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Numerical Solution of the Modified Kawahara Equation Using Sinc Collocation Method

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Abstract: This paper investigates numerical solution of the modified Kawahara equation by using finite d-ifference method and sinc collocation method. The semi-discrete scheme is obtained by approximating time derivative with θ-weighted scheme. Then, a fully discrete scheme is established by a collocation method using sinc basis. The stability of proposed method is analyzed by expressing the fully discrete scheme in matrix form. Finally, some numerical experiments illustrate the efficiency and superiority of present method, and two invariants are evaluated to determine the conservativeness of the method.

Keywords: modified Kawahara equation; sinc collocation method; finite difference method

1 Introduction

Many phenomena in mathematical physics and some engineering fields such as plasma physics, optical fibers, solid state physics, chemical kinematics, fluid mechanics and chemical physics can be modeled by nonlinear differential equations [1, 2]. The Kawahara equation is one of the most known nonlinear differential equations, which was first introduced by Kawahara [3]. As a differential equation with both third-order and fifth-order dispersion and nonlinear terms, the Kawahara equation is a generalized nonlinear dispersive equation and can be used as a model for many problems, such as magneto-acoustic waves in a plasma, capillary-gravity waves and solitary-wave propagation in dispersive media [4].

In this paper, we focus our attention on the following form of the modified Kawahara equation [5]

$$u_t + u_x + \alpha u^2 u_x + \beta u_{3x} + \gamma u_{5x} = 0, \tag{1}$$

where λ, β, γ are nonzero arbitrary constants and γ is called the dispersion coefficient. The modified Kawahara equation is known as singularly perturbed KdV equation, which plays an important role in the theory of shallow water waves with surface tension [6].

So far, the modified Kawahara equation has been studied by many researchers. Bridges et al. [7] considered the linear stability problem for solitary wave states of the modified Kawahara equation. Zhang [8] presented doubly periodic solutions of the modified Kawahara equation. Araruna et al. [9] analyzed the eventual dissipation of solutions for the modified Kawahara equation. Dereli [2] obtained the numerical solutions of the modified Kawahara equation by radial basis functions. There are also some other powerful methods like Dual-Petrov-Galerkin method [10] and Fourier splitting method [11] to solve the modified Kawahara equations numerically.

Sinc method was originally proposed by Stenger [12]. Thereafter, it is widely used to solve various types of problems, such as the initial value problems [13], the multi-point boundary value problems [14], the generalized regularized long wave equation [15], the Volterra partial integro-differential equation [16], the fuzzy Fredholm integral equation [17] and the astrophysics equations [18]. There are many advantages to apply sinc collocation method to solve equations. On the one hand, the rate of convergence is exponential. On the other hand, approximation by sinc functions handles singularities in the problem [19]. Therefore, a rapidly convergent scheme can be constructed for solving the problem.

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Motivated by Başhan's recent work on numerical solutions of the modified Kawahara equation [5], we will suggest sinc collocation method to solve the modified Kawahara equation numerically and acquire travelling wave solution. In order to use sinc collocation method, we first discrete the time derivative term by using finite difference method. More precisely, we obtain the semi-discrete scheme with θ-weighted scheme [20]. When θ is 0, $\frac{1}{2}$, 1, the semi-discrete scheme can be regarded as explicit, Crank-Nicholson and implicit scheme, respectively. After that, we employ sinc collocation method to approximate the space derivatives and derive the fully discrete scheme. The stability analysis is investigated by the matrix analysis technique. According to the numerical results and relative changes of two invariants, we can conclude that sinc collocation method is effective and can be used as an alternative to solve this type of partial differential equation.

The layout of the paper is as follows. In Section II, we briefly review the basic knowledge of sinc function required for our subsequent discussion. Section III is concerned with the discrete scheme of the modified Kawahara. In Section IV, we discuss the stability of the discrete scheme based on matrix analysis. Some numerical examples are presented in Section V. Section VI concludes.

2 Preliminaries

In this section, some basic formulation of sinc function is presented. These properties can be seen in [21–23]

The sinc function is defined on the whole real line by [23]

$$sinc(x) = \begin{cases} \dfrac{sin(\pi x)}{\pi x}, & x \leq 0, \\ 0, & x = 0, \end{cases} \tag{2}$$

and the j-th translate of the sinc function is defined by

$$S(j,h)(x) = sinc(\frac{x - jh}{h}), \tag{3}$$

where j is an integer and h is a positive number.

