The Generalized Adjacency, Laplacian and Signless Laplacian Spectra of the Mixed Weighted Corona

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Abstract: In recent years, the study of spectral analysis of corona graphs has received a great deal of attention from different scientific communities. A large number of researchers have studied vertex corona, edge corona, subdivision-vertex and subdivision-edge neighbourhood corona. In this paper, a new weighted network model, the mixed weighted corona model, is further studied by combining vertex corona and subdivided edge neighborhood corona. In order to obtain all the eigenvalues of the Laplacian matrix, we construct the super mixed corona matrix. Then, we give a complete description of the generalized adjacency spectra, Laplacian and signless Laplacian spectra of the mixed weighted corona.

Keywords: Mixed weighted corona, Generalized adjacency spectra, Laplacian spectra, Signless Laplacian spectra.

1 Introduction

Calculating the spectra of graphs is a fundamental and very meaningful work in spectral graph theory [1–4]. Many graph operations [5–7] have been introduced by researchers and some research results have been obtained [8–10]. Barik et al. studied the Laplacian spectra of some variants of corona by giving a complete description of the eigenvalues and the eigenvectors of graphs with super corona and super neighbourhood corona matrices [11]. Dai et al. studied the generalized adjacency and Laplacian spectra of the weighted corona graphs [12]. Liu et al. studied the spectra of subdivision-vertex and subdivision-edge neighbourhood corona [13]. Motivated by the work above, this paper defines a new graph operation.

Throughout this paper we consider undirected and simple graphs [14]. Let $G = (V, E)$ be a graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E = \{e_1, e_2, \ldots, e_m\}$. Let $\alpha_{ij}$ be the weight of edge linking vertex $v_i$ and vertex $v_j$. The generalized adjacency matrix of $G$ is denoted by $A(G) = \{a_{ij}\}$, where $a_{ij} = a_{ji} = \alpha_{ij}$ if nodes $v_i$ and $v_j$ are adjacent, $a_{ij} = a_{ji} = 0$ otherwise. The incidence matrix [15] of $G$ is denoted by $M(G) = (m_{ij})$, where $m_{ij}=1$ if the vertex $v_i$ and edge $e_j$ are incident in $G$ and 0 otherwise. Let $D(G)$ be the diagonal degree matrix of $G$, where the main entries of $D(G)$ are the vertex degrees of $G$ [11, 16, 17]. The Laplacian matrix of $G$ is defined as $L(G) = D(G) - A(G)$. The smallest Laplacian eigenvalue of $G$ is always 0 and the corresponding eigenvector is $1$, the vector of all ones [11]. The signless Laplacian matrix of $G$ is defined as $Q(G) = D(G) + A(G)$. A graph $G$ is $d$-regular if all its vertices have a degree $d$ [18, 19]. Let $\mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$, $0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ and $\delta_1(G) \leq \delta_2(G) \leq \cdots \leq \delta_n(G)$ denote the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$, respectively. The set of all the eigenvalues of a matrix together with its multiplicities is known as the spectrum of that matrix [12, 20–27]. It is well known that the spectrum of a graph can describe a lot of structural properties about the graph [13, 28].

In this paper, we use the following notations. $J_n$ (respectively, $0_n$) denotes the $n \times 1$ vector with each entry 1 (respectively, 0). The Kronecker product $R \otimes S$ of two matrices $R = [r_{ij}]$ and $S$ is defined as partitioned matrix $[r_{ij}S]$. $I_n$ is the identity matrix of order $n$, $E_i$ is the vector with $i$-th entry equal to the identity matrix and all other entries zero. $\epsilon_i$ is the vector with $i$-th entry equal to one and all other entries zero.

The organization of this paper is as follows. In next section, we construct the model of the mixed weighted corona of $G, G_1$ and $G_2$. In section 3, we deduce the generalized adjacency eigenvalues of the mixed weighted corona. In Section

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4. Firstly, we introduce the super mixed corona matrix, then we deduce all the eigenvalues as well as their multiplicities and the corresponding eigenvectors of this matrix, next we use the result to obtain the Laplacian eigenvalues. In Section 5, the analytical expressions of the signless Laplacian spectra are given. Finally, we draw the conclusion.

Figure 1: (a) Initial graph $C_5$, copy graphs $T_4$ and $C_4$. (b) The mixed weighted corona $C_5^{(M)} \bullet \{T_4, C_4\}$.
2 Model

Let $G$ be a $d$-regular graph consisting of $n$ vertices and $m$ edges with unit weight, then we denote the vertices by $v_1, v_2, \cdots, v_n$, denote the edges by $e_1, e_2, \cdots, e_m$. Let $G_1$ be a $d_1$-regular graph consisting of $n_1$ vertices with weight $r_1$, $G_2$ be a $d_2$-regular graph consisting of $n_2$ vertices with weight $r_2$, $0 < r_1, r_2 \leq 1$.

**Definition 1.1** The mixed weighted corona of $G$, $G_1$ and $G_2$, denoted by $G^M \bullet \{G_1, G_2\}$, is the graph obtained as follows.

