\ &\lim_{z_1 \to +\infty} {X_0}'(z_1) \quad \text{exists}, \end{aligned} \tag{18}$$

$$\begin{cases} {X_m}''(z_1) - m^2\pi^2 X_m(z_1) = f_m(z_1), \\ \lim_{z_1 \to -\infty} X_m(z_1) = 0, \\ \lim_{z_1 \to +\infty} {X_m}'(z_1) \quad \text{exists.} \end{cases} \tag{19}$$

Solving these ordinary differential equations directly yield

$$X_0(z_1) = \int_{-\infty}^{z_1} \int_{-\infty}^{t} f_0(\xi) d\xi dt, \tag{20}$$

$$X_m(z_1) = e^{m\pi z_1} \int_{+\infty}^{z_1} e^{-2m\pi t} \int_{-\infty}^{t} e^{m\pi\xi} f_m(\xi) d\xi dt, \quad m \geq 1, \tag{21}$$

We now analyze the expressions in (20)-(21). This will be divided into two parts.

Part1. Estimate of $X_0(z_1)$. By using the expression of $X_0(z_1)$ in (20) and integrating by parts, one has

$$X_0(z_1) = t \int_{-\infty}^{t} f_0(\xi) d\xi |_{-\infty}^{z_1} - \int_{-\infty}^{z_1} t f_0(t) dt. \tag{22}$$

By $f(z) \in H_{1,\alpha}^{(\delta_0)}(\tilde{\Omega})$, we have

$$|f_0(z_1)| \leq |f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)} e^{-\delta_0 |z_1|}. \tag{23}$$

Thus

$$\lim_{t \to -\infty} t \int_{-\infty}^{t} f_0(\xi) d\xi = 0.$$

This and (22), yield

$$X_0(z_1) = z_1 \int_{-\infty}^{z_1} f_0(t) dt - \int_{-\infty}^{z_1} t f_0(t) dt. \tag{24}$$

For $z_1 < 0$, it follows from (20) and (23) that

$$|X_0(z_1)| \leq \int_{-\infty}^{z_1} \int_{-\infty}^{t} |f_0(\xi)| d\xi dt \leq |f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)} \int_{-\infty}^{z_1} \int_{-\infty}^{t} e^{\delta_0 \xi} d\xi dt \leq \frac{1}{\delta_0^2} |f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)} e^{\delta_0 z_1}.$$

For $z_1 > 0$, by using (23)-(24), we have

$$\begin{aligned} |X_0(z_1)| &\leq z_1 \int_{-\infty}^{z_1} |f_0(t)| dt + \int_{-\infty}^{z_1} |t f_0(t)| dt \\ &\leq |f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)} (z_1 \int_{-\infty}^{0} e^{\delta_0 t} dt + z_1 \int_{0}^{z_1} e^{-\delta_0 t} dt - \int_{-\infty}^{0} t e^{\delta_0 t} dt + \int_{0}^{z_1} t e^{-\delta_0 t} dt) \\ &\leq C(1 + z_1) |f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}. \end{aligned}$$

This means

$$\|X_0(z_1)\|_{0,0}^{(\delta_0)} \leq C |f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}. \tag{25}$$

Next we estimate ${X_0}'(z_1)$. Note that

$${X_0}'(z_1) = \int_{-\infty}^{z_1} f_0(t) dt.$$

If $z_1 < 0$, then one has

$$|{X_0}'(z_1)| \leq |f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)} \int_{-\infty}^{z_1} e^{\delta_0 t} dt \leq \frac{1}{\delta_0} |f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)} e^{\delta_0 z_1}.$$

If $z_1 > 0$, then

$$|{X_0}'(z_1)| \leq |f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)} (\int_{-\infty}^{0} e^{\delta_0 t} dt + \int_{0}^{z_1} e^{-\delta_0 t} dt) \leq \frac{2}{\delta_0} |f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}. \tag{26}$$

Thus, we arrive at

$$\|{X_0}'(z_1)\|_{0,0}^{(\delta_0)} \leq C |f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}.$$

and

$$\lim_{z_1\to+\infty} {X_0}'(z_1) = \int_{-\infty}^{+\infty}\int_0^1 f(t,z_2)dz_2dt. \tag{27}$$