If a function f is defined on the whole real line, then for $h > 0$ the series [15]

$$C(f,h)(x) = \sum_{j=-\propto}^{\propto} f(jh)S(j,h)(x) \tag{4}$$

is called the Whittaker cardinal expansion of f, whenever this series converges. We also need derivatives of composite sinc functions evaluated at the node x_i. The expressions required for this paper are collected as follows [15]

$$\delta^0_{ji} = [S(j,h)(x)] \mid_{x=x_i} = \begin{cases} 1, & j = i, \\ 0, & j \leq i, \end{cases} \tag{5}$$

$$\delta^1_{ji} = \frac{d}{dx}[S(j,h)(x)]|_{x=x_i} = \begin{cases} 0, & j = i, \\ \dfrac{(-1)^{i-j}}{(i-j)h}, & j \neq i, \end{cases} \tag{6}$$

$$\delta^3_{ji} = \frac{d^3}{dx^3}[S(j,h)(x)]|_{x=x_i} = \begin{cases} 0, & j = i, \\ \dfrac{(-1)^{i-j}}{(i-j)^3h^3}[6 - \pi^2(i-j)^2], & j \neq i, \end{cases} \tag{7}$$

$$\delta^5_{ji} = \frac{d^5}{dx^5}[S(j,h)(x)]|_{x=x_i} = \begin{cases} 0, & j = i, \\ \dfrac{(-1)^{i-j}}{(i-j)^5h^5}[120 - 20\pi^2(i-j)^2 + \pi^4(i-j)^4], & j \neq i. \end{cases} \tag{8}$$

For the higher-order derivatives of sinc functions at the nodes, we refer to [15].

3 Discrete schemes

In this paper, we consider the modified Kawahara equation as

$$u_t + u_x + \alpha u^2 u_x + \beta u_{3x} + \gamma u_{5x} = 0,\ a \le x \le b,\ 0 \le t \le T, \tag{9}$$

with the following initial condition

$$u(x,0) = f(x),\ a \le x \le b, \tag{10}$$

and the boundary conditions

$$u(a,t) = g_a(t),\ \ u(b,t) = g_b(t),\ \ 0 \le t \le T, \tag{11}$$

where λ, β, γ are nonzero arbitrary constants, $f(x)$, $g_a(t)$, $g_b(t)$ are given functions.

Next, the semi-discrete and fully discrete scheme of (9), (10) and (11) are derived by θ-weighted and a collocation method using sinc basis.

3.1 The semi-discrete scheme

Take two positive integers N and M. Set $h = \frac{b-a}{N-1}$, $\tau = \frac{T}{M}$, $x_i = a + (i-1)h,\ 1 \le i \le N$, and $t_k = (k-1)\tau,\ 1 \le k \le M+1$.

We discrete time derivative of the modified Kawahara equation (9) by θ-weighted scheme as

$$\frac{u^{k+1} - u^k}{\tau} + \theta(u_x^{k+1} + \alpha(u^2u_x)^{k+1} + \beta u_{3x}^{k+1} + \gamma u_{5x}^{k+1}) + (1-\theta)(u_x^k + \alpha(u^2u_x)^k + \beta u_{3x}^k + \gamma u_{5x}^k) = 0 \tag{12}$$

where $\theta \in [0,1]$, u^{k+1} and u_x^{k+1} denote $u(x, t_{k+1})$ and $u_x(x, t_{k+1})$, respectively.

Remark When $\theta = 0$, semi-discrete scheme (12) can be considered as explicit scheme, and when $\theta = 1$, scheme (12) degenerates to implicit scheme. For the special case of $\theta = \frac{1}{2}$, scheme (12) is just Crank-Nicholson scheme.

In the following, we use Taylor expansion to approximate the nonlinear term $(u^2u_x)^{k+1}$ in (12). According to Taylor expansion, we obtain

$$\begin{aligned}
&(u^2)^{k+1} = (u^2)^k + \tau[(u^2)_t]^k + O(\tau^2),\\
&u_x^{k+1} = u_x^k + \tau(u_x^k)_t + O(\tau^2) \approx u_x^k + \tau\frac{u_x^{k+1} - u_x^k}{\tau} + O(\tau^2).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
(u^2u_x)^{k+1} &\approx (u^2u_x)^k + \tau(u^2)_t^k u_x^k + \tau(u^2)^k u_{xt}^k\\
&\approx (u^2)^k u_x^k + \tau[\frac{(u^2)^{k+1} - (u^2)^k}{\tau}u_x^k + (u^2)^k\frac{u_x^{k+1} - u_x^k}{\tau}]\\
&= (u^2)^{k+1}u_x^k + (u^2)^k u_x^{k+1} - (u^2)^k u_x^k\\
&\approx (u^2)^k u_x^{k+1} + 2u^k u_x^k u^{k+1} - 2(u^2)^k u_x^k.
\end{aligned}$$

By substituting the above approximation of $(u^2u_x)^{k+1}$ into (12), we can get the semi-discrete scheme of the modified Kawahara equation (9) as

$$\begin{aligned}
&u^{k+1} + \theta\tau(u_x^{k+1} + \alpha(u^2)^k u_x^{k+1} + 2\alpha u^k u_x^k u^{k+1} + \beta u_{3x}^{k+1} + \gamma u_{5x}^{k+1})\\
=&u^k + (\theta-1)\tau u_x^k + (3\theta-1)\alpha\tau(u^2)^k u_x^k + (\theta-1)\beta\tau u_{3x}^k + (\theta-1)\gamma\tau u_{5x}^k.
\end{aligned} \tag{13}$$