1. Let $G_1^{(1)}, G_1^{(2)}, \cdots, G_1^{(n)}$ be the copies of $G_1$. For vertex $v_i$ ($i = 1, 2, \cdots, n$), we link each vertex in $G_1^{(i)}$ to $v_i$ forming a new edge with weight $r_1$.

2. Let $G_2^{(1)}, G_2^{(2)}, \cdots, G_2^{(m)}$ be the copies of $G_2$. For edge $e_j$ ($j = 1, 2, \cdots, m$), we link each vertex in $G_2^{(j)}$ to two endpoints of $e_j$ forming two new edges with weight $r_2$.

3. For edge $e_j$ ($j = 1, 2, \cdots, m$), we insert one and only one vertex.

Then the mixed weighted corona $G^M \bullet \{G_1, G_2\}$ has been set up.

**Example 1.2** Let $G = C_5$, $G_1 = T_4$ and $G_2 = C_4$, the mixed weighted corona $G^M \bullet \{G_1, G_2\}$ is shown in Fig. 1.

3 Generalized adjacency spectra of $G^M \bullet \{G_1, G_2\}$

According to the construction of $G^M \bullet \{G_1, G_2\}$, the generalized adjacency matrix of $G^M \bullet \{G_1, G_2\}$ can be easily obtained as below.

$$A(G^M \bullet \{G_1, G_2\}) = \begin{pmatrix}
0 & M(G) & r_1J_{n_1}^T \otimes I_n & r_2J_{n_2}^T \otimes M(G) \\
M(G)^T & 0 & 0 & 0 \\
r_1J_{n_1} \otimes I_n & 0 & r_1 A(G_1) \otimes I_n & 0 \\
r_2J_{n_2} \otimes M(G)^T & 0 & 0 & r_2 A(G_2) \otimes I_m
\end{pmatrix}. \quad (1)$$

Let $\sigma(G) = \{\mu_1, \mu_2, \cdots, \mu_n = d\}$, $\sigma(G_1) = \{\mu_1^{(1)}, \mu_2^{(1)}, \cdots, \mu_n^{(1)} = d_1\}$ and $\sigma(G_2) = \{\mu_1^{(2)}, \mu_2^{(2)}, \cdots, \mu_m^{(2)} = d_2\}$. Then the generalized adjacency spectrum of $G^M \bullet \{G_1, G_2\}$ will be determined. Let $X = [X_1, X_2, \cdots, X_{n_1+n_2+2}]^T$ be the eigenvector of $A(G^M \bullet \{G_1, G_2\})$ which corresponding to the eigenvalue $\mu$, where $X_1, X_2, \cdots, X_{n_1+n_2+2} \in R^n, X_2, X_{n_1+3}, X_{n_1+4}, \cdots, X_{n_1+n_2+2} \in R^m$. Then,

$$A(G^M \bullet \{G_1, G_2\})X = \mu X. \quad (2)$$

Next we calculate the eigenvalues of $A(G^M \bullet \{G_1, G_2\})$ except for $\mu = r_1d_1$ and $\mu = r_2d_2$. We discuss the following two cases.

**Case 1** Nonzero vector $X_1$

According to Eqs. (1) and (2), we have

$$M(G)X_2 + r_1(X_3 + X_4 + \cdots + X_{n_1+2}) + r_2M(G)(X_{n_1+3} + X_{n_1+4} + \cdots + X_{n_1+n_2+2}) = \mu X_1, \quad (3)$$

and

$$M(G)^TX_1 = \mu X_2, \quad (4)$$

and

$$\begin{cases}
r_1X_1 + r_1E_1[A(G_1) \otimes I_n][X_3 \cdots X_{n_1+2}]^T = \mu X_3, \\
r_1X_1 + r_1E_2[A(G_1) \otimes I_n][X_3 \cdots X_{n_1+2}]^T = \mu X_4, \\
\vdots \\
r_1X_1 + r_1E_{n_1}[A(G_1) \otimes I_n][X_3 \cdots X_{n_1+2}]^T = \mu X_{n_1+2},
\end{cases} \quad (5)$$

and

$$\begin{cases}
r_2M(G)^TX_1 + r_2E_1[A(G_2) \otimes I_m][X_{n_1+3} \cdots X_{n_1+n_2+2}]^T = \mu X_{n_1+3}, \\
r_2M(G)^TX_1 + r_2E_2[A(G_2) \otimes I_m][X_{n_1+3} \cdots X_{n_1+n_2+2}]^T = \mu X_{n_1+4}, \\
\vdots \\
r_2M(G)^TX_1 + r_2E_{n_2}[A(G_2) \otimes I_m][X_{n_1+3} \cdots X_{n_1+n_2+2}]^T = \mu X_{n_1+n_2+2}.
\end{cases} \quad (6)$$

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By adding all equations in Eqs. (5), we obtain
\[ n_1 r_1 X_1 + r_1 d_1 (X_3 + X_4 + \cdots + X_{n+2}) = \mu (X_3 + X_4 + \cdots + X_{n+2}). \] (7)

By adding all equations in Eqs. (6), we obtain
\[ n_2 r_2 M(G)^T X_1 + r_2 d_2 (X_{n+3} + X_{n+4} + \cdots + X_{n+n+2}) = \mu (X_{n+3} + X_{n+4} + \cdots + X_{n+n+2}). \] (8)