Part2. Estimate of $X_m(z_1)$ with $m \geq 1$ by (21) and integration by parts, we have

$$\begin{aligned} X_m(z_1) &= -\frac{1}{2m\pi}e^{m\pi z_1}(e^{-2m\pi t}\int_{-\infty}^{t} e^{m\pi\xi}f_m(\xi)d\xi|_{+\infty}^{z_1} - \int_{+\infty}^{z_1} e^{-m\pi t}f_m(t)dt) \\ &= -\frac{1}{2m\pi}(e^{-m\pi z_1}\int_{-\infty}^{z_1} e^{m\pi t}f_m(t)dt + e^{m\pi z_1}\int_{z_1}^{+\infty} e^{-m\pi t}f_m(t)dt). \end{aligned} \tag{28}$$

where we have used that

$$\lim_{t\to+\infty}\frac{\int_{-\infty}^{t} e^{m\pi\xi}f_m(\xi)d\xi}{e^{2m\pi t}} = \lim_{t\to+\infty}\frac{f_m(t)}{2m\pi e^{m\pi t}} = 0.$$

It is noted that by Lemma 3.1 and the proof of Lemma 3.1, we have

(i) If $z_1 < 0$, then

$$|e^{-m\pi z_1}\int_{-\infty}^{z_1} e^{m\pi t}f_m(t)dt| \leq \frac{C}{m^2}|f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}e^{-m\pi z_1}\int_{-\infty}^{z_1} e^{(m\pi+\delta_0)t}dt \leq \frac{C}{m^2(m\pi+\delta_0)}|f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}e^{\delta_0 z_1}. \tag{29}$$

and

$$\begin{aligned} |e^{m\pi z_1}\int_{z_1}^{+\infty} e^{-m\pi t}f_m(t)dt| &\leq \frac{C}{m^2}|f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}e^{m\pi z_1}(\int_{z_1}^{0} e^{(\delta_0-m\pi)t}dt + \int_0^{+\infty} e^{-(m\pi+\delta_0)t}dt) \\ &\leq \frac{C}{m^2(m\pi-\delta_0)}|f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}e^{\delta_0 z_1}. \end{aligned} \tag{30}$$

(ii)If $z_1 > 0$, then

$$\begin{aligned} |e^{-m\pi z_1}\int_{-\infty}^{z_1} e^{m\pi t}f_m(t)dt| &\leq \frac{C}{m^2}|f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}e^{-m\pi z_1}(\int_{-\infty}^{0} e^{(m\pi+\delta_0)t}dt + \int_0^{z_1} e^{(m\pi-\delta_0)t}dt) \\ &\leq \frac{C}{m^2(m\pi-\delta_0)}|f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}e^{-\delta_0 z_1}. \end{aligned} \tag{31}$$

and

$$\begin{aligned} |e^{m\pi z_1}\int_{z_1}^{+\infty} e^{-m\pi t}f_m(t)dt| &\leq \frac{C}{m^2}|f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}e^{m\pi z_1}\int_{z_1}^{+\infty} e^{-(m\pi+\delta_0)t}dt \\ &\leq \frac{C}{m^2(m\pi+\delta_0)}|f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}e^{-\delta_0 z_1}. \end{aligned} \tag{32}$$

Substituting (29)-(30) and (31)-(32) in (28) yields

$$|X_m(z_1)| \leq \frac{C}{m^3(m\pi-\delta_0)}|f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}e^{-\delta_0|z_1|}.$$

Namely,

$$|X_m(z_1)|_{0,0}^{(\delta_0)} \leq \frac{C}{m^3(m\pi-\delta_0)}|f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}. \tag{33}$$

Next, we estimate ${X_m}'(z_1)$, Since

$${X_m}'(z_1) = \frac{1}{2}(e^{-m\pi z_1}\int_{-\infty}^{z_1} e^{m\pi t}f_m(t)dt - e^{m\pi z_1}\int_{z_1}^{+\infty} e^{-m\pi t}f_m(t)dt),$$

by using (29)-(30) and (31)-(32), we have

$$|{X_m}'(z_1)| \leq \frac{C}{m^2(m\pi-\delta_0)}|f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}e^{-\delta_0|z_1|}.$$

This means

$$|X_m{}'(z_1)|_{0,0}^{(\delta_0)} \leq \frac{C}{m^2(m\pi-\delta_0)}|f|_{1,\alpha;\tilde{\Omega}}^{(\delta_0)}, \tag{34}$$

$$\lim_{z_1\to+\infty} X_m{}'(z_1)=0. \tag{35}$$

Combining (25)-(27) and (33)-(35) yield (12).

On the other hand, it follows from (27) and (35) that

$$\lim_{z_1\to+\infty}\partial_{z_1}u(z)=\int_{-\infty}^{+\infty}\int_0^1 \tilde{f}(t,z_2)dz_2dt-\int_{-\infty}^{+\infty}(\tilde{g_2}(z_1)-\tilde{g_1}(z_1))dz_1, \tag{36}$$

$$\lim_{z_1\to+\infty}\partial_{z_2}u(z)=0. \tag{37}$$

The proof is complete.