3.2 The fully discrete scheme

For convenience, we denote

$$S_j(x) = sinc(\frac{x - (j-1)h - a}{h}), \tag{14}$$

and

$$u(x_i, t_k) = u^k(x_i) \simeq \sum_{j=1}^{N} S_j(x_i)u_j^k. \tag{15}$$

Substituting (15) into (13), we obtain the fully discrete scheme of equation (9) as follows

$$\begin{aligned}&\sum_{j=1}^{N} S_j(x_i)u_j^{k+1} + \theta\tau\sum_{j=1}^{N} S'_j(x_i)u_j^{k+1} + \theta\tau\alpha[\sum_{j=1}^{N} S_j(x_i)u_j^{k}][\sum_{j=1}^{N} S_j(x_i)u_j^{k}][\sum_{j=1}^{N} S'_j(x_i)u_j^{k+1}]\\&+ 2\theta\tau\alpha[\sum_{j=1}^{N} S_j(x_i)u_j^{k}][\sum_{j=1}^{N} S'_j(x_i)u_j^{k}][\sum_{j=1}^{N} S_j(x_i)u_j^{k+1}] + \theta\tau\beta\sum_{j=1}^{N} S'''_j(x_i)u_j^{k+1} + \theta\tau\gamma\sum_{j=1}^{N} S'''''_j(x_i)u_j^{k+1}\\&= \sum_{j=1}^{N} S_j(x_i)u_j^{k} + (\theta-1)\tau\sum_{j=1}^{N} S'_j(x_i)u_j^{k} + (3\theta-1)\tau\alpha[\sum_{j=1}^{N} S_j(x_i)u_j^{k}][\sum_{j=1}^{N} S_j(x_i)u_j^{k}][\sum_{j=1}^{N} S'_j(x_i)u_j^{k}]\\&+ (\theta-1)\tau\beta\sum_{j=1}^{N} S'''_j(x_i)u_j^{k} + (\theta-1)\tau\gamma\sum_{j=1}^{N} S'''''_j(x_i)u_j^{k},\ \ i = 2,3\ldots,N-1.\end{aligned} \tag{16}$$

Due to boundary conditions (11), we can get that

$$\begin{aligned}&\sum_{j=1}^{N} S_j(x_1)u_j^{k+1} = g_a(t^{k+1}),\\&\sum_{j=1}^{N} S_j(x_N)u_j^{k+1} = g_b(t^{k+1}).\end{aligned} \tag{17}$$

4 Stability anslysis

In this section, we discuss the stability of the fully discrete scheme obtained from Section 3.

System (16) and (17) can be rewritten in the matrix form as

$$\mathbf{A}\mathbf{U}^{k+1} = \mathbf{B}\mathbf{U}^{k} + \mathbf{F}^{k+1}, \tag{18}$$

where

$$\begin{aligned}&\mathbf{U}^{k} = (u_1^k, u_2^k, \cdots, u_N^k)^T.\\&\mathbf{A} = \mathbf{I} + \theta\tau(\mathbf{C} + \alpha\mathbf{H} + 2\alpha\mathbf{L} + \beta\mathbf{D} + \gamma\mathbf{E}),\\&\mathbf{B} = \hat{\mathbf{I}} + \tau[(\theta-1)(\mathbf{C} + \beta\mathbf{D} + \gamma\mathbf{E}) + (3\theta-1)\alpha\mathbf{H}],\\&\mathbf{F}^{k+1} = [g_a(t^{k+1}), 0, \ldots, 0, g_b(t^{k+1})]^T,\end{aligned}$$

in which

$$\begin{aligned}&\mathbf{I} = [\delta_{ji}^0 : i = 1,\ldots,N, j = 1,\ldots,N\ and\ 0\ elsewhere]_{N\times N},\\&\hat{\mathbf{I}} = [\delta_{ji}^0 : i = 2,\ldots,N-1, j = 1,\ldots,N\ and\ 0\ elsewhere]_{N\times N},\\&\mathbf{C} = [\delta_{ji}^1 : i = 2,\ldots,N-1, j = 1,\ldots,N\ and\ 0\ elsewhere]_{N\times N},\\&\mathbf{D} = [\delta_{ji}^3 : i = 2,\ldots,N-1, j = 1,\ldots,N\ and\ 0\ elsewhere]_{N\times N},\\&\mathbf{E} = [\delta_{ji}^5 : i = 2,\ldots,N-1, j = 1,\ldots,N\ and\ 0\ elsewhere]_{N\times N},\\&\mathbf{H} = \hat{\mathbf{U}}^k\hat{\mathbf{U}}^k\mathbf{C}.\end{aligned}$$

Here $\hat{\mathbf{U}}^k$ denotes a diagonal matrix, and its diagonal elements are the elements in $\mathbf{U}^k$. Besides, $\mathbf{L}$ is also a diagonal matrix with diagonal entries are the elements of $\hat{\mathbf{U}}^k\mathbf{U}_\mathbf{x}$ in order, where $\mathbf{U}_\mathbf{x}$ is computed by

$$\mathbf{U}_\mathbf{x} = \mathbf{C}\mathbf{U}^k.$$

Let $u_{exa}(x_i, t_k)$ and $u_{num}(x_i, t_k)$ denote the exact and numerical solution at point (x_i, t_k). The error at kth time level is defined by

$$\mathbf{e}^k = |\mathbf{U}_{exa}^k - \mathbf{U}_{num}^k|,$$

where

$$\mathbf{U}_{exa}^{k} = (u_{exa}(x_1,t_k), u_{exa}(x_2,t_k), \cdots, u_{exa}(x_N,t_k))^T,$$
$$\mathbf{U}_{num}^{k} = (u_{num}(x_1,t_k), u_{num}(x_2,t_k), \cdots, u_{num}(x_N,t_k))^T.$$