Then
\[ (X_3 + X_4 + \cdots + X_{n+2}) = \frac{n_1 r_1 X_1}{\mu - r_1 d_1}, \] (9)
and
\[ (X_{n+3} + X_{n+4} + \cdots + X_{n+n+2}) = \frac{n_2 r_2 M(G)^T X_1}{\mu - r_2 d_2}. \] (10)

From Eq. (4), it gives
\[ X_2 = \frac{1}{\mu} M(G)^T X_1. \] (11)

Substituting Eqs. (9), (10) and (11) to Eq. (3), we obtain
\[ \frac{1}{\mu} A(G) X_1 + \frac{d}{\mu} X_1 + \frac{n_1 r_1^2}{\mu - r_1 d_1} X_1 + \frac{n_2 r_2^2}{\mu - r_2 d_2} A(G) X_1 + \frac{n_2 r_2^2 d}{\mu - r_2 d_2} X_1 = \mu X_1, \] (12)

namely,
\[ \left( \frac{1}{\mu} - \frac{n_2 r_2^2}{\mu - r_2 d_2} \right) A(G) X_1 = \left( \mu - d \right) \frac{X_1}{\mu - r_1 d_1} - \frac{n_2 r_2^2 d}{\mu - r_2 d_2} X_1. \] (13)

Notice that the spectrum of \( A(G) \) is \( \sigma(G) = \{ \mu_1 | \mu_1 \leq \mu_2 \leq \cdots < \mu_n = d \} \). Based on Eq. (13) and the definition of the eigenvalue and eigenvector, we obtain
\[ \mu^4 - (r_1 d_1 + r_2 d_2) \mu^3 + (r_1 r_2 d_1 d_2 - d - n_1 r_1^2 - n_2 r_2^2 - (1 + n_2 r_2^2) \mu_1) \mu^2 + (d r_1 d_1 + r_1 r_2 d_2 + n_1 r_1^2 r_2 d_2 + n_2 d r_2^2 r_1 d_1 + (r_1 d_1 + r_2 d_2 + n_2 r_2^2 r_1 d_1) \mu_3 - d r_1 r_2 d_1 d_2 - r_1 r_2 d_2 d_1 \mu_1 = 0. \] (14)

It is known that all the roots of the above quartic equation in the variable \( \mu \) are the eigenvalues of \( A(G^{(M)} \cdot \{G_1, G_2 \}) \).

**Case(2) Zero vector \( X_1 \)**

According to Eqs. (1) and (2), we have
\[ r_1 (X_3 + X_4 + \cdots + X_{n+2}) + r_2 M(G) (X_{n+3} + X_{n+4} + \cdots + X_{n+n+2}) = 0, \] (15)

and
\[ \begin{cases}  
 r_1 [A(G_1) \otimes I_m] [X_3 \cdots X_{n+2}]^T = \mu [X_3 \cdots X_{n+2}]^T, 
  
 r_2 [A(G_2) \otimes I_m] [X_{n+3} \cdots X_{n+n+2}]^T = \mu [X_{n+3} \cdots X_{n+n+2}]^T. 
\end{cases} \] (16)

Notice that the spectra of \( A(G_1) \) and \( A(G_2) \) are \( \sigma(G_1) = \{ \mu^{(1)}_1 | \mu^{(1)}_1 \leq \mu^{(1)}_2 \leq \cdots < \mu^{(1)}_n = d_1 \} \) and \( \sigma(G_2) = \{ \mu^{(2)}_1 | \mu^{(2)}_1 \leq \mu^{(2)}_2 \leq \cdots < \mu^{(2)}_n = d_2 \} \), respectively. So we can straightforward find out that
\[ \begin{cases}  
 \mu = r_1 \mu^{(1)}_j, & j = 1, 2, \cdots, n_1 - 1, 
  
 \mu = r_2 \mu^{(2)}_j, & j = 1, 2, \cdots, n_2 - 1. 
\end{cases} \] (17)

According to Eq. (16), it gives that the multiplicity of \( \mu = r_1 \mu^{(1)}_j \) is \( n \), the multiplicity of \( \mu = r_2 \mu^{(2)}_j \) is \( m \).

From Eqs. (14) and (17), we have already obtained \( 4n + n(n_1 - 1) + m(n_2 - 1) \) eigenvalues of \( A(G^{(M)} \cdot \{G_1, G_2 \}) \). Thus, \( \mu = r_1 d_1 \) and \( \mu = r_2 d_2 \) with multiplicity \( m - n \) are also the eigenvalues.

Through the above steps, the following theorem can be obtained.