Based on Lemma 3.2, we now give the estimate of $\|u\|_{4,\alpha;\tilde{\Omega}}^{(\delta_0)}$, for the solution of (11).

Corollary 6 *Under the assumptions of Lemma 3.2, the solution $u(z)$ of (11) satisfies the estimate*

$$\|u\|_{4,\alpha;\tilde{\Omega}}^{(\delta_0)} \leq C(|\tilde{f}|_{2,\alpha;\tilde{\Omega}}^{(\delta_0)}+\sum_{i=1}^{2}|\tilde{g_i}|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)}). \tag{38}$$

Proof. To prove (38), by use of Lemma 3.2, it only suffices to prove

$$\begin{aligned}|\partial_{z_2}u|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)} &\leq C(|\tilde{f}|_{2,\alpha;\tilde{\Omega}}^{(\delta_0)}+\sum_{i=1}^{2}|\tilde{g_i}|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)}),\\ |\partial_{z_1}^2u|_{2,\alpha;\tilde{\Omega}}^{(\delta_0)} &\leq C(|\tilde{f}|_{2,\alpha;\tilde{\Omega}}^{(\delta_0)}+\sum_{i=1}^{2}|\tilde{g_i}|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)}).\end{aligned} \tag{39}$$

Since these properties can be directly verified by using the coordinate transformation and introducing weighted $H\ddot{o}lder$ norms we used before,then we omit them.

Based on Corollary 3.3, we now derive uniform estimates on the solution $\dot{u}(z)$ to problem (3,1).

Lemma 7 *Suppose that the assumption (10) holds, and $\dot{u}\in C(\bar{\tilde{\Omega}})$ is a solution of (9). Then there exists a positive constant δ_0 such that for any $\dot{f}\in H_{2,\alpha}^{(\delta_0)}(\tilde{\Omega})$, $\tilde{g}_i\in H_{3,\alpha}^{(\delta_0)}(\tilde{\Omega})$ $(i=1,2)$, we have $\dot{u}\in \mathbf{H}_{4,\alpha}^{(\delta_0)}(\tilde{\Omega})$ with*

$$\|\dot{u}\|_{4,\alpha;\tilde{\Omega}}^{(\delta_0)} \leq C(|\dot{f}|_{2,\alpha;\tilde{\Omega}}^{(\delta_0)}+\sum_{i=1}^{2}|\dot{g}_i|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)}), \tag{40}$$

where $C>0$ depends only on the constants Λ and λ in (10).

Proof. Firstly, we introduce the coordinate transformation

$$\begin{aligned}\tilde{z}_1&=k_1z_1,\\ \tilde{z}_2&=k_2z_2\end{aligned} \tag{41}$$

with $k_1=\frac{1}{\sqrt{c^2(p_0)-q_0^2}}$ and $k_2=\frac{1}{c(p_0)}$. Under this transformation, the domain $\tilde{\Omega}$ is changed into the domain $Q\equiv(-\infty,+\infty)\times[0,\frac{1}{c(p_0)}]$, and the equation (9) can be rewritten as

$$\begin{cases}\triangle\dot{u}=\bar{f} \quad \text{in} \quad Q,\\ \partial_{\tilde{z_2}}\dot{u}=\tilde{g}_1 \quad \text{on} \quad \tilde{z_2}=0,\\ \partial_{\tilde{z_2}}\dot{u}=\tilde{g}_2 \quad \text{on} \quad \tilde{z_2}=l,\\ \lim_{\tilde{z_1}\to-\infty}\dot{u}=0,\\ \lim_{\tilde{z_1}\to+\infty}\bigtriangledown\tilde{z} \qquad\qquad \dot{u} \quad \text{exists.}\end{cases} \tag{42}$$

where $l = \frac{1}{c(p_0)}$, and

$$\bar{f} = \dot{f} + \sum_{i=1}^{2}(1 - k_i^2(c^2(\nabla v) - \partial_{z_1}^2 v))\partial_{\tilde{z}_i}^2 \dot{u} - 2k_1 k_2 \partial_{z_1} v \partial_{z_2} v \partial_{\tilde{z}_1 \tilde{z}_2} \dot{u}, \quad \tilde{g}_i = c(p_0)\dot{g}_i, \quad i = 1, 2. \tag{43}$$

For simplicity and without loss of generality, we assume $l = 1$ in (42). By the assumption $\|v - q_0 z_1\|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} < \varepsilon$, we have

$$|\bar{f}|^{(\delta_0)}_{2,\alpha;\tilde{\Omega}} \leq O(\varepsilon)\|\dot{u}\|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} + |\dot{f}|^{(\delta_0)}_{2,\alpha;\tilde{\Omega}}. \tag{44}$$

On the other hand, by using Corollary 3.3, one has

$$\|\dot{u}\|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} \leq C(|\bar{f}|^{(\delta_0)}_{2,\alpha;\tilde{\Omega}} + \sum_{i=1}^{2}|\dot{g}_i|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}}). \tag{45}$$

Substituting (43) into (44) yields (40).