Therefore, the evolution of error can be expressed as

$$\mathbf{A}\mathbf{e}^{k+1} = \mathbf{B}\mathbf{e}^{k}.$$

Assuming $\mathbf{G} = \mathbf{A}^{-1}\mathbf{B}$ together with the results in [20, 24], we have the necessary condition for the stability of (18). The condition is

$$\rho(\mathbf{G}) \le 1 + c\tau, \tag{19}$$

where $\rho(\mathbf{G})$ denotes the spectral radius of $\mathbf{G}$ and c is a positive number. If the modulus of all eigenvalues of G is less than $1 + c\tau$, condition (19) holds. Now, we discuss the eigenvalues of G.

Let $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6$ denote the eigenvalues of $\hat{\mathbf{I}}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{L}$, respectively. It is obvious that the eigenvalues of $\mathbf{I}$ are all 1 and the eigenvalues of $\mathbf{L}$ are all real numbers.

For convenience, we set

$$\begin{aligned}
&\lambda_2^R = \tau Re(\lambda_2),\ \lambda_2^I = \tau Im(\lambda_2),\\
&\lambda_3^R = \beta\tau Re(\lambda_3),\ \lambda_3^I = \beta\tau Im(\lambda_3),\\
&\lambda_4^R = \gamma\tau Re(\lambda_4),\ \lambda_4^I = \gamma\tau Im(\lambda_4),\\
&\lambda_5^R = \alpha\tau Re(\lambda_5),\ \lambda_5^I = \alpha\tau Im(\lambda_5),
\end{aligned}$$

and

$$\widetilde{\lambda_6} = \alpha\tau\lambda_6.$$

To ensure the stability, it follows from (19) that

$$\Big|\frac{\lambda_1 + (\theta-1)(\lambda_2^R+\lambda_3^R+\lambda_4^R) + (3\theta-1)\lambda_5^R + i[(\theta-1)(\lambda_2^I+\lambda_3^I+\lambda_4^I) + (3\theta-1)\lambda_5^I]}{1 + \theta(\lambda_2^R+\lambda_3^R+\lambda_4^R+\lambda_5^R) + 2\theta\widetilde{\lambda_6} + i[\theta(\lambda_2^I+\lambda_3^I+\lambda_4^I+\lambda_5^I)]}\Big| \le 1,$$

i.e.,

$$\begin{aligned}
&\lambda_1^2 - 1 + (1-2\theta)(|\lambda_2|^2+|\lambda_3|^2+|\lambda_4|^2) + (2\theta-1)(4\theta-1)|\lambda_5|^2 - 4\theta^2(\widetilde{\lambda_6})^2\\
&\le 2\theta(\lambda_2^R+\lambda_3^R+\lambda_4^R+\lambda_5^R+2\widetilde{\lambda_6}) + 2(1-\theta)\lambda_1(\lambda_2^R+\lambda_3^R+\lambda_4^R) + 2(1-3\theta)\lambda_1\lambda_5^R\\
&-(4\theta^2-8\theta+2)(\lambda_2^R+\lambda_3^R+\lambda_4^R)\lambda_5^R - (4\theta^2-8\theta+2)(\lambda_2^I+\lambda_3^I+\lambda_4^I)\lambda_5^I\\
&+4\theta^2(\lambda_2^R+\lambda_3^R+\lambda_4^R+\lambda_5^R)\widetilde{\lambda_6} + 2(2\theta-1)(\lambda_2^R\lambda_3^R+\lambda_2^R\lambda_4^R+\lambda_3^R\lambda_4^R+\lambda_2^I\lambda_3^I+\lambda_2^I\lambda_4^I+\lambda_3^I\lambda_4^I),
\end{aligned} \tag{20}$$

where $|\lambda_2|$, $|\lambda_3|$, $|\lambda_4|$ and $|\lambda_5|$ represent the modulus of λ_2, λ_3, λ_4, λ_5, respectively.

5 Numerical examples

The modified Kawahara equation (9) has travelling wave solution in the following form [5]

$$U(x,t) = \pm\frac{3\beta}{\sqrt{-10\alpha\gamma}}sech^2[\frac{1}{2}\sqrt{\frac{\beta}{-5\gamma}}(x - \frac{25\gamma-4\beta^2}{25\gamma}t)],$$

for $\beta\cdot\gamma < 0$. Since equation (9) has both positive and negative amplitudes solutions, we take the same initial condition and boundary conditions as [5]. The initial condition is

$$f(x) = \pm\frac{3\beta}{\sqrt{-10\alpha\gamma}}sech^2[\frac{1}{2}\sqrt{\frac{\beta}{-5\gamma}}x],$$

and the boundary conditions are taken as

$$g_a(t) = \pm\frac{3\beta}{\sqrt{-10\alpha\gamma}}sech^2[\frac{1}{2}\sqrt{\frac{\beta}{-5\gamma}}(a - \frac{25\gamma - 4\beta^2}{25\gamma}t)],$$
$$g_b(t) = \pm\frac{3\beta}{\sqrt{-10\alpha\gamma}}sech^2[\frac{1}{2}\sqrt{\frac{\beta}{-5\gamma}}(b - \frac{25\gamma - 4\beta^2}{25\gamma}t)].$$