**Theorem 2.1** Let the generalized adjacency spectra of \( G, G_1 \) and \( G_2 \) are \( \sigma(G) = \{ \mu_1 | \mu_1 \leq \mu_2 \leq \cdots < \mu_n = d \} \), \( \sigma(G_1) = \{ \mu^{(1)}_1 | \mu^{(1)}_1 \leq \mu^{(1)}_2 \leq \cdots < \mu^{(1)}_n = d_1 \} \) and \( \sigma(G_2) = \{ \mu^{(2)}_1 | \mu^{(2)}_1 \leq \mu^{(2)}_2 \leq \cdots < \mu^{(2)}_n = d_2 \} \), respectively. Then the eigenvalues of \( A(G^{(M)} \cdot \{G_1, G_2 \}) \) can be expressed as
Let $p, q \in N, p < q, r, s \in N \cup \{0\}, 0 < p_1, p_2 \leq 1$, where $N$ is the set of positive integers, and $c \in \{1, -1\}$. Let $A, B, C, D, E$ be real square matrix of size $p, q, r, s$, respectively and $B$ be a $p \times q$ real matrix. Focus on the following real matrix of order $p(r + 1) + q(s + 1)$:

$$S_M = \begin{pmatrix}
A & B & cp_1J_r^T \otimes I_p & cp_2J_s^T \otimes J_r \\
B^T & C & 0_{q \times pr} & 0_{q \times qs} \\
0_{pr \times q} & D \otimes I_P & 0_{pr \times qs} \\
0_{s \times pr} & 0_{sq \times qs} & E \otimes I_q
\end{pmatrix}. $$

Let $1_r = \tilde{U}_1, \tilde{U}_2, \cdots, \tilde{U}_r$ be the eigenvectors of $D$ which corresponding to the eigenvalues $d_1, d_2, \cdots, d_r$, respectively. Let $1_s = \tilde{V}_1, \tilde{V}_2, \cdots, \tilde{V}_s$ be the eigenvectors of $E$ which corresponding to the eigenvalues $e_1, e_2, \cdots, e_s$, respectively. $S_M$ is called the super mixed corona matrix [11], if the following conditions are met:

1. If $BY_i = b_iX_i$ and $B^TX_i = b_iY_i$ for $i = 1, 2, \cdots, p$, then $AX_i = a_iX_i$ and $CY_i = c_iY_i$, where $a_i$ and $c_i$ are the eigenvalues of $A$ and $C$, respectively;

2. If $BY_j = 0_q$ for $j = 1, 2, \cdots, q - p$ (this is true as $p < q$), then $C\tilde{Y}_j = \lambda_j \tilde{Y}_j$ for $j = 1, 2, \cdots, q - p$, where $\lambda_j$ are eigenvalues of $C$.

Next the eigenvalues of $S_M$ are calculated. Consider the matrix equation

$$S_M \begin{pmatrix}
h_1X_i & h_2Y_i & h_3J_r \otimes X_i & J_s \otimes Y_i
\end{pmatrix}^T = \lambda \begin{pmatrix}
h_1X_i & h_2Y_i & h_3J_r \otimes X_i & J_s \otimes Y_i
\end{pmatrix}^T,$$

where $h_1, h_2, h_3$ and $\lambda$ are undetermined constants. Comparing both sides, there are

$$\begin{cases}
h_1a_i + h_2b_i + h_3cp_1r + cp_2sb_i = h_1\lambda, \\
h_1b_i + h_2c_i = h_2\lambda, \\
h_1cp_1 + h_3d_1 = h_3\lambda, \\
h_1cp_2b_i + e_i = \lambda.
\end{cases}$$

Eliminating $h_1, h_2$ and $h_3$, it gives

$$((\lambda - d_1)(\lambda - a_i)(\lambda - c_i) - (\lambda - d_1)(\lambda - e_1)b_i^2 - (\lambda - e_1)(\lambda - c_i)c^2p_1^2r$$

$$- (\lambda - d_1)(\lambda - c_i)c^2p_2^2b_i^2 = 0.$$
As $c^2 = 1$, obtain

$$
\lambda^4 - (d_1 + e_1 + a_i + c_i)\lambda^3 + ((e_1 + a_i)(d_1 + c_i) + d_1 e_1 + e_1 a_i - (b_i^2 + p_2^2 r + p_2^2 s b_i^2))\lambda^2 \\
- ((d_1 a_i - p_2^2 r)(e_1 + c_i) + e_1 c_i(d_1 + a_i) - (d_1 + e_1) b_i^2 - (d_1 + c_i)p_2^2 s b_i^2)\lambda \\
+ d_1 e_1 (a_i c_i - b_i^2) - e_1 c_i p_2^2 r - d_1 c_i p_2^2 s b_i^2 = 0.
$$

All roots of the above quartic equation in the variable $\lambda$ are the eigenvalues of $S_M$.

It can be observed that

$$
S_M \left( \begin{array}{ccc}
0 & p & 0 & 0 \\
0 & q & 0 & 0 \\
U_j & \otimes & \epsilon_i & 0 \\
0 & 0 & 0 & 0
\end{array} \right) = d_j \left( \begin{array}{ccc}
0 & p & 0 & 0 \\
0 & q & 0 & 0 \\
U_j & \otimes & \epsilon_i & 0 \\
0 & 0 & 0 & 0
\end{array} \right)^T,
$$

for $j = 2, 3, \cdots, r$ and $i = 1, 2, \cdots, p$.

It gives that $d_j$ for $j = 2, 3, \cdots, r$ with multiplicity $p$ are the eigenvalues of $S_M$.

