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Generalized Order (α, β) Based Some Growth Analysis of Composite Analytic Functions in the Unit Disc

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Abstract: In this paper we introduce the idea of generalized order (α, β) and generalized lower order (α, β) of an analytic function in the unit disc. Hence we study some growth properties relating to the composition of two analytic function in the unit disc on the basis of generalized order (α, β) and generalized lower order (α, β) as compared to the growth of their corresponding left and right factors.

Keywords: Growth; analytic function; composition; unit disc; generalized order (α, β); generalized lower order (α, β).

1 Introduction, Definitions and Notations

Let $f(z) = {}_{n=0}^{\infty} c_n z^n$ be analytic in the unit disc $U = \{z : |z| < 1\}$ and $M_f(r)$ be the maximum of $|f(z)|$ on $|z| = r$. In [4], Sons was define the order $\rho(f)$ and the lower order $\lambda(f)$ as

$$\begin{matrix} \rho(f) \\ \lambda(f) \end{matrix} = \lim_{r\to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[2]} M_f(r)}{-\log(1-r)}.$$

Now let L be a class of continuous non-negative on $(-\infty, +\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. Further we assume that throughout the present paper $\alpha, \alpha_1, \alpha_2, \alpha_3, \beta \in L$. Now considering this, we introduce the definition of the generalized order (α, β) and generalized lower order (α, β) of an analytic function f in the unit disc U which are as follows:

Definition 1 *The generalized order (α, β) denoted by $\rho^{(\alpha,\beta)}[f]$ and generalized lower order (α, β) denoted by $\lambda^{(\alpha,\beta)}[f]$ of an analytic function f in the unit disc U are defined as:*

$$\begin{matrix} \rho^{(\alpha,\beta)}[f] \\ \lambda^{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(M_f(r))}{\beta\left(\frac{1}{1-r}\right)}.$$

Clearly $\rho^{(\log\log r, \log r)}[f] = \rho(f)$ and $\lambda^{(\log\log r, \log r)}[f] = \lambda(f)$.

Now one may give the definitions of generalized hyper order (α, β) and generalized logarithmic order (α, β) of an analytic function f in the unit disc U in the following way:

Definition 2 *The generalized hyper order (α, β) denoted by $\overline{\rho}^{(\alpha,\beta)}[f]$ and generalized hyper lower order (α, β) denoted by $\overline{\lambda}^{(\alpha,\beta)}[f]$ of an analytic function f in the unit disc U are defined as:*

$$\begin{matrix} \overline{\rho}^{(\alpha,\beta)}[f] \\ \overline{\lambda}^{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M_f(r))}{\beta\left(\frac{1}{1-r}\right)}.$$

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Definition 3 *The generalized logarithmic order* (α,β) *denoted by* $\rho_{\log}^{(\alpha,\beta)}[f]$ *and generalized logarithmic lower order* (α,β) *denoted by* $\lambda_{\log}^{(\alpha,\beta)}[f]$ *of an analytic function* f *in the unit disc* U *are defined as:*

$$\begin{matrix}\rho_{\log}^{(\alpha,\beta)}[f] \\ \lambda_{\log}^{(\alpha,\beta)}[f]\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(M_f(r))}{\beta\left(\log\left(\frac{1}{1-r}\right)\right)}.$$

In this paper we study some growth properties relating to the composition of two analytic function of in the unit disc on the basis of generalized order (α,β), generalized hyper order (α,β) and generalized logarithmic order (α,β) as compared to the growth of their corresponding left and right factors. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [1], [2] and [3].

2 Theorems

In this section we present the main results of the paper.

Theorem 1 *Let* f *and* g *be any two non-constant analytic functions in* U *such that* $0 < \lambda^{(\alpha_1,\beta)}[f\circ g] \le \rho^{(\alpha_1,\beta)}[f\circ g] < \infty$ *and* $0 < \lambda^{(\alpha_2,\beta)}[f] \le \rho^{(\alpha_2,\beta)}[f] < \infty$. *Then*

$$\frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_2,\beta)}[f]} \le \liminf_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \le \frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_2,\beta)}[f]}$$
$$\le \limsup_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \le \frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_2,\beta)}[f]}.$$

Proof. From the definitions of $\rho^{(\alpha_2,\beta)}[f]$ and $\lambda^{(\alpha_1,\beta)}[f\circ g]$, we have for arbitrary positive ε and for all sufficiently large values of $\frac{1}{1-r}$ that

$$\alpha_1(M_{f\circ g}(r)) \geqslant \left(\lambda^{(\alpha_1,\beta)}[f\circ g]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right) \tag{1}$$

and

$$\alpha_2(M_f(r)) \le \left(\rho^{(\alpha_2,\beta)}[f]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right). \tag{2}$$