Next, we conduct two numerical examples to prove the efficiency of proposed method for solving the modified Kawahara equation. Numerical accuracy is shown by calculating L_2 and L_∞ error norms, as defined in [5]

$$L_2 = \sqrt{h\sum_{j=1}^{N}|u_j^{exact} - u_j^{numerical}|^2},$$
$$L_\infty = \max_{1\le j\le N}|u_j^{exact} - u_j^{numerical}|.$$

The conserved quantities I_1 and I_2 defined as follows (see [20, 25]) will be also examined

$$I_1 = \int_a^b u dx \simeq h\sum_{j=1}^{N}u_j^k,$$
$$I_2 = \frac{1}{2}\int_a^b u^2 dx \simeq \frac{h}{2}\sum_{j=1}^{N}(u_j^k)^2.$$

The relative changes of invariants are computed as [5]

$$I_s^* = \frac{I_s^{final} - I_s^{initial}}{I_s^{initial}},\ s = 1, 2.$$

Example 1. In the first example, we take $\alpha = 1$, $\beta = 0.01$, $\gamma = -1$. The value of the amplitudes for the travelling wave keeps positive.

We choose $\theta = \frac{1}{2}$, time step $\tau = 1$ and space step $h = 5$. The numerical simulations of the travelling wave solution are plotted in Fig. 1 when t takes different values. Fig. 2 shows the absolute error. Besides, the results of L_2, L_∞, I_1 and I_2 up to time $T = 100$ are shown in Table 1. As can be seen from the table, relative change of invariants I_1^* and I_2^* at time $t = 100$ are 1.40×10^{-5} and 0, respectively. It indicates the conservation of our scheme is satisfactory. The comparison results of the absolute errors at time $t = 100$ by our proposed method with these of four different methods namely multiquadric (MQ), Gaussian (G), inversequadric (IQ), inverse multiquadric (IMQ) in [2] are listed in Table 2.

On the other hand, we take $\theta = \frac{1}{2}$, $N = 40$ and the error norms L_2 and L_∞ with comparison of the differential quadrature method (DQM) [5] are given in Table 3. At the last simulation time $t = 100$, the relative change of invariants I_1^* and I_2^* are 6.48×10^{-5} and 4.73×10^{-8}, respectively. Actually, our method provides more accurate solutions by less spatial decomposition.

Table 1: Error norms and two invariants for Example 1

t	$L_2 \times 10^5$	$L_\infty \times 10^5$	I_1	I_2
0	0	0	0.8484282	0.0026833
20	0.775225	0.187629	0.8485111	0.0026833
40	1.173483	0.188603	0.8485419	0.0026833
60	1.604883	0.217865	0.8485434	0.0026833
80	2.072109	0.293151	0.8485170	0.0026833
100	2.609060	0.373076	0.8484401	0.0026833

Example 2. In this case, we test our method by selecting $\alpha = 1$, $\beta = 1$, $\gamma = -1$, $\theta = \frac{1}{2}$, spatial division $N = 1001$, over the region $[-200, 200]$.

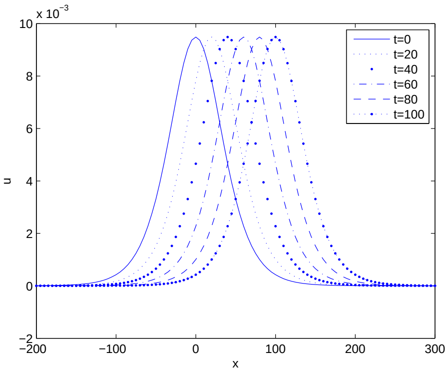

Figure 1: Travelling wave solution for Example 1

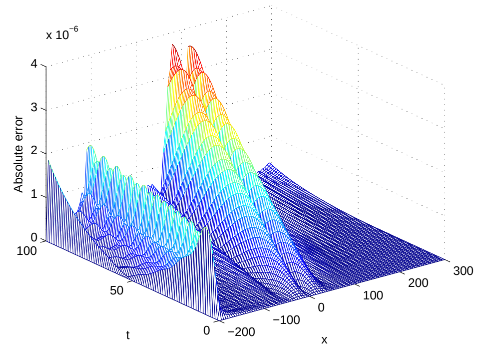

Figure 2: Absolute error for Example 1

Table 2: Comparisons of errors at time $t = 100$ for Example 1

Method	$L_2 \times 10^5$	$L_\infty \times 10^5$	Amplitude	x
Present method	2.609060	0.373076	0.009487	100.00
MQ [2]	3.120253	0.494228	0.009487	100.00
G [2]	3.120140	0.495053	0.009487	100.00
IQ [2]	3.367458	0.495159	0.009487	100.00
IMQ [2]	3.248706	0.599387	0.009486	100.00