Similarly, we observe that

$$
S_M \left( \begin{array}{ccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
V_j & \otimes & \epsilon_i & 0 \\
0 & 0 & 0 & 0
\end{array} \right) = e_j \left( \begin{array}{ccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
V_j & \otimes & \epsilon_i & 0 \\
0 & 0 & 0 & 0
\end{array} \right)^T,
$$

for $j = 2, 3, \cdots, s$ and $i = 1, 2, \cdots, q$.

It gives that $e_j$ for $j = 2, 3, \cdots, s$ with multiplicity $q$ are the eigenvalues of $S_M$.

Now we considering the orthogonal vectors $\tilde{Y}_j$ for $j = 1, 2, \cdots, q - p$ such that $B \tilde{Y}_j = 0$ and $C \tilde{Y}_j = \tilde{c}_j \tilde{Y}_j$. We can easily observe that the vectors

$$
\left( \begin{array}{ccc}
0 & \tilde{Y}_j & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right)^T \text{ and } \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right)^T
$$

are the eigenvectors of $S_M$ which corresponding to the eigenvalues $\tilde{c}_j$ and $e_1$, respectively.

Through the above steps, the following theorem can be obtained.

**Theorem 3.1** Let $S_M$ be a super mixed corona matrix as defined above. Then the eigenvalues of $S_M$ can be expressed as

1. Four roots of the quartic equation

$$
\lambda^4 - (d_1 + e_1 + a_i + c_i)\lambda^3 + ((e_1 + a_i)(d_1 + c_i) + d_1 e_1 + e_1 a_i - (b_i^2 + p_2^2 r + p_2^2 s b_i^2))\lambda^2 \\
- ((d_1 a_i - p_2^2 r)(e_1 + c_i) + e_1 c_i(d_1 + a_i) - (d_1 + e_1) b_i^2 - (d_1 + c_i)p_2^2 s b_i^2)\lambda \\
+ d_1 e_1 (a_i c_i - b_i^2) - e_1 c_i p_2^2 r - d_1 c_i p_2^2 s b_i^2 = 0,
$$

with multiplicity 1, for $i = 1, 2, \cdots, p$;

2. $\tilde{c}_j$ for $j = 1, 2, \cdots, q - p$;

3. $e_1$ repeated $q - p$ times;

4. $d_j$ repeated $p$ times, for $j = 2, 3, \cdots, r$;

5. $e_j$ repeated $q$ times, for $j = 2, 3, \cdots, s$.

**4.2 Laplacian spectra of $G^{(M)} \bullet \{G_1, G_2\}$**

According to the construction of $G^{(M)} \bullet \{G_1, G_2\}$, the Laplacian matrix of $G^{(M)} \bullet \{G_1, G_2\}$ can be easily obtained as below.

$$
L(G^{(M)} \bullet \{G_1, G_2\}) =
\begin{pmatrix}
(d + n_1 r_1 + n_2 r_2) I_n & - M(G) & - r_1 J_{n_1}^T \otimes I_n & - r_2 J_{n_2}^T \otimes M(G) \\
-M(G)^T & 2I_n & 0 & 0 \\
r_1 J_{n_1} \otimes I_n & 0 & r_1 (L(G_1) + I_{n_1}) \otimes I_n & 0 \\
r_2 J_{n_2} \otimes M(G)^T & 0 & 0 & r_2 (L(G_2) + 2I_{n_2}) \otimes I_m
\end{pmatrix}.
$$

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Let \( \ell(G) = \{ \lambda_i | 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \} \), \( \ell(G_1) = \{ \lambda^{(1)}_i | 0 = \lambda^{(1)}_1 \leq \lambda^{(1)}_2 \leq \cdots \leq \lambda^{(1)}_n \} \) and \( \ell(G_2) = \{ \lambda^{(2)}_i | 0 = \lambda^{(2)}_1 \leq \lambda^{(2)}_2 \leq \cdots \leq \lambda^{(2)}_n \} \).

Comparing with the super mixed corona \( S_{M} \), it gives that \( p = n, q = m, r = n_1, s = n_2, c = -1, p_1 = r_1, p_2 = r_2, A = (d + n_1 r_1 + d n_2 r_2) I_{n_1}, B = -M(G), C = 2I_m, D = r_1 (L(G_1) + I_{n_1}), E = r_2 (L(G_2) + 2I_{n_2}) \). We know \( M(G) M(G)^T = 2d I_n - L(G) \) and \( L(G_1) I_{n_1} = 0_{n_1}, L(G_2) I_{n_2} = 0_{n_2} \), thus we obtain

\[
\begin{align*}
\lambda_i &= d + n_1 r_1 + d n_2 r_2; b_i^2 &= 2d - \lambda_i; e_i = 2 \quad \text{for } i = 1, 2, \cdots, n; \tilde{e}_j = 2 \quad \text{for } j = 1, 2, \cdots, m - n; \\
d_i &= r_1 (\lambda^{(1)}_i + 1) \quad \text{for } i = 2, 3, \cdots, n_1; \\
\end{align*}
\]

By substituting these values in Theorem 3.1, the following theorem can be obtained.