Now from (1) and (2) it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \geqslant \frac{\left(\lambda^{(\alpha_1,\beta)}[f\circ g]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}{\left(\rho^{(\alpha_2,\beta)}[f]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \geqslant \frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_2,\beta)}[f]}. \tag{3}$$

Again for a sequence of values of $\frac{1}{1-r}$ tending to infinity,

$$\alpha_1(M_{f\circ g}(r)) \le \left(\lambda^{(\alpha_1,\beta)}[f\circ g]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right) \tag{4}$$

and for all sufficiently large values of $\frac{1}{1-r}$,

$$\alpha_2(M_f(r)) \geqslant \left(\lambda^{(\alpha_2,\beta)}[f]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right). \tag{5}$$

Combining (4) and (5) we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \leq \frac{\left(\lambda^{(\alpha_1,\beta)}[f\circ g]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}{\left(\lambda^{(\alpha_2,\beta)}[f]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \leq \frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_2,\beta)}[f]}. \tag{6}$$

Also for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\alpha_2(M_f(r)) \leq \left(\lambda^{(\alpha_2,\beta)}[f]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right). \tag{7}$$

Now from (1) and (7) we obtain for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \geq \frac{\left(\lambda^{(\alpha_1,\beta)}[f\circ g]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}{\left(\lambda^{(\alpha_2,\beta)}[f]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \geq \frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_2,\beta)}[f]}. \tag{8}$$

Also for all sufficiently large values of $\frac{1}{1-r}$,

$$\alpha_1(M_{f\circ g}(r)) \leq \left(\rho^{(\alpha_1,\beta)}[f\circ g]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right). \tag{9}$$

Now, it follows from (5) and (9) , for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \leq \frac{\left(\rho^{(\alpha_1,\beta)}[f\circ g]+\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}{\left(\lambda^{(\alpha_2,\beta)}[f]-\varepsilon\right)\beta\left(\frac{1}{1-r}\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \leq \frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_2,\beta)}[f]}. \tag{10}$$

Thus the theorem follows from (3) , (6) , (8) and (10) . ■

The following theorem can be proved in the line of Theorem 1 and so the proof is omitted.

Theorem 2 *Let f and g be any two non-constant analytic functions in U such that* $0<\lambda^{(\alpha_1,\beta)}[f\circ g]\leq\rho^{(\alpha_1,\beta)}[f\circ g]<\infty$ *and* $0<\lambda^{(\alpha_3,\beta)}[g]\leq\rho^{(\alpha_3,\beta)}[g]<\infty$. *Then*

$$\frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_3,\beta)}[g]} \leq \liminf_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_3(M_g(r))} \leq \frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_3,\beta)}[g]}$$
$$\leq \limsup_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_3(M_g(r))} \leq \frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_3,\beta)}[g]}.$$

Theorem 3 *Let f and g be any two non-constant analytic functions in U such that $0 < \rho^{(\alpha_1,\beta)}[f \circ g] < \infty$ and $0 < \rho^{(\alpha_2,\beta)}[f] < \infty$. Then*

$$\liminf_{r\to 1} \frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \leq \frac{\rho^{(\alpha_1,\beta)}[f \circ g]}{\rho^{(\alpha_2,\beta)}[f]} \leq \limsup_{r\to 1} \frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))}.$$

Proof. From the definition of $\rho^{(\alpha_2,\beta)}[f]$, we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\alpha_2(M_f(r)) \geqslant \left(\rho^{(\alpha_2,\beta)}[f] - \varepsilon\right) \beta\left(\frac{1}{1-r}\right). \tag{11}$$

Now from (9) and (11) , it follows for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \leq \frac{\left(\rho^{(\alpha_1,\beta)}[f \circ g] + \varepsilon\right) \beta\left(\frac{1}{1-r}\right)}{\left(\rho^{(\alpha_2,\beta)}[f] - \varepsilon\right) \beta\left(\frac{1}{1-r}\right)}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1} \frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \leq \frac{\rho^{(\alpha_1,\beta)}[f \circ g]}{\rho^{(\alpha_2,\beta)}[f]}. \tag{12}$$

Again for a sequence of values of $\frac{1}{1-r}$ tending to infinity,

$$\alpha_1(M_{f\circ g}(r)) \geqslant \left(\rho^{(\alpha_1,\beta)}[f \circ g] - \varepsilon\right) \beta\left(\frac{1}{1-r}\right). \tag{13}$$

So combining (2) and (13) , we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \geqslant \frac{\left(\rho^{(\alpha_1,\beta)}[f \circ g] - \varepsilon\right) \beta\left(\frac{1}{1-r}\right)}{\left(\rho^{(\alpha_2,\beta)}[f] + \varepsilon\right) \beta\left(\frac{1}{1-r}\right)}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\limsup_{r\to 1} \frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \geqslant \frac{\rho^{(\alpha_1,\beta)}[f \circ g]}{\rho^{(\alpha_2,\beta)}[f]}. \tag{14}$$

Thus the theorem follows from (12) and (14) . ∎

The following theorem can be carried out in the line of Theorem 3 and therefore we omit its proof.