Table 3: Comparisons of errors and two invariants for Example 1

Method	N	τ	t	$L_2 \times 10^5$	$L_\infty \times 10^5$	I_1	I_2
			0	0	0	0.848445	0.002683
			20	0.678576	0.123727	0.848512	0.002683
Present	40	1	40	1.113155	0.141538	0.848541	0.002683
			60	1.567010	0.215443	0.848548	0.002683
			80	2.043041	0.268093	0.848538	0.002683
			100	2.561128	0.346649	0.848503	0.002683
			0	-	-	0.848432	0.002683
			20	0.6789	0.1256	0.848465	0.002683
DQM [5]	75	1	40	1.1183	0.1451	0.848473	0.002683
			60	1.5949	0.2136	0.848466	0.002683
			80	2.1657	0.3065	0.848439	0.002683
			100	3.0938	0.3848	0.848370	0.002683

Fig. 3 and Fig. 4 show the numerical solutions when $\tau = 0.0002$ with positive amplitude and negative amplitude up to time $T = 0.5$. Absolute error is graphed in Fig. 5. In Table 4, we compare the error norms with DQM [5], Dual-Petrov-Galerkin method [10] and multi-symplectic Fourier pseudospectral method [26]. As can be seen from the table, our method has higher accuracy. Besides, I_1, I_2 of positive amplitude and negative amplitude are also given in Table 4. By Table 4, it is obvious that the relative changes of invariants I_1^*, I_2^* are close to 0. It means our method has a good conservation.

Furthermore, the numerical simulations of travelling wave with positive amplitude and negative amplitude and the absolute error in a larger time range $T = 30$ with $\tau = 0.0002$ are illustrated in Fig. 6, Fig. 7 and Fig. 8, respectively. The error norms and conserved quantities are compared with DQM in Table 5. It can be seen that our method has smaller error norms, and the relative changes of invariants I_1^*, I_2^* are 0.