**Theorem 3.2** Let the Laplacian spectra of \( G, G_1 \) and \( G_2 \) are \( \ell(G) = \{ \lambda_i | 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \} \), \( \ell(G_1) = \{ \lambda^{(1)}_i | 0 = \lambda^{(1)}_1 \leq \lambda^{(1)}_2 \leq \cdots \leq \lambda^{(1)}_n \} \) and \( \ell(G_2) = \{ \lambda^{(2)}_i | 0 = \lambda^{(2)}_1 \leq \lambda^{(2)}_2 \leq \cdots \leq \lambda^{(2)}_n \} \), respectively. Then the eigenvalues of \( L(G^{(M)} \bullet \{ G_1, G_2 \}) \) can be expressed as

1. Four roots of the quartic equation

\[
\begin{align*}
\lambda^4 - (d + n_1 r_1 + d n_2 r_2 + r_1 + 2 r_2 + 2) \lambda^3 + ((d + n_1 r_1 + d n_2 r_2 + 2 r_1) r_1 + 2 r_1 + 2 r_2 (d + n_1 r_1 + d n_2 r_2 + r_1) - 2 (2d - \lambda_i + r_1^2 n_1 + r_2^2 n_2 (2d - \lambda_i))) \lambda^2 \\
- ((d + n_1 r_1 + d n_2 r_2) r_1 - r_1^2 n_1) (2 r_2 + 2) + 4 r_2 ((d + n_1 r_1 + d n_2 r_2) r_1) - (r_1 + 2 r_2) (2d - \lambda_i) - (r_1 + 2 r_2) r_2 n_2 (2d - \lambda_i) \lambda + 2 r_1 r_2 (2d + n_1 r_1 + d n_2 r_2) \\
- (2d - \lambda_i) - 4 r_2 r_1^2 n_1 - 2 r_1 r_2^2 n_2 (2d - \lambda_i) = 0,
\end{align*}
\]

with multiplicity 1, for \( i = 1, 2, \cdots, p \);

2. \( 2r \), repeated \( m - n \) times;

3. \( 2r_2 \), repeated \( n \) times;

4. \( r_1 \lambda^{(1)}_i + 1 \) repeated \( n_1 \) times, for \( i = 2, 3, \cdots, n_1 \);

5. \( 2r_1 \lambda^{(2)}_i + 2 \) repeated \( m \) times, for \( i = 2, 3, \cdots, n_2 \).

## 5 Signless Laplacian spectra of \( G^{(M)} \bullet \{ G_1, G_2 \} \)

According to the construction of \( G^{(M)} \bullet \{ G_1, G_2 \} \), it is easy to obtain the signless Laplacian matrix of \( G^{(M)} \bullet \{ G_1, G_2 \} \) as below.

\[
Q(G^{(M)} \bullet \{ G_1, G_2 \}) = \begin{pmatrix}
(d + n_1 r_1 + d n_2 r_2) I_n & M(G) & \tilde{r}_1 J_{n_1}^T & I_{n_1} & 0 & 0 \\
M(G)^T & 2 I_m & 0 & r_1 (Q(G_1) + I_{n_1}) \otimes I_{n_1} & r_2 J_{n_2}^T \otimes M(G) & 0 \\
r_1 J_{n_1} \otimes I_n & 0 & r_1 (Q(G_1) + I_{n_1}) \otimes I_{n_1} & 0 & 0 & 0 \\
r_2 J_{n_2} \otimes M(G)^T & 0 & 0 & r_2 (Q(G_2) + 2 I_{n_2}) \otimes I_m & 0 & 0
\end{pmatrix}.
\tag{18}
\]

Let \( \tau(G) = \{ \delta_i | \delta_1 \leq \delta_2 \leq \cdots \leq \delta_n = 2d \} \), \( \tau(G_1) = \{ \delta^{(1)}_i | \delta^{(1)}_1 \leq \delta^{(1)}_2 \leq \cdots < \delta^{(1)}_n = 2d_1 \} \) and \( \tau(G_2) = \{ \delta^{(2)}_i | \delta^{(2)}_1 \leq \delta^{(2)}_2 \leq \cdots < \delta^{(2)}_n = 2d_2 \} \). Then the signless Laplacian spectrum of \( G^{(M)} \bullet \{ G_1, G_2 \} \) will be determined. Let \( X = [X_1, X_2, \cdots, X_{n_1 + n_2 + 2}]^T \) be the eigenvector of \( Q(G^{(M)} \bullet \{ G_1, G_2 \}) \) which corresponding to the eigenvalue \( \delta \), where \( X_1, X_3, X_4, \cdots, X_{n_1 + 2} \in R^n, X_2, X_{n_1 + 3}, X_{n_1 + 4}, \cdots, X_{n_1 + n_2 + 2} \in R^m \). Then,

\[
Q(G^{(M)} \bullet \{ G_1, G_2 \}) X = \delta X.
\tag{19}
\]

Next we calculate the eigenvalues of \( Q(G^{(M)} \bullet \{ G_1, G_2 \}) \) except for \( \delta = r_1 (2d_1 + 1) \) and \( \delta = r_2 (2d_2 + 2) \). We discuss the following two cases.