Theorem 4 *Let f and g be any two non-constant analytic functions in U such that $0 < \rho^{(\alpha_1,\beta)}[f \circ g] < \infty$ and $0 < \rho^{(\alpha_3,\beta)}[g] < \infty$. Then*

$$\liminf_{r\to 1} \frac{\alpha_1(M_{f\circ g}(r))}{\alpha_3(M_g(r))} \leq \frac{\rho^{(\alpha_1,\beta)}[f \circ g]}{\rho^{(\alpha_3,\beta)}[g]} \leq \limsup_{r\to 1} \frac{\alpha_1(M_{f\circ g}(r))}{\alpha_3(M_g(r))}.$$

The following theorem is a natural consequence of Theorem 1 and Theorem 3.

Theorem 5 *Let f and g be any two non-constant analytic functions in U such that $0 < \lambda^{(\alpha_1,\beta)}[f\circ g] \leq \rho^{(\alpha_1,\beta)}[f\circ g] < \infty$ and $0 < \lambda^{(\alpha_2,\beta)}[f] \leq \rho^{(\alpha_2,\beta)}[f] < \infty$. Then*

$$\liminf_{r\to 1} \frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \leq \min\left\{\frac{\lambda^{(\alpha_1,\beta)}[f \circ g]}{\lambda^{(\alpha_2,\beta)}[f]}, \frac{\rho^{(\alpha_1,\beta)}[f \circ g]}{\rho^{(\alpha_2,\beta)}[f]}\right\}$$
$$\leq \max\left\{\frac{\lambda^{(\alpha_1,\beta)}[f \circ g]}{\lambda^{(\alpha_2,\beta)}[f]}, \frac{\rho^{(\alpha_1,\beta)}[f \circ g]}{\rho^{(\alpha_2,\beta)}[f]}\right\} \leq \limsup_{r\to 1} \frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))}.$$

The proof is omitted.

Analogously one may state the following theorem without its proof.

Theorem 6 *Let f and g be any two non-constant analytic functions in U such that* $0 < \lambda^{(\alpha_1,\beta)}[f\circ g] \leq \rho^{(\alpha_1,\beta)}[f\circ g] < \infty$ *and* $0 < \lambda^{(\alpha_3,\beta)}[g] \leq \rho^{(\alpha_3,\beta)}[g] < \infty$. *Then*

$$\liminf_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_3(M_g(r))} \leq \min\left\{\frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_3,\beta)}[g]}, \frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_3,\beta)}[g]}\right\} \leq \max\left\{\frac{\lambda^{(\alpha_1,\beta)}[f\circ g]}{\lambda^{(\alpha_3,\beta)}[g]}, \frac{\rho^{(\alpha_1,\beta)}[f\circ g]}{\rho^{(\alpha_3,\beta)}[g]}\right\} \leq \limsup_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_3(M_g(r))}.$$

We may now state the following two theorems without proof based on Definition 2.

Theorem 7 *Let f and g be any two non-constant analytic functions in U such that* $0 < \overline{\lambda}^{(\alpha_1,\beta)}[f\circ g] \leq \overline{\rho}^{(\alpha_1,\beta)}[f\circ g] < \infty$ *and* $0 < \overline{\lambda}^{(\alpha_2,\beta)}[f] \leq \overline{\rho}^{(\alpha_2,\beta)}[f] < \infty$. *Then*

$$\frac{\overline{\lambda}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\rho}^{(\alpha_2,\beta)}[f]} \leq \liminf_{r\to 1}\frac{\alpha_1(\log(M_{f\circ g}(r)))}{\alpha_2(\log(M_f(r)))} \leq \min\left\{\frac{\overline{\lambda}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\lambda}^{(\alpha_2,\beta)}[f]}, \frac{\overline{\rho}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\rho}^{(\alpha_2,\beta)}[f]}\right\} \leq \max\left\{\frac{\overline{\lambda}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\lambda}^{(\alpha_2,\beta)}[f]}, \frac{\overline{\rho}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\rho}^{(\alpha_2,\beta)}[f]}\right\} \leq \limsup_{r\to 1}\frac{\alpha_1(\log(M_{f\circ g}(r)))}{\alpha_2(\log(M_f(r)))} \leq \frac{\overline{\rho}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\lambda}^{(\alpha_2,\beta)}[f]}.$$