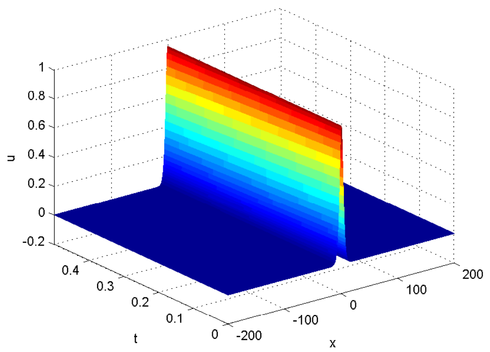

Figure 3: Numerical solution with positive amplitude

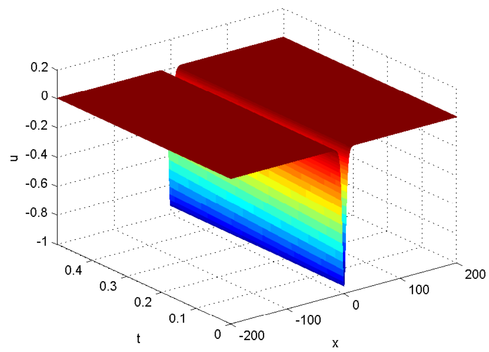

Figure 4: Numerical solution with negative amplitude

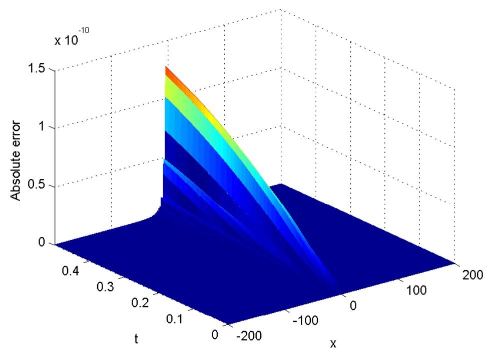

Figure 5: Absolute error for $T = 0.5$

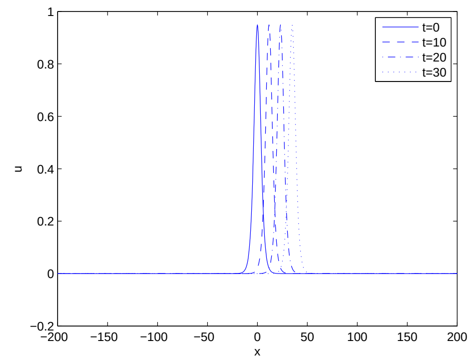

Figure 6: Travelling wave solution with positive amplitude

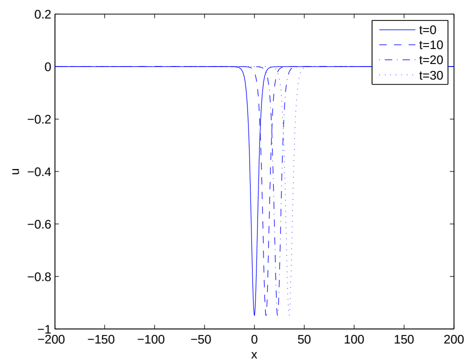

Figure 7: Travelling wave solution with negative amplitude

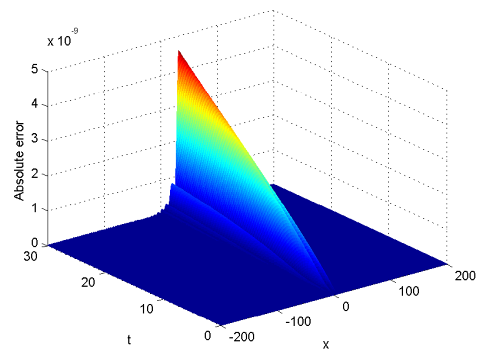

Figure 8: Absolute error for $T = 30$

6 Conclusions

The numerical scheme for the modified Kawahara equation was proposed by the finite difference and sinc collocation method. The stability analysis is given by expressing the fully discrete scheme in matrix form. It shows the scheme is stable if the growth matrix satisfies Von Neumann condition. By comparing the results with these of other methods, we can conclude that the present method is valid and has higher accuracy. Besides, two invariants are also analyzed to illustrate the conservation of present method. All of the numerical results reveal that sinc collocation method can be considered as an efficient numerical method for solving the modified Kawahara equation.

Table 4: Comparisons of error norms and two invariants for Example 2

Method	N	τ	t	L_2	L_∞	I_1 Posi.	I_2 Posi.	I_1 Nega.	I_2 Nega.
Present	1001	0.0001	0.0	0	0	8.485281	2.683282	-8.485281	2.683282
			0.1	1.539×10^{-11}	8.562×10^{-12}	8.485281	2.683282	-8.485281	2.683282
			0.2	2.996×10^{-11}	1.612×10^{-11}	8.485281	2.683282	-8.485281	2.683282
			0.3	4.371×10^{-11}	2.262×10^{-11}	8.485281	2.683282	-8.485281	2.683282
			0.4	5.683×10^{-11}	2.783×10^{-11}	8.485281	2.683282	-8.485281	2.683282
			0.5	6.947×10^{-11}	3.210×10^{-11}	8.485281	2.683282	-8.485281	2.683282
DQM [5]	1001	0.0001	0.0	-	-	8.485276	2.683282	-8.485276	2.683282
			0.1	1.1×10^{-7}	5×10^{-8}	8.485282	2.683281	-8.485282	2.683281
			0.2	1.5×10^{-7}	7×10^{-8}	8.485282	2.683281	-8.485282	2.683281
			0.3	1.9×10^{-7}	8×10^{-8}	8.485281	2.683281	-8.485281	2.683281
			0.4	2.1×10^{-7}	8×10^{-8}	8.485281	2.683282	-8.485281	2.683282
			0.5	2.4×10^{-7}	1×10^{-7}	8.485278	2.683281	-8.485278	2.683281
[10]	1000	0.0001	0.1	1.770×10^{-5}	-	-	-	-	-
			0.2	2.930×10^{-5}	-	-	-	-	-
			0.4	5.190×10^{-5}	-	-	-	-	-
			0.5	6.330×10^{-5}	-	-	-	-	-
MSFPM[26]	1000	0.0001	0.1	5.32×10^{-6}	-	-	-	-	-
			0.2	9.77×10^{-6}	-	-	-	-	-
			0.4	1.827×10^{-5}	-	-	-	-	-
			0.5	2.250×10^{-5}	-	-	-	-	-

Table 5: Error norms and two invariants

Method	N	τ	t	$L_2 \times 10^5$	$L_\infty \times 10^5$	I_1 Posi.	I_2 Posi.	I_1Nega.	I_2 Nega.
Present	1001	0.0001	0	0	0	8.48528137	2.68328157	-8.48528137	2.68328157
			0.5	0.00000695	0.00000321	8.48528137	2.68328157	-8.48528137	2.68328157
			1	0.00001289	0.00000568	8.48528137	2.68328157	-8.48528137	2.68328157
			5	0.00005852	0.00002334	8.48528137	2.68328157	-8.48528137	2.68328157
			10	0.00011445	0.00004263	8.48528137	2.68328157	-8.48528137	2.68328157
			15	0.00016809	0.00006218	8.48528137	2.68328157	-8.48528137	2.68328157
			20	0.00021998	0.00008086	8.48528137	2.68328157	-8.48528137	2.68328157
			25	0.00027066	0.00009851	8.48528137	2.68328157	-8.48528137	2.68328157
			30	0.00032046	0.00011623	8.48528137	2.68328157	-8.48528137	2.68328157
DQM [5]	1001	0.0001	0	-	-	8.485276	2.683282	-8.485276	2.683282
			0.5	0.0236	0.0096	8.485278	2.683281	-8.485278	2.683281
			1	0.0319	0.0123	8.485280	2.683281	-8.485280	2.683281
			5	0.0587	0.0197	8.485278	2.683281	-8.485278	2.683281
			10	0.0818	0.0224	8.485277	2.683282	-8.485277	2.683282
			15	0.0998	0.0229	8.485280	2.683281	-8.485280	2.683281
			20	0.1096	0.0224	8.485277	2.683282	-8.485277	2.683282
			25	0.1296	0.0251	8.485281	2.683282	-8.485281	2.683282
			30	0.1478	0.0422	8.485277	2.683282	-8.485277	2.683282

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References

[1] T. Kakutani. Axially symmetric stagnation-point flow of an electrically conducting fluid under transverse magnetic field. *Journal of the Physical Society of Japan*, 15(1960)(4):688–695.