**Case(1) Nonzero vector \( X_1 \)**

IJNS homepage: http://www.nonlinearscience.org.uk/
According to Eqs. (18) and (19), we have
\[(d + n_1r_1 + dn_2r_2)X_1 + M(G)X_2 + r_1(X_3 + X_4 + \cdots + X_{n_1+2}) + r_2M(G)(X_{n_1+3} + X_{n_1+4} + \cdots + X_{n_1+n_2+2}) = \delta X_1, \tag{20}\]
\[M(G)^T X_1 + 2X_2 = \delta X_2, \tag{21}\]
and
\[
\begin{cases}
    r_1X_1 + r_1E_1[(Q(G) + I_n) \otimes I_n][X_3 \cdots X_{n_1+n_2+2}]^T = \delta X_3, \\
n_1r_1X_1 + r_1E_2[(Q(G) + I_n) \otimes I_n][X_3 \cdots X_{n_1+n_2+2}]^T = \delta X_4, \\
\vdots \\
n_1r_1X_1 + r_1E_{n_1}[(Q(G) + I_n) \otimes I_n][X_3 \cdots X_{n_1+n_2+2}]^T = \delta X_{n_1+2},
\end{cases} \tag{22}\]
and
\[
\begin{cases}
n_1r_1X_1 + r_1E_1[(Q(G) + 2I_n) \otimes I_n][X_{n_1+3} \cdots X_{n_1+n_2+2}]^T = \delta X_{n_1+3}, \\
n_1r_1X_1 + r_1E_2[(Q(G) + 2I_n) \otimes I_n][X_{n_1+3} \cdots X_{n_1+n_2+2}]^T = \delta X_{n_1+4}, \\
\vdots \\
n_1r_1X_1 + r_1E_{n_1}[(Q(G) + 2I_n) \otimes I_n][X_{n_1+3} \cdots X_{n_1+n_2+2}]^T = \delta X_{n_1+n_2+2}.
\end{cases} \tag{23}\]
By adding all equations in Eqs. (22), we obtain
\[n_1r_1X_1 + r_1(2d_1 + 1)(X_3 + X_4 + \cdots + X_{n_1+2}) = \delta(X_3 + X_4 + \cdots + X_{n_1+2}). \tag{24}\]
By adding all equations in Eqs. (23), we obtain
\[n_1r_1X_1 + r_1(2d_1 + 2)(X_{n_1+3} + X_{n_1+4} + \cdots + X_{n_1+n_2+2}) = \delta(X_{n_1+3} + X_{n_1+4} + \cdots + X_{n_1+n_2+2}). \tag{25}\]
Then
\[(X_3 + X_4 + \cdots + X_{n_1+2}) = \frac{n_1r_1X_1}{\delta - r_1(2d_1 + 1)}, \tag{26}\]
and
\[(X_{n_1+3} + X_{n_1+4} + \cdots + X_{n_1+n_2+2}) = \frac{n_1r_1X_1}{\delta - r_2(2d_2 + 2)}. \tag{27}\]
From Eq. (21), it gives
\[X_2 = \frac{1}{\delta - 2} M(G)^T X_1. \tag{28}\]
Substituting Eqs. (26), (27) and (28) to Eq. (20), we get
\[(d + n_1r_1 + dn_2r_2)X_1 + \frac{1}{\delta - 2} Q(G)X_1 + \frac{n_1r_1^2}{\delta - r_1(2d_1 + 1)} X_1 + \frac{n_2r_2^2}{\delta - 2r_2(2d_2 + 2)} Q(G)X_1 = \delta X_1, \tag{29}\]
namely,
\[
\left(\frac{1}{\delta - 2} + \frac{n_2r_2^2}{\delta - 2r_2(2d_2 + 1)}\right) Q(G)X_1 = (\delta - (d + n_1r_1 + dn_2r_2) - \frac{n_1r_1^2}{\delta - r_1(2d_1 + 1)}) X_1. \tag{30}\]
Notice that the spectrum of $Q(G)$ is $\tau(G) = \{\delta_1, \delta_2, \leq \cdots \leq \delta_n = 2d\}$. Based on Eq. (30) and the definition of the eigenvalue and eigenvector, we obtain
\[
\begin{align*}
\delta^4 & - (2r_2(2d_2 + 1) + (2 + r_1(2d_1 + 1))(d + n_1r_1 + dn_2r_2))\delta^3 + (2r_2(2 + r_1(2d_1 + 1)) + 2(2d_1 + 1)) \\
&\quad - n_1r_1^2 - (1 + n_2r_2^2 + 2r_2(2d_2 + 1) + 2r_1(2d_1 + 1))(d + n_1r_1 + dn_2r_2))\delta^2 \\
&\quad + (4r_1r_2(2d_1 + 1)(d_2 + 1) - (d + n_1r_1 + dn_2r_2)(2r_1(2d_1 + 1)(d_2 + 1) + 2r_1(2d_1 + 1)) \\
&\quad - n_1r_1^2(2 + 2r_2(2d_2 + 1) - r_1(2d_1 + 1) + 2r_2(2d_2 + 1) + n_2r_2^2(2 + 2r_2(2d_2 + 1))))\delta \\
&\quad - 4r_1r_2(2d_1 + 1)(d_2 + 1)(d + n_1r_1 + dn_2r_2) - 4r_2(d_2 + 1) + 2n_1r_1r_2^2(2d_1 + 1))\delta_1 = 0.
\end{align*}
\tag{31}
It is known that all the roots of the above quartic equation in the variable $\delta$ are the eigenvalues of $Q(G^{(M)} \bullet \{G_1, G_2\})$.