Moreover, from (36) and (37), it follows that

$$\lim_{z_1 \to +\infty} |\partial_{z_1} \dot{u}| \leq C(|\dot{f}|^{(\delta_0)}_{2,\alpha;\tilde{\Omega}} + \sum_{i=1}^{2}|\dot{g}_i|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}}), \qquad \lim_{z_1 \to +\infty} \partial_{z_2} \dot{u} = 0. \tag{46}$$

Therefore, the proof is complete.

Based on Lemmas 3.2 and 3.4, from the standard continuity method (see [12]) we have the following result.

Theorem 8 *There exists a unique solution* $\dot{u} \in \mathbf{H}^{(\delta_0)}_{4,\alpha}(\tilde{\Omega})$ *to problem (9) for some* $\delta_0 > 0$*,which admits the following estimate*

$$\|\dot{u}\|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} \leq C(|\dot{f}|^{(\delta_0)}_{2,\alpha;\tilde{\Omega}} + \sum_{i=1}^{2}|\dot{g}_i|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}}). \tag{47}$$

Proof of Theorem 2.2. Then by standard nonlinear iterative, applying the contraction mapping principle, problem (8) has a unique solution in $\mathbf{H}^{(\delta_0)}_{4,\alpha}(\tilde{\Omega})$.

Next, we show $\lim_{z_1 \to +\infty} \nabla_z \varphi(z)$ exists. Since for $Z_1 > Z_2 > 0$, we have

$$\begin{aligned} |\partial_{z_1}\varphi(Z_1, z_2) - \partial_{z_1}\varphi(Z_2, z_2)| &= (Z_1 - Z_2)|\int_0^1 \partial_{z_1}^2 \varphi(\theta Z_1 + (1-\theta)Z_2, z_2)d\theta| \\ &\leq C(Z_1 - Z_2)|\int_0^1 e^{-\delta_0(\theta Z_1 + (1-\theta)Z_2)}d\theta| \leq Ce^{-\delta_0 Z_2}. \end{aligned}$$

This means that there exists a function $q(z_2)$ such that $\partial_{z_1}\varphi(z_1, z_2)$ converges to $q(z_2)$ uniformly as $z_1 \to +\infty$. On the other hand, $|\partial^2_{z_1 z_2}\varphi(z_1, z_2)| \leq Ce^{-\delta_0 z_1}$, this implies that $\partial^2_{z_1 z_2}\varphi(z_1, z_2)$ converges to 0 uniformly as $z_1 \to +\infty$. Therefore, we can arrive at $q'(z_2) \equiv 0$, namely, $q(z_2) \equiv q$, here q is a constant which will be determined later on. In addition, $|\partial_{z_2}\varphi(z)| \leq Ce^{-\delta_0|z_1|}$, then $\lim_{z_1 \to \pm\infty} \partial_{z_2}\varphi(z) = 0$. From the analysis above, we can also obtain under the $x-$ coordinates,

$$\lim_{x_1 \to \infty} \partial_{x_1}\varphi = q, \lim_{x_1 \to \pm\infty} \partial_{x_2}\varphi = 0. \tag{48}$$

We now show that $q = q_0$ holds. Integrating the mass conservation equation $\sum_{j=1}^{2} \partial_{x_j}(\rho(|\nabla\varphi|)\partial_{x_j}\varphi) = 0$ in $\Omega_R = \Omega \cap \{x : -R < x_1 < R\}$ yields

$$0 = -\int_{x_1=-R} \rho(\nabla\varphi)\partial_{x_1}\varphi d\sigma + \int_{x_1=R} \rho(\nabla\varphi)\partial_{x_1}\varphi d\sigma. \tag{49}$$

Using (48) and letting $R \to +\infty$ in (49), we arrive at

$$\rho(q)q = \rho(q_0)q_0. \tag{50}$$

In addition, it is easy to verify that $\rho(q)q$ is an increasing function of q, for $q < c(\rho_0)$, then we derive $q = q_0$. From the analysis above, we complete the proof.

Since the proofs of Theorem 1.1 and 2.1 come directly from Theorem 2.2, then we omit them.

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