[2] Y. Dereli and I. Dağ. Numerical solutions of the Kawahara type equations using radial basis functions. *Numerical Methods for Partial Differential Equations*, 28(2012)(2):542–553.

[3] T. Kawahara. Oscillatory solitary waves in dispersive media. *Journal of the Physical Society of Japan*, 33(1972)(1):260–264.

[4] A. Korkmaz and D. Dağ. Crank-Nicolson-Differential quadrature algorithms for the Kawahara equation. *Chaos Solitons and Fractals*, 42(2009)(1):65–73.

[5] A. Başhan. Highly efficient approach to numerical solutions of two different forms of the modified Kawahara equation via contribution of two effective methods. *Mathematics and Computers in Simulation*, 179(2021):111-125.

[6] J. K. Hunter and J. Scheurle. Existence of perturbed solitary wave solutions to a model equation for water waves. *Physica D: Nonlinear Phenomena*, 32(1988)(2):253–268.

[7] T. J. Bridges and G. Derks. Linear instability of solitary wave solutions of the Kawahara equation and its generalizations. *SIAM Journal on Mathematical Analysis*, 33(2002)(6):1356–1378.

[8] D. Zhang. Doubly periodic solutions of the modified Kawahara equation. *Chaos Solitons & Fractals*, 25(2005)(5):1155–1160.

[9] F. D. Araruna, R. A. Capistrano-Filho and G. G. Doronin. Energy decay for the modified Kawahara equation posed in a bounded domain. *Journal of Mathematical Analysis and Applications*, 385(2012)(2):743–756.

[10] J. M. Yuan, J. Shen and J. Wu. A Dual-Petrov-Galerkin method for the Kawahara-type equations. *Journal of Scientific Computing*, 34(2008)(1):48–63.

[11] P. U. Suárez. and J. H. Morales,. Fourier splitting method for Kawahara type equations. *Journal of Computational Methods in Physics*, (2014),Article ID 894956, 4 pages.

[12] F. Stenger. A "Sinc-Galerkin" method of solution of boundary value problems. *Mathematics of Computation*, 33(1979)(145):85–109.

[13] T. S. Carlson, J. Dockery and J. Lund. A sinc–collocation method for initial value problems. *Mathematics of Computation*, 66(1997)(217):215–235.

[14] A. Saadatmandi and M. Dehghan. The use of sinc-collocation method for solving multi-point boundary value problems. *Communications in Nonlinear Science and Numerical Simulation*, 17(2012)(2):593–601.

[15] R. Mokhtari and M. Mohammadi. Numerical solution of GRLW equation using Sinc-collocation method. *Computer Physics Communications*, 181(2010)(7):1266–1274.

[16] A. Fahim et al. Numerical solution of Volterra partial integro-differential equations based on sinc-collocation method. *Advances in Difference Equations*, 2017(2017)(362).

[17] S. Noeiaghdam, M. A. Fariborzi Araghi and S. Abbasbandy. Valid implementation of Sinc-collocation method to solve the fuzzy Fredholm integral equation. *Journal of Computational and Applied Mathematics*, 370(2020):112632.

[18] K. Parand and A. Pirkhedri. Sinc-collocation method for solving astrophysics equations. *New Astronomy*, 15(2010)(6):533–537.

[19] M. Dehghan and A. Saadatmandi. The numerical solution of a nonlinear system of second-order boundary value problems using the sinc-collocation method. *Mathematical and Computer Modelling*, 46(2007)(11-12):1434–1441.

[20] D. Kong, Y. Xu and Z. Zheng. A hybrid numerical method for the KdV equation by finite difference and sinc collocation method. *Applied Mathematics & Computation*, 355(2019):61–72.

[21] J. Lund and K. L. Bowers. Sinc methods for quadrature and differential equations. SIAM. 1992.

[22] K. Maleknejad, R. Mollapourasl and A. Ostadi. Convergence analysis of Sinc-collocation methods for nonlinear Fredholm integral equations with a weakly singular kernel. *Journal of Computational and Applied Mathematics*, 278(2015):1–11.

[23] E. Hashemizadeh and M. Rostami. Numerical solution of Hammerstein integral equations of mixed type using the Sinc-collocation method. *Journal of Computational and Applied Mathematics*, 279(2015)(1):31–39.

[24] J. C. Strikwerda. Finite difference schemes and partial differential equations. SIAM. 2004.

[25] R. P. Malik. On fifth order KdV-type equation. *Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Moscow Region, Russia*,141980(1997):1–12.

[26] Y. Gong, J. Cai and Y. Wang. Multi-symplectic Fourier pseudospectral method for the Kawahara equation. *Communications in Computational Physics*, 16(2014)(1):35–55.