Theorem 8 *Let f and g be any two non-constant analytic functions in U such that* $0 < \overline{\lambda}^{(\alpha_1,\beta)}[f\circ g] \leq \overline{\rho}^{(\alpha_1,\beta)}[f\circ g] < \infty$ *and* $0 < \overline{\lambda}^{(\alpha_3,\beta)}[g] \leq \overline{\rho}^{(\alpha_3,\beta)}[g] < \infty$. *Then*

$$\frac{\overline{\lambda}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\rho}^{(\alpha_3,\beta)}[g]} \leq \liminf_{r\to 1}\frac{\alpha_1(\log(M_{f\circ g}(r)))}{\alpha_3(\log(M_g(r)))} \leq \min\left\{\frac{\overline{\lambda}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\lambda}^{(\alpha_3,\beta)}[g]}, \frac{\overline{\rho}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\rho}^{(\alpha_3,\beta)}[g]}\right\} \leq \max\left\{\frac{\overline{\lambda}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\lambda}^{(\alpha_3,\beta)}[g]}, \frac{\overline{\rho}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\rho}^{(\alpha_3,\beta)}[g]}\right\} \leq \limsup_{r\to 1}\frac{\alpha_1(\log(M_{f\circ g}(r)))}{\alpha_3(\log(M_g(r)))} \leq \frac{\overline{\rho}^{(\alpha_1,\beta)}[f\circ g]}{\overline{\lambda}^{(\alpha_3,\beta)}[g]}.$$

We may now state the following two theorems without proof based on Definition 3.

Theorem 9 *Let f and g be any two non-constant analytic functions in U such that* $0 < \overline{\lambda}^{(\alpha_1,\beta)}[f\circ g] \leq \overline{\rho}^{(\alpha_1,\beta)}[f\circ g] < \infty$

and $0 < \overline{\lambda}^{(\alpha_2,\beta)}[f] \leq \overline{\rho}^{(\alpha_2,\beta)}[f] < \infty$. *Then*

$$\frac{\lambda_{\log}^{(\alpha_1,\beta)}[f\circ g]}{\rho_{\log}^{(\alpha_2,\beta)}[f]} \leq \liminf_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \leq \min\left\{\frac{\lambda_{\log}^{(\alpha_1,\beta)}[f\circ g]}{\lambda_{\log}^{(\alpha_2,\beta)}[f]}, \frac{\rho_{\log}^{(\alpha_1,\beta)}[f\circ g]}{\rho_{\log}^{(\alpha_2,\beta)}[f]}\right\} \leq \max\left\{\frac{\lambda_{\log}^{(\alpha_1,\beta)}[f\circ g]}{\lambda_{\log}^{(\alpha_2,\beta)}[f]}, \frac{\rho_{\log}^{(\alpha_1,\beta)}[f\circ g]}{\rho_{\log}^{(\alpha_2,\beta)}[f]}\right\} \leq \limsup_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_f(r))} \leq \frac{\rho_{\log}^{(\alpha_1,\beta)}[f\circ g]}{\lambda_{\log}^{(\alpha_2,\beta)}[f]}.$$

Theorem 10 *Let f and g be any two non-constant analytic functions in U such that* $0 < \overline{\lambda}^{(\alpha_1,\beta)}[f\circ g] \leq \overline{\rho}^{(\alpha_1,\beta)}[f\circ g] < \infty$ *and* $0 < \overline{\lambda}^{(\alpha_3,\beta)}[g] \leq \overline{\rho}^{(\alpha_3,\beta)}[g] < \infty$. *Then*

$$\frac{\lambda_{\log}^{(\alpha_1,\beta)}[f\circ g]}{\rho_{\log}^{(\alpha_3,\beta)}[g]} \leq \liminf_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_3(M_g(r))} \leq \min\left\{\frac{\lambda_{\log}^{(\alpha_1,\beta)}[f\circ g]}{\lambda_{\log}^{(\alpha_3,\beta)}[g]}, \frac{\rho_{\log}^{(\alpha_1,\beta)}[f\circ g]}{\rho_{\log}^{(\alpha_3,\beta)}[g]}\right\} \leq \max\left\{\frac{\lambda_{\log}^{(\alpha_1,\beta)}[f\circ g]}{\lambda_{\log}^{(\alpha_3,\beta)}[g]}, \frac{\rho_{\log}^{(\alpha_1,\beta)}[f\circ g]}{\rho_{\log}^{(\alpha_3,\beta)}[g]}\right\} \leq \limsup_{r\to 1}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_3(M_g(r))} \leq \frac{\rho_{\log}^{(\alpha_1,\beta)}[f\circ g]}{\lambda_{\log}^{(\alpha_3,\beta)}[g]}.$$

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