Case(2) Zero vector $X_1$

According to Eqs. (18) and (19), we have

$$r_1(X_{n+3} + X_{n+4} + \cdots + X_{n+n_2+2}) + r_2M(G)(X_{n+3} + X_{n+4} + \cdots + X_{n+n_2+2}) = 0.$$  \hspace{1cm} (32)

and

$$\begin{cases}
    r_1[(Q(G_1) + I_{n_1}) \otimes I_{n_1}]X_{n+3} \cdots X_{n+n_2+2} = \delta X_{n+3} \cdots X_{n+n_2+2}, \\
    r_2[Q(G_2) + 2I_{n_2}] \otimes I_{n_1}X_{n+3} \cdots X_{n+n_2+2} = \delta X_{n+3} \cdots X_{n+n_2+2}.
\end{cases}$$  \hspace{1cm} (33)

Notice that the spectra of $Q(G_1)$ and $Q(G_2)$ are $\{\delta_1^{(1)} \leq \delta_2^{(1)} \leq \cdots < \delta_{n_1}^{(1)} = 2d_1\}$ and $\{\delta_1^{(2)} \leq \delta_2^{(2)} \leq \cdots < \delta_{n_2}^{(2)} = 2d_2\}$, respectively. So we can straightforward find out that

$$\begin{cases}
    \delta = r_1(\delta_j^{(1)} + 1), j = 1, 2, \cdots, n_1 - 1, \\
    \delta = r_2(\delta_j^{(2)} + 2), j = 1, 2, \cdots, n_2 - 1.
\end{cases}$$  \hspace{1cm} (34)

According to Eq. (33), it gives that the multiplicity of $\delta = r_1(\delta_j^{(1)} + 1)$ is $n$, the multiplicity of $\delta = r_2(\delta_j^{(2)} + 2)$ is $m$.

From Eqs. (31) and (34), we have already obtained $4n + n(n_1 - 1) + m(n_2 - 1)$ eigenvalues of $Q(G^{(M)} \bullet \{G_1, G_2\})$.

Theorem 4.1

Let the signless Laplacian spectra of $G, G_1$ and $G_2$ are $\tau(G) = \{\delta_1 \leq \delta_2 \leq \cdots < \delta_n = 2d\}$, $\tau(G_1) = \{\delta_1^{(1)} \leq \delta_2^{(1)} \leq \cdots < \delta_{n_1}^{(1)} = 2d_1\}$ and $\tau(G_2) = \{\delta_1^{(2)} \leq \delta_2^{(2)} \leq \cdots < \delta_{n_2}^{(2)} = 2d_2\}$, respectively. Then the eigenvalues of $Q(G^{(M)} \bullet \{G_1, G_2\})$ can be expressed as

1. Four roots of the quartic equation

$$\begin{align*}
    &\delta^4 - (2r_2(d_2 + 1) + (2 + r_1(2d_1 + 1))(d + n_1r_1 + dn_2r_2))\delta^3 + (2r_2(2 + r_1(2d_1 + 1)) \\
    &+ 2(2d_1 + 1) - n_1r_1^2 - (1 + 2r_2(2d_1 + 1) + 2 + r_1(2d_1 + 1))(d + n_1r_1 + dn_2r_2))\delta^2 \\
    &+ (4r_1r_2(2d_1 + 1)(d_2 + 1) - (d + n_1r_1 + dn_2r_2)(2r_2(2d_1 + 1)) + 2r_1(2d_1 + 1)) \\
    &- n_1r_1^2(2 + 2r_2(d_2 + 1)) - (r_1(2d_1 + 1) + 2r_2(d_2 + 1) + n_2r_2^2(2 + r_2(2d_1 + 1))\delta_1^2 \\
    &- 4r_1r_2(2d_1 + 1)(d_2 + 1)(d + n_1r_1 + dn_2r_2) - 4r_2(d_2 + 1)n_1r_1^2 - (2r_1r_2(2d_1 + 1))d_2 + 1) \\
    &+ 2n_1r_1r_2^2(d_2 + 1))\delta_1 = 0,
\end{align*}$$

with multiplicity $1$, for $j = 1, 2, \cdots, n$;

2. $r_1(\mu_j^{(1)} + 1)$ repeated $n$ times, for $j = 1, 2, \cdots, n_1 - 1$;

3. $r_2(\mu_j^{(2)} + 2)$ repeated $m$ times, for $j = 1, 2, \cdots, n_2 - 1$;

4. $r_1(2d_1 + 1)$ and $r_2(2d_2 + 2)$ repeated $m - n$ times respectively (if possible).

6 Conclusions

In conclusion, this paper studied the mixed weighted corona $G^{(M)} \bullet \{G_1, G_2\}$ and a special form of matrix called super mixed corona matrix. We computed the Laplacian spectrum from the super mixed corona matrix by guessing eigenvectors of the matrix if the composing graphs are regular graphs. Then we obtained the generalized adjacency, Laplacian and signless Laplacian spectra for the case of $G, G_1$ and $G_2$ being regular graphs.